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Dynamic Programming for the Subset Sum Problem[1]

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Summary. The subset sum problem is a basic problem in the field of theoretical computer science, especially in the complexity theory [3]. The input is a sequence of positive integers and a target positive integer. The task is to determine if there exists a subsequence of the input sequence with sum equal to the target integer. It is known that the problem is NP-hard [2] and can be solved by dynamic programming in pseudo-polynomial time [1]. In this article we formalize the recurrence relation of the dynamic programming.

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1. Preliminaries

Let x be a finite sequence and I be a set. The functor $\mathrm{Seq}(x, I)$ yielding a finite sequence is defined by the term

(Def. 1) $\mathrm{Seq}(x \restriction I)$.

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Let D be a set and x be a D-valued finite sequence. One can check that $\mathrm{Seq}(x, I)$ is D-valued.

Let x be a real-valued finite sequence. Let us observe that $\mathrm{Seq}(x, I)$ is real-valued.

Let D be a set, x be a D-valued finite sequence, and i be a natural number. Let us observe that $x{\restriction}i$ is D-valued as a finite sequence-like function.

Let x be a real-valued finite sequence. One can verify that $x{\restriction}i$ is real-valued as a finite sequence-like function.

2. Summing Up Finite Sequences

Let x be an $\mathbb{R}$-valued finite sequence and a be a real number. We say that the sum of x is equal to a if and only if

(Def. 2) there exists a set I such that $I \subseteq \mathrm{dom}\, x$ and $\sum \mathrm{Seq}(x, I) = a$.

The functor Q_x yielding a function from $\mathrm{Seg}\,\mathrm{len}\, x \times \mathbb{R}$ into *Boolean* is defined by

(Def. 3) for every natural number i and for every real number s such that $1 \leqslant i \leqslant \mathrm{len}\, x$ holds if the sum of $x{\restriction}i$ is equal to s, then $it(i, s) = true$ and if the sum of $x{\restriction}i$ is not equal to s, then $it(i, s) = false$.

Let A be a subset of $\mathbb{N}$, i be a natural number, s be a real number, and f be a function from $A \times \mathbb{R}$ into *Boolean*. Let us note that $f(i, s)$ is Boolean.

Let a, b be objects. The functor $a =_{\Sigma} b$ yielding an object is defined by the term

(Def. 4) $(a = b \rightarrow true, false)$.

Note that $a =_{\Sigma} b$ is Boolean.

Let a, b be extended reals. The functor $a \leqslant_{\Sigma} b$ yielding an object is defined by the term

(Def. 5) $(a > b \rightarrow false, true)$.

Let us note that $a \leqslant_{\Sigma} b$ is Boolean.

Now we state the propositions:

(1) Let us consider a real number s, and an $\mathbb{R}$-valued finite sequence x. Suppose $1 \leqslant \mathrm{len}\, x$. Then $\mathrm{Q}_x(1, s) = (x(1) =_{\Sigma} s) \vee (s =_{\Sigma} 0)$.

(2) Let us consider functions f, g, and sets X, Y. Suppose $\mathrm{rng}\, g \subseteq X$. Then $(f{\restriction}(X \cup Y)) \cdot g = (f{\restriction}X) \cdot g$.

Proof: For every object i, $i \in \mathrm{dom}((f{\restriction}(X \cup Y)) \cdot g)$ iff $i \in \mathrm{dom}\, g$ and $g(i) \in \mathrm{dom}(f{\restriction}X)$. For every object i such that $i \in \mathrm{dom}((f{\restriction}(X \cup Y)) \cdot g)$ holds $((f{\restriction}(X \cup Y)) \cdot g)(i) = (f{\restriction}X)(g(i))$. $\square$

(3) Let us consider an $\mathbb{R}$-valued finite sequence x, a natural number i, and a set I_0. Suppose $I_0 \subseteq \operatorname{Seg} i$ and $\operatorname{Seg}(i+1) \subseteq \operatorname{dom} x$. Then $\operatorname{Seq}(x \restriction (i+1), I_0 \cup \{i+1\}) = \operatorname{Seq}(x \restriction i, I_0) \frown \langle x(i+1) \rangle$. The theorem is a consequence of (2).

(4) Let us consider a real-valued finite sequence x. If $x \neq \emptyset$ and x is positive, then $0 < \sum x$.

(5) Let us consider a real-valued finite sequence x, and a natural number i. Suppose x is positive and $1 \leqslant i \leqslant \operatorname{len} x$. Then

  (i) $x \restriction i$ is positive, and

  (ii) $x \restriction i \neq \emptyset$.

PROOF: For every natural number j such that $j \in \operatorname{dom}(x \restriction i)$ holds $0 < (x \restriction i)(j)$ by [4, (112)]. $\square$

(6) Let us consider a real-valued finite sequence x, and a set I. Suppose x is positive and $I \subseteq \operatorname{dom} x$ and $I \neq \emptyset$. Then

  (i) $\operatorname{Seq}(x, I)$ is positive, and

  (ii) $\operatorname{Seq}(x, I) \neq \emptyset$.

PROOF: For every natural number j such that $j \in \operatorname{dom}(\operatorname{Seq}(x, I))$ holds $0 < (\operatorname{Seq}(x, I))(j)$. $\square$

3. RECURRENCE RELATION OF DYNAMIC PROGRAMMING FOR THE SUBSET SUM PROBLEM

Now we state the proposition:

(7) Let us consider an $\mathbb{R}$-valued finite sequence x. Suppose x is positive. Let us consider a natural number i, and a real number s. Suppose $1 \leqslant i < \operatorname{len} x$. Then $\mathrm{Q}_x(i+1, s) = \mathrm{Q}_x(i, s) \vee (x(i+1) \leqslant_\Sigma s) \wedge \mathrm{Q}_x(i, s - x(i+1))$.
PROOF: $\mathrm{Q}_x(i+1, s) = \mathit{true}$ iff $\mathrm{Q}_x(i, s) \vee (x(i+1) \leqslant_\Sigma s) \wedge \mathrm{Q}_x(i, s - x(i+1)) = \mathit{true}$. $\square$

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REFERENCES

[1] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman & Co., New York, NY, USA, 1979. ISBN 0716710447.

[2] Richard M. Karp. Reducibility among combinatorial problems. In Miller et al. [3], pages 85–103. ISBN 978-1-4684-2001-2. doi:10.1007/978-1-4684-2001-2_9.

[3] Raymond E. Miller, James W. Thatcher, and Jean D. Bohlinger, editors. *Complexity of Computer Computations*, 1972. Springer US. ISBN 978-1-4684-2001-2. doi:10.1007/978-1-4684-2001-2_9.

[4] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

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