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Elementary Number Theory Problems. Part I

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Summary. In this paper we demonstrate the feasibility of formalizing *recreational mathematics* in Mizar ([1], [2]) drawing examples from W. Sierpinski's book "250 Problems in Elementary Number Theory" [4]. The current work contains proofs of initial ten problems from the chapter devoted to the divisibility of numbers. Included are problems on several levels of difficulty.

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1. Problem 1

One can verify that there exists an integer which is positive.

Now we state the propositions:

(1) Let us consider a positive integer n. Then $n+1 \mid n^{\mathbf{2}}+1$ if and only if $n=1$.

PROOF: If $n+1 \mid n^{\mathbf{2}}+1$, then $n=1$ by [6, (2)]. $\square$

(2) Let us consider integers i, n. If $|i|=n$, then $i=n$ or $i=-n$.

(3) Let us consider a natural number n. Suppose $n \mid 24$. Then

(i) $n=1$, or

(ii) $n=2$, or

(iii) $n=3$, or

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(iv) $n = 4$, or
(v) $n = 6$, or
(vi) $n = 8$, or
(vii) $n = 12$, or
(viii) $n = 24$.

(4) Let us consider an integer t. Suppose $t \mid 24$. Then
(i) $t = -1$, or
(ii) $t = 1$, or
(iii) $t = -2$, or
(iv) $t = 2$, or
(v) $t = -3$, or
(vi) $t = 3$, or
(vii) $t = -4$, or
(viii) $t = 4$, or
(ix) $t = -6$, or
(x) $t = 6$, or
(xi) $t = -8$, or
(xii) $t = 8$, or
(xiii) $t = -12$, or
(xiv) $t = 12$, or
(xv) $t = -24$, or
(xvi) $t = 24$.

The theorem is a consequence of (3) and (2).

2. Problem 2

Now we state the proposition:

(5) Let us consider an integer x. Suppose $x - 3 \mid x^3 - 3$. Then
(i) $x = -21$, or
(ii) $x = -9$, or
(iii) $x = -5$, or
(iv) $x = -3$, or

(v) $x = -1$, or

(vi) $x = 0$, or

(vii) $x = 1$, or

(viii) $x = 2$, or

(ix) $x = 4$, or

(x) $x = 5$, or

(xi) $x = 6$, or

(xii) $x = 7$, or

(xiii) $x = 9$, or

(xiv) $x = 11$, or

(xv) $x = 15$, or

(xvi) $x = 27$.

The theorem is a consequence of (4).

3. Problem 3

Now we state the proposition:

(6) $\{n$, where n is a positive integer : $5 \mid 4 \cdot (n^2) + 1$ and $13 \mid 4 \cdot (n^2) + 1\}$ is infinite.

Proof: Set $S = \{n$, where n is a positive integer : $5 \mid 4 \cdot (n^2) + 1$ and $13 \mid 4 \cdot (n^2) + 1\}$. Define $\mathcal{F}$(natural number) $= 65 \cdot \$_1 + 56$. Consider f being a many sorted set indexed by $\mathbb{N}$ such that for every element n of $\mathbb{N}$, $f(n) = \mathcal{F}(n)$. Set $R = \operatorname{rng} f$. $R \subseteq S$. For every element m of $\mathbb{N}$, there exists an element n of $\mathbb{N}$ such that $n \geqslant m$ and $n \in R$. $\square$

4. Problem 4

Now we state the proposition:

(7) Let us consider a positive integer n. Then $169 \mid 3^{3 \cdot n+3} - 26 \cdot n - 27$.

Proof: Reconsider $k = n$ as a natural number. Define $\mathcal{P}$[natural number] $\equiv$ $169 \mid 3^{3 \cdot \$_1+3} - 26 \cdot \$_1 - 27$. For every natural number k such that $1 \leqslant k$ holds $\mathcal{P}[k]$. $\square$

5. Problem 5

Now we state the proposition:

(8) Let us consider a natural number k. Then $19 \mid 2^{2^{6 \cdot k+2}} + 3$.

6. Problem 6 (due to Kraitchik)

Now we state the proposition:

(9) $13 \mid 2^{70} + 3^{70}$.

7. Problem 7

Now we state the propositions:

(10) $11 \cdot 31 \cdot 61 \mid 20^{15} - 1$.

(11) Let us consider an integer a, and a natural number m. Then $a-1 \mid a^m - 1$.
Proof: Define $\mathcal{P}$[natural number] $\equiv a - 1 \mid a^{\$_1} - 1$. For every natural number k, $\mathcal{P}[k]$. $\square$

(12) Let us consider a natural number a, a positive integer m, and a finite 0-sequence f of $\mathbb{Z}$. Suppose $a > 1$ and $\operatorname{len} f = m - 1$ and for every natural number i such that $i \in \operatorname{dom} f$ holds $f(i) = a^{i+1} - 1$. Then $a^m - 1 \operatorname{div}(a - 1) = \sum f + m$.
Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite 0-sequence f of $\mathbb{Z}$ such that $\operatorname{len} f = \$_1$ and for every natural number i such that $i \in \operatorname{dom} f$ holds $f(i) = a^{i+1} - 1$ holds $a^{\$_1+1} - 1 \operatorname{div}(a - 1) = \sum f + (\$_1 + 1)$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. $\square$

8. Problem 8

Now we state the proposition:

(13) Let us consider a positive integer m, and a natural number a. Suppose $a > 1$. Then $\gcd(a^m - 1 \operatorname{div}(a - 1), a - 1) = \gcd(a - 1, m)$.
Proof: Reconsider $m_0 = m$ as a natural number. Reconsider $m_1 = m_0 - 1$ as a natural number. Define $\mathcal{F}$(natural number) $= a^{\$_1+1} - 1$. Consider f being a finite 0-sequence such that $\operatorname{len} f = m_1$ and for every natural number i such that $i \in m_1$ holds $f(i) = \mathcal{F}(i)$ from [5, Sch.2]. $\operatorname{rng} f \subseteq \mathbb{Z}$. $a^m - 1 \operatorname{div}(a - 1) = \sum f + m$. $\square$

9. Problem 9

Now we state the propositions:

(14) Let us consider finite 0-sequences s_1, s_2 of $\mathbb{N}$, and a natural number n. Suppose $\operatorname{len} s_1 = n+1$ and for every natural number i such that $i \in \operatorname{dom} s_1$ holds $s_1(i) = i^5$ and $\operatorname{len} s_2 = n+1$ and for every natural number i such that $i \in \operatorname{dom} s_2$ holds $s_2(i) = i^3$. Then $\sum s_2 \mid 3 \cdot (\sum s_1)$.

Proof: Define $\mathcal{F}(\text{natural number}) = {\$_1}^3$. Consider S_2 being a sequence of real numbers such that for every natural number i, $S_2(i) = \mathcal{F}(i)$. Define $\mathcal{G}(\text{natural number}) = {\$_1}^5$.

Consider S_1 being a sequence of real numbers such that for every natural number i, $S_1(i) = \mathcal{G}(i)$. $\square$

(15) Let us consider integers a, b, and a positive natural number m. Then $\sum\langle\binom{m}{0}a^0b^m, \ldots, \binom{m}{m}a^mb^0\rangle = a^m + b^m + \sum(\langle\binom{m}{0}a^0b^m, \ldots, \binom{m}{m}a^mb^0\rangle\restriction m)_{\downharpoonleft 1}$.

(16) Let us consider natural numbers n, k. If n is odd, then $n \mid k^n + (n-k)^n$. The theorem is a consequence of (15).

10. Problem 10

Now we state the proposition:

(17) Let us consider a finite sequence s of elements of $\mathbb{N}$, and a natural number n. Suppose $n > 1$ and $\operatorname{len} s = n - 1$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $s(i) = i^n$. If n is odd, then $n \mid \sum s$.

Proof: $\operatorname{rng}(s + \operatorname{Rev}(s)) \subseteq \mathbb{N}$. If n is odd, then $n \mid \sum s$ by [3, (3)]. $\square$

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