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Klein-Beltrami model. Part IV

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Summary. Timothy Makarios (with Isabelle/HOL[1]) and John Harrison (with HOL-Light[2]) shown that "the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski's axioms except his Euclidean axiom" [2],[3],[4, 5].

With the Mizar system [1] we use some ideas taken from Tim Makarios's MSc thesis [10] to formalize some definitions and lemmas necessary for the verification of the independence of the parallel postulate. In this article, which is the continuation of [8], we prove that our constructed model satisfies the axioms of segment construction, the axiom of betweenness identity, and the axiom of Pasch due to Tarski, as formalized in [11] and related Mizar articles.

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1. Preliminaries

Let us consider real numbers a, b. Now we state the propositions:

(1) If $a \neq b$, then $1 - \frac{a}{a-b} = -\frac{b}{a-b}$.

(2) If $0 < a \cdot b$, then $0 < \frac{a}{b}$.

Now we state the propositions:

(3) Let us consider real numbers a, b, c. Suppose $0 \leqslant a \leqslant 1$ and $0 < b \cdot c$. Then $0 \leqslant \frac{a \cdot c}{(1-a) \cdot b + a \cdot c} \leqslant 1$.

[1] `https://www.isa-afp.org/entries/Tarskis_Geometry.html`

[2] `https://github.com/jrh13/hol-light/blob/master/100/independence.ml`

(4) Let us consider real numbers a, b, c. Suppose $(1-a)\cdot b+a\cdot c\neq 0$. Then $1-\frac{a\cdot c}{(1-a)\cdot b+a\cdot c}=\frac{(1-a)\cdot b}{(1-a)\cdot b+a\cdot c}$.

(5) Let us consider real numbers a, b, c, d. If $b\neq 0$, then $\frac{\frac{a\cdot b}{c}\cdot d}{b}=\frac{a\cdot d}{c}$.

(6) Let us consider an element u of $\mathcal{E}^3_{\mathrm{T}}$. Then $u=[u(1),u(2),u(3)]$.

(7) Let us consider an element P of the BK-model. Then BK-to-REAL2$(P)\in$ TarskiEuclid2Space.

Let P be a point of BK-model-Plane. The functor BKtoT2(P) yielding a point of TarskiEuclid2Space is defined by

(Def. 1) there exists an element p of the BK-model such that $P=p$ and $it=$ BK-to-REAL2(p).

Let P be a point of TarskiEuclid2Space. Assume $\hat{P}\in$ the inside of circle(0,0,1). The functor T2toBK(P) yielding a point of BK-model-Plane is defined by

(Def. 2) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $it=$ the direction of u and $(u)_{\mathbf{3}}=1$ and $\hat{P}=[(u)_{\mathbf{1}},(u)_{\mathbf{2}}]$.

Let us consider a point P of BK-model-Plane. Now we state the propositions:

(8) $\widehat{\mathrm{BKtoT2}(P)}\in$ the inside of circle(0,0,1).

(9) T2toBK(BKtoT2(P)) $=P$.

(10) Let us consider a point P of TarskiEuclid2Space. Suppose $\hat{P}$ is an element of the inside of circle(0,0,1). Then BKtoT2(T2toBK(P)) $=P$.

(11) Let us consider a point P of BK-model-Plane, and an element p of the BK-model. Suppose $P=p$. Then

  (i) BKtoT2$(P)=$ BK-to-REAL2(p), and

  (ii) $\widehat{\mathrm{BKtoT2}(P)}=$ BK-to-REAL2(p).

(12) Let us consider points P, Q, R of BK-model-Plane, and points P_2, Q_2, R_2 of TarskiEuclid2Space. Suppose $P_2=$ BKtoT2(P) and $Q_2=$ BKtoT2(Q) and $R_2=$ BKtoT2(R). Then Q lies between P and R if and only if Q_2 lies between P_2 and R_2. The theorem is a consequence of (11).

(13) Let us consider elements P, Q of $\mathcal{E}^2_{\mathrm{T}}$. If $P\neq Q$, then $P(1)\neq Q(1)$ or $P(2)\neq Q(2)$.

(14) Let us consider real numbers a, b, and elements P, Q of $\mathcal{E}^2_{\mathrm{T}}$. If $P\neq Q$ and $(1-a)\cdot P+a\cdot Q=(1-b)\cdot P+b\cdot Q$, then $a=b$. The theorem is a consequence of (13).

(15) Let us consider points P, Q of BK-model-Plane. If $\widehat{\mathrm{BKtoT2}(P)}=\widehat{\mathrm{BKtoT2}(Q)}$, then $P=Q$. The theorem is a consequence of (11).

Let P, Q, R be points of BK-model-Plane. Assume Q lies between P and R and $P\neq R$. The functor length(P,Q,R) yielding a real number is defined by

(Def. 3) $0 \leqslant it \leqslant 1$ and $\mathrm{BKto\hat{T}2}(Q) = (1 - it) \cdot (\mathrm{BKto\hat{T}2}(P)) + it \cdot (\mathrm{BKto\hat{T}2}(R))$.

Let us consider points P, Q of BK-model-Plane. Now we state the propositions:

(16) (i) P lies between P and Q, and

(ii) Q lies between P and Q.

The theorem is a consequence of (12).

(17) If $P \neq Q$, then $\mathrm{length}(P, P, Q) = 0$ and $\mathrm{length}(P, Q, Q) = 1$. The theorem is a consequence of (16).

(18) Let us consider a square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $N = \langle\langle 2, 0, -1\rangle, \langle 0, \sqrt{3}, 0\rangle, \langle 1, 0, -2\rangle\rangle$. Then

(i) $\mathrm{Det}\, N = (-3) \cdot \sqrt{3}$, and

(ii) N is invertible.

(19) Let us consider elements a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32}, b_{33}, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 of $\mathbb{R}_{\mathrm{F}}$, and square matrices A, B over $\mathbb{R}_{\mathrm{F}}$ of dimension 3.

Suppose $A = \langle\langle a_{11}, a_{12}, a_{13}\rangle, \langle a_{21}, a_{22}, a_{23}\rangle, \langle a_{31}, a_{32}, a_{33}\rangle\rangle$ and $B = \langle\langle b_{11}, b_{12}, b_{13}\rangle, \langle b_{21}, b_{22}, b_{23}\rangle, \langle b_{31}, b_{32}, b_{33}\rangle\rangle$ and $a_1 = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}$ and $a_2 = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}$ and $a_3 = a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33}$ and $a_4 = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31}$.

Suppose $a_5 = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32}$ and $a_6 = a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33}$ and $a_7 = a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31}$ and $a_8 = a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32}$ and $a_9 = a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33}$.

Then $A \cdot B = \langle\langle a_1, a_2, a_3\rangle, \langle a_4, a_5, a_6\rangle, \langle a_7, a_8, a_9\rangle\rangle$.

Let us consider square matrices N_1, N_2 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Now we state the propositions:

(20) Suppose $N_1 = \langle\langle 2, 0, -1\rangle, \langle 0, \sqrt{3}, 0\rangle, \langle 1, 0, -2\rangle\rangle$ and $N_2 = \langle\langle \frac{2}{3}, 0, -\frac{1}{3}\rangle, \langle 0, \frac{1}{\sqrt{3}}, 0\rangle, \langle \frac{1}{3}, 0, -\frac{2}{3}\rangle\rangle$. Then $N_1 \cdot N_2 = \langle\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle\rangle$. The theorem is a consequence of (19).

(21) Suppose $N_2 = \langle\langle 2, 0, -1\rangle, \langle 0, \sqrt{3}, 0\rangle, \langle 1, 0, -2\rangle\rangle$ and $N_1 = \langle\langle \frac{2}{3}, 0, -\frac{1}{3}\rangle, \langle 0, \frac{1}{\sqrt{3}}, 0\rangle, \langle \frac{1}{3}, 0, -\frac{2}{3}\rangle\rangle$. Then $N_1 \cdot N_2 = \langle\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle\rangle$. The theorem is a consequence of (19).

(22) Suppose $N_1 = \langle\langle 2, 0, -1\rangle, \langle 0, \sqrt{3}, 0\rangle, \langle 1, 0, -2\rangle\rangle$ and $N_2 = \langle\langle \frac{2}{3}, 0, -\frac{1}{3}\rangle, \langle 0, \frac{1}{\sqrt{3}}, 0\rangle, \langle \frac{1}{3}, 0, -\frac{2}{3}\rangle\rangle$. Then N_1 is inverse of N_2. The theorem is a consequence of (20) and (21).

Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Now we state the propositions:

(23) Suppose $N = \langle\langle \frac{2}{3}, 0, -\frac{1}{3}\rangle, \langle 0, \frac{1}{\sqrt{3}}, 0\rangle, \langle \frac{1}{3}, 0, -\frac{2}{3}\rangle\rangle$. Then (the homography of N)$^\circ$(the absolute) $\subseteq$ the absolute.
PROOF: (The homography of N)$^\circ$(the absolute) $\subseteq$ the absolute by [7, (89)], [9, (7)]. $\square$

(24) Suppose $N = \langle\langle 2, 0, -1\rangle, \langle 0, \sqrt{3}, 0\rangle, \langle 1, 0, -2\rangle\rangle$. Then (the homography of N)$^\circ$(the absolute) = the absolute.
PROOF: (The homography of N)$^\circ$(the absolute) $\subseteq$ the absolute.
The absolute $\subseteq$ (the homography of N)$^\circ$(the absolute) by [6, (19)], (22), (23). $\square$

(25) Let us consider real numbers a, b, r, and elements P, Q, R of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $Q \in \mathcal{L}(P, R)$ and P, $R \in$ the inside of circle(a,b,r). Then $Q \in$ the inside of circle(a,b,r).

(26) Let us consider non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose the direction of u = the direction of v and $u(3) \neq 0$ and $u(3) = v(3)$. Then $u = v$.

(27) Let us consider an element R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, elements P, Q of the BK-model, non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$, and a real number r. Suppose $0 \leqslant r \leqslant 1$ and P = the direction of u and Q = the direction of v and R = the direction of w and $u(3) = 1$ and $v(3) = 1$ and $w = r \cdot u + (1 - r) \cdot v$. Then R is an element of the BK-model.
PROOF: Consider u_2 being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u_2 = P$ and $u_2(3) = 1$ and BK-to-REAL2(P) = $[u_2(1), u_2(2)]$. $u = u_2$. Reconsider $r_4 = [u_2(1), u_2(2)]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. Consider v_2 being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $v_2 = Q$ and $v_2(3) = 1$ and BK-to-REAL2(Q) = $[v_2(1), v_2(2)]$. $v = v_2$. Reconsider $r_6 = [v_2(1), v_2(2)]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. Reconsider $r_8 = [w(1), w(2)]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. $r_8 = r \cdot r_4 + (1 - r) \cdot r_6$. Consider R_3 being an element of $\mathcal{E}^2_{\mathrm{T}}$ such that $R_3 = r_8$ and REAL2-to-BK(r_8) = the direction of $[(R_3)_{\mathbf{1}}, (R_3)_{\mathbf{2}}, 1]$. $\square$

(28) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, points P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and P = the direction of u and Q = the direction of v and Q = (the homography of N)(P) and $u(3) = 1$. Then there exists a non zero real number a such that

(i) $v(1) = a \cdot (n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13})$, and

(ii) $v(2) = a \cdot (n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23})$, and

(iii) $v(3) = a \cdot (n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33})$.

(29) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element

P of the BK-model, a point Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $P =$ the direction of u and $Q =$ the direction of v and $Q =$ (the homography of N)(P) and $u(3) = 1$ and $v(3) = 1$. Then

(i) $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \neq 0$, and

(ii) $v(1) = \frac{n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}$, and

(iii) $v(2) = \frac{n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}$.

The theorem is a consequence of (28).

(30) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, an element h of the subgroup of K-isometries, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element P of the BK-model, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $h =$ the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $P =$ the direction of u and $u(3) = 1$. Then $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \neq 0$. The theorem is a consequence of (29).

(31) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element P of the absolute, a point Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $P =$ the direction of u and $Q =$ the direction of v and $Q =$ (the homography of N)(P) and $u(3) = 1$ and $v(3) = 1$. Then

(i) $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \neq 0$, and

(ii) $v(1) = \frac{n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}$, and

(iii) $v(2) = \frac{n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}}{n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}}$.

The theorem is a consequence of (28).

(32) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, an element h of the subgroup of K-isometries, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element P of the absolute, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $h =$ the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $P =$ the direction of u and $u(3) = 1$. Then $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \neq 0$. The theorem is a consequence of (31).

(33) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, an element h of the subgroup of K-isometries, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element P of the BK-model, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $h =$ the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $P =$ the direction

of u and $u(3) = 1$. Then (the homography of N)(P) = the direction of $[\frac{n_{11}\cdot u(1)+n_{12}\cdot u(2)+n_{13}}{n_{31}\cdot u(1)+n_{32}\cdot u(2)+n_{33}}, \frac{n_{21}\cdot u(1)+n_{22}\cdot u(2)+n_{23}}{n_{31}\cdot u(1)+n_{32}\cdot u(2)+n_{33}}, 1]$. The theorem is a consequence of (29).

(34) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, an element h of the subgroup of K-isometries, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, an element P of the absolute, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose h = the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and P = the direction of u and $u(3) = 1$. Then (the homography of N)(P) = the direction of $[\frac{n_{11}\cdot u(1)+n_{12}\cdot u(2)+n_{13}}{n_{31}\cdot u(1)+n_{32}\cdot u(2)+n_{33}}, \frac{n_{21}\cdot u(1)+n_{22}\cdot u(2)+n_{23}}{n_{31}\cdot u(1)+n_{32}\cdot u(2)+n_{33}}, 1]$. The theorem is a consequence of (31).

(35) Let us consider a subset A of $\mathcal{E}^3_{\mathrm{T}}$, a convex, non empty subset B of $\mathcal{E}^2_{\mathrm{T}}$, a real number r, and an element x of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $A = \{x$, where x is an element of $\mathcal{E}^3_{\mathrm{T}}$: $[(x)_{\mathbf{1}}, (x)_{\mathbf{2}}] \in B$ and $(x)_{\mathbf{3}} = r\}$. Then A is non empty and convex.

(36) Let us consider elements n_1, n_2, n_3 of $\mathbb{R}_{\mathrm{F}}$, and elements n, u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $n = \langle n_1, n_2, n_3\rangle$ and $u(3) = 1$. Then $|(n, u)| = n_1 \cdot u(1) + n_2 \cdot u(2) + n_3$.

(37) Let us consider a convex, non empty subset A of $\mathcal{E}^3_{\mathrm{T}}$, and elements n, u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose for every element w of $\mathcal{E}^3_{\mathrm{T}}$ such that $w \in A$ holds $|(n, w)| \neq 0$ and u, $v \in A$. Then $0 < |(n, u)| \cdot |(n, v)|$.

PROOF: Set $x = |(n, u)|$. Set $y = |(n, v)|$. Reconsider $l = \frac{x}{x-y}$ as a non zero real number. Reconsider $w = l \cdot v + (1 - l) \cdot u$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. $x \neq y$. $1 - l = -\frac{y}{x-y}$. $|(n, w)| = 0$. □

Let us consider elements n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$ and elements u, v of $\mathcal{E}^2_{\mathrm{T}}$. Now we state the propositions:

(38) Suppose u, $v \in$ the inside of circle(0,0,1) and for every element w of $\mathcal{E}^2_{\mathrm{T}}$ such that $w \in$ the inside of circle(0,0,1) holds $n_{31}\cdot w(1)+n_{32}\cdot w(2)+n_{33} \neq 0$. Then $0 < (n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}) \cdot (n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33})$. The theorem is a consequence of (35), (36), and (37).

(39) Suppose $u \in$ the inside of circle(0,0,1) and $v \in$ circle$(0, 0, 1)$ and for every element w of $\mathcal{E}^2_{\mathrm{T}}$ such that $w \in$ the closed inside of circle(0,0,1) holds $n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33} \neq 0$. Then $0 < (n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}) \cdot (n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33})$. The theorem is a consequence of (35), (36), and (37).

(40) Let us consider real numbers l, r, elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$, and real numbers n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}, m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9.

Suppose $m_3 \neq 0$ and $m_6 \neq 0$ and $m_9 \neq 0$ and $r = \frac{l \cdot m_6}{(1-l) \cdot m_3 + l \cdot m_6}$ and $(1-l) \cdot m_3 + l \cdot m_6 \neq 0$ and $w = (1-l) \cdot u + l \cdot v$ and $m_1 = n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}$ and $m_2 = n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}$ and $m_3 = n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}$ and $m_4 = n_{11} \cdot v(1) + n_{12} \cdot v(2) + n_{13}$.

Suppose $m_5 = n_{21} \cdot v(1) + n_{22} \cdot v(2) + n_{23}$ and $m_6 = n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}$ and $m_7 = n_{11} \cdot w(1) + n_{12} \cdot w(2) + n_{13}$ and $m_8 = n_{21} \cdot w(1) + n_{22} \cdot w(2) + n_{23}$ and $m_9 = n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33}$.

Then $(1-r) \cdot [\frac{m_1}{m_3}, \frac{m_2}{m_3}, 1] + r \cdot [\frac{m_4}{m_6}, \frac{m_5}{m_6}, 1] = [\frac{m_7}{m_9}, \frac{m_8}{m_9}, 1]$. The theorem is a consequence of (4) and (5).

(41) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, an element h of the subgroup of K-isometries, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, and an element P of the BK-model. Suppose $h =$ the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$. Then (the homography of $N)(P) =$ the direction of $[\frac{n_{11} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{1}} + n_{12} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{2}} + n_{13}}{n_{31} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{1}} + n_{32} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{2}} + n_{33}}, \frac{n_{21} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{1}} + n_{22} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{2}} + n_{23}}{n_{31} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{1}} + n_{32} \cdot (\text{BK-to-REAL2}(P))_{\mathbf{2}} + n_{33}}, 1]$. The theorem is a consequence of (33).

(42) Let us consider an element h of the subgroup of K-isometries, an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, and an element u_2 of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $h =$ the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $u_2 \in$ the inside of circle(0,0,1). Then $n_{31} \cdot u_2(1) + n_{32} \cdot u_2(2) + n_{33} \neq 0$. The theorem is a consequence of (30).

(43) Let us consider a positive real number r, and an element u of $\mathcal{E}^2_{\mathrm{T}}$. If $u \in \text{circle}(0, 0, r)$, then u is not zero.

(44) Let us consider an element h of the subgroup of K-isometries, an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} of $\mathbb{R}_{\mathrm{F}}$, and an element u_2 of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $h =$ the homography of N and $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $u_2 \in$ the closed inside of circle(0,0,1). Then $n_{31} \cdot u_2(1) + n_{32} \cdot u_2(2) + n_{33} \neq 0$. The theorem is a consequence of (30), (43), and (32).

(45) Let us consider real numbers a, b, c, d, e, f, r. Suppose $(1-r) \cdot [a, b, 1] + r \cdot [c, d, 1] = [e, f, 1]$. Then $(1-r) \cdot [a, b] + r \cdot [c, d] = [e, f]$.

(46) Let us consider points P, Q, R, P', Q', R' of BK-model-Plane, elements p, q, r, p', q', r' of the BK-model, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $h =$ the homography of N and Q lies between P and R and $P = p$ and $Q = q$ and $R = r$ and $p' =$ (the homography of $N)(p)$ and $q' =$ (the homography of $N)(q)$ and $r' =$ (the homography of $N)(r)$ and

$P' = p'$ and $Q' = q'$ and $R' = r'$. Then Q' lies between P' and R'.

PROOF: Consider n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} being elements of $\mathbb{R}_{\mathrm{F}}$ such that $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$. Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = p$ and $u(3) = 1$ and BK-to-REAL2$(p) = [u(1), u(2)]$. Consider v being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $v = r$ and $v(3) = 1$ and BK-to-REAL2$(r) = [v(1), v(2)]$. Consider w being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $w = q$ and $w(3) = 1$ and BK-to-REAL2$(q) = [w(1), w(2)]$.

Reconsider $m_1 = n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}$, $m_2 = n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}$, $m_3 = n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}$, $m_4 = n_{11} \cdot v(1) + n_{12} \cdot v(2) + n_{13}$, $m_5 = n_{21} \cdot v(1) + n_{22} \cdot v(2) + n_{23}$, $m_6 = n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}$, $m_7 = n_{11} \cdot w(1) + n_{12} \cdot w(2) + n_{13}$, $m_8 = n_{21} \cdot w(1) + n_{22} \cdot w(2) + n_{23}$, $m_9 = n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33}$ as a real number. BKtoT2(P) = BK-to-REAL2(p) and BKto$\hat{\mathrm{T}}$2(P) = BK-to-REAL2(p) and BKtoT2(Q) = BK-to-REAL2(q) and BKto$\hat{\mathrm{T}}$2(Q) = BK-to-REAL2(q) and BKtoT2(R) = BK-to-REAL2(r) and BKto$\hat{\mathrm{T}}$2(R) = BK-to-REAL2(r). Consider l being a real number such that $0 \leqslant l \leqslant 1$ and BKto$\hat{\mathrm{T}}$2$(Q) = (1-l) \cdot ($BKto$\hat{\mathrm{T}}$2$(P)) + l \cdot ($BKto$\hat{\mathrm{T}}$2$(R))$.

Set $r = \frac{l \cdot m_6}{(1-l) \cdot m_3 + l \cdot m_6}$. $(1-r) \cdot [\frac{m_1}{m_3}, \frac{m_2}{m_3}, 1] + r \cdot [\frac{m_4}{m_6}, \frac{m_5}{m_6}, 1] = [\frac{m_7}{m_9}, \frac{m_8}{m_9}, 1]$. $0 \leqslant r \leqslant 1$. BKtoT2$(P')$ = BK-to-REAL2(p') and BKto$\hat{\mathrm{T}}$2(P') = BK-to-REAL2(p') and BKtoT2(Q') = BK-to-REAL2(q') and BKto$\hat{\mathrm{T}}$2(Q') = BK-to-REAL2(q') and BKtoT2(R') = BK-to-REAL2(r') and BKto$\hat{\mathrm{T}}$2(R') = BK-to-REAL2(r'). $\square$

Let P be a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. We say that P is point at ∞ if and only if

(Def. 4) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that P = the direction of u and $(u)_{\mathbf{3}} = 0$.

Now we state the proposition:

(47) Let us consider a point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that P = the direction of u and $(u)_{\mathbf{3}} \neq 0$. Then P is not point at ∞.

Note that there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is point at ∞ and there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is non point at ∞.

Let P be a non point at ∞ point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor RP3toREAL2$(P)$ yielding an element of $\mathcal{R}^2$ is defined by

(Def. 5) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that P = the direction of u and $(u)_{\mathbf{3}} = 1$ and $it = [(u)_{\mathbf{1}}, (u)_{\mathbf{2}}]$.

The functor RP3toT2(P) yielding a point of TarskiEuclid2Space is defined by the term

(Def. 6) RP3toREAL2(P).

Now we state the propositions:

(48) Let us consider non point at ∞ elements P, Q, R, P', Q', R' of the projective space over $\mathcal{E}^3_{\rm T}$, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\rm F}$ of dimension 3.

Suppose $h =$ the homography of N and $P, Q \in$ the BK-model and $R \in$ the absolute and $P' =$ (the homography of N)(P) and $Q' =$ (the homography of N)(Q) and $R' =$ (the homography of N)(R) and RP3toT2(Q) lies between RP3toT2(P) and RP3toT2(R).

Then RP3toT2(Q') lies between RP3toT2(P') and RP3toT2(R').

Proof: Consider n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} being elements of $\mathbb{R}_{\rm F}$ such that $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$. Consider u being a non zero element of $\mathcal{E}^3_{\rm T}$ such that $P =$ the direction of u and $(u)_{\mathbf{3}} = 1$ and RP3toREAL2$(P) = [(u)_{\mathbf{1}}, (u)_{\mathbf{2}}]$. Consider v being a non zero element of $\mathcal{E}^3_{\rm T}$ such that $R =$ the direction of v and $(v)_{\mathbf{3}} = 1$ and RP3toREAL2$(R) = [(v)_{\mathbf{1}}, (v)_{\mathbf{2}}]$. Consider w being a non zero element of $\mathcal{E}^3_{\rm T}$ such that $Q =$ the direction of w and $(w)_{\mathbf{3}} = 1$ and RP3toREAL2$(Q) = [(w)_{\mathbf{1}}, (w)_{\mathbf{2}}]$.

Reconsider $m_1 = n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}$, $m_2 = n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}$, $m_3 = n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}$, $m_4 = n_{11} \cdot v(1) + n_{12} \cdot v(2) + n_{13}$, $m_5 = n_{21} \cdot v(1) + n_{22} \cdot v(2) + n_{23}$, $m_6 = n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}$, $m_7 = n_{11} \cdot w(1) + n_{12} \cdot w(2) + n_{13}$, $m_8 = n_{21} \cdot w(1) + n_{22} \cdot w(2) + n_{23}$, $m_9 = n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33}$ as a real number.

Consider l being a real number such that $0 \leqslant l \leqslant 1$ and RP3to$\hat{\ }$T2$(Q) = (1-l) \cdot ($RP3to$\hat{\ }$T2$(P)) + l \cdot ($RP3to$\hat{\ }$T2$(R))$. Set $r = \frac{l \cdot m_6}{(1-l) \cdot m_3 + l \cdot m_6}$. $(1-r) \cdot [\frac{m_1}{m_3}, \frac{m_2}{m_3}, 1] + r \cdot [\frac{m_4}{m_6}, \frac{m_5}{m_6}, 1] = [\frac{m_7}{m_9}, \frac{m_8}{m_9}, 1]$. $0 \leqslant r \leqslant 1$. $\square$

(49) Let us consider real numbers a, b, c, and elements u, v, w of $\mathcal{E}^3_{\rm T}$. Suppose $a \neq 0$ and $a + b + c = 0$ and $a \cdot u + b \cdot v + c \cdot w = 0_{\mathcal{E}^3_{\rm T}}$. Then $u \in {\rm Line}(v, w)$.

(50) Let us consider non point at ∞ points P, Q, R of the projective space over $\mathcal{E}^3_{\rm T}$, and non zero elements u, v, w of $\mathcal{E}^3_{\rm T}$. Suppose $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $(u)_{\mathbf{3}} = 1$ and $(v)_{\mathbf{3}} = 1$ and $(w)_{\mathbf{3}} = 1$. Then P, Q and R are collinear if and only if u, v and w are collinear. The theorem is a consequence of (49).

(51) Let us consider elements u, v, w of $\mathcal{E}^3_{\rm T}$. Suppose $u \in \mathcal{L}(v, w)$. Then $[(u)_{\mathbf{1}}, (u)_{\mathbf{2}}] \in \mathcal{L}([(v)_{\mathbf{1}}, (v)_{\mathbf{2}}], [(w)_{\mathbf{1}}, (w)_{\mathbf{2}}])$.

(52) Let us consider elements u, v, w of $\mathcal{E}^2_{\rm T}$. Suppose $u \in \mathcal{L}(v, w)$. Then $[(u)_{\mathbf{1}}, (u)_{\mathbf{2}}, 1] \in \mathcal{L}([(v)_{\mathbf{1}}, (v)_{\mathbf{2}}, 1], [(w)_{\mathbf{1}}, (w)_{\mathbf{2}}, 1])$.

PROOF: Consider r being a real number such that $0 \leqslant r$ and $r \leqslant 1$ and $u = (1-r) \cdot v + r \cdot w$. Reconsider $u' = [(u)_{\mathbf{1}}, (u)_{\mathbf{2}}, 1]$, $v' = [(v)_{\mathbf{1}}, (v)_{\mathbf{2}}, 1]$, $w' = [(w)_{\mathbf{1}}, (w)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. $u' = (1-r) \cdot v' + r \cdot w'$. $\square$

(53) Let us consider non point at ∞ points P, Q, R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Then P, Q and R are collinear if and only if RP3toT2(P), RP3toT2(Q) and RP3toT2(R) are collinear. The theorem is a consequence of (50), (51), and (52).

(54) Let us consider elements u, v, w of $\mathcal{E}^2_{\mathrm{T}}$. Suppose u, v and w are collinear. Then $[(u)_{\mathbf{1}}, (u)_{\mathbf{2}}, 1]$, $[(v)_{\mathbf{1}}, (v)_{\mathbf{2}}, 1]$ and $[(w)_{\mathbf{1}}, (w)_{\mathbf{2}}, 1]$ are collinear. The theorem is a consequence of (52).

(55) Let us consider non point at ∞ elements P, Q, P_1 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose P, $Q \in$ the BK-model and $P_1 \in$ the absolute. Then RP3toT2(P_1) does not lie between RP3toT2(Q) and RP3toT2(P). The theorem is a consequence of (52) and (27).

The functor Dir001 yielding a non point at ∞ element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 7) the direction of $[0, 0, 1]$.

The functor Dir101 yielding a non point at ∞ element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 8) the direction of $[1, 0, 1]$.

Now we state the propositions:

(56) Let us consider non point at ∞ elements P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose P, $Q \in$ the absolute. Then $\overline{\text{RP3toT2(Dir001) RP3toT2}(P)} \cong \overline{\text{RP3toT2(Dir001) RP3toT2}(Q)}$.

(57) Let us consider non point at ∞ elements P, Q, R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose P, $Q \in$ the absolute and $P \neq Q$ and $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $(u)_{\mathbf{3}} = 1$ and $(v)_{\mathbf{3}} = 1$ and $w = [\frac{(u)_{\mathbf{1}}+(v)_{\mathbf{1}}}{2}, \frac{(u)_{\mathbf{2}}+(v)_{\mathbf{2}}}{2}, 1]$. Then $R \in$ the BK-model.
PROOF: Reconsider $u' = [u(1), u(2)]$, $v' = [v(1), v(2)]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. $u' \neq v'$. Reconsider $r_8 = [(w)_{\mathbf{1}}, (w)_{\mathbf{2}}]$ as an element of the inside of circle(0,0,1). Consider R_3 being an element of $\mathcal{E}^2_{\mathrm{T}}$ such that $R_3 = r_8$ and REAL2-to-BK(r_8) = the direction of $[(R_3)_{\mathbf{1}}, (R_3)_{\mathbf{2}}, 1]$. $\square$

(58) Let us consider points R_1, R_2 of TarskiEuclid2Space. Suppose $\hat{R}_1$, $\hat{R}_2 \in$ circle$(0, 0, 1)$ and $R_1 \neq R_2$. Then there exists an element P of BK-model-Plane such that BKtoT2(P) lies between R_1 and R_2. The theorem is a consequence of (47), (57), and (26).

(59) Let us consider non point at ∞ elements P, Q of the projective space

over $\mathcal{E}^3_{\mathrm{T}}$. If RP3toT2($P$) = RP3toT2($Q$), then $P = Q$.

(60) Let us consider non point at ∞ elements R_1, R_2 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose R_1, $R_2 \in$ the absolute and $R_1 \neq R_2$. Then there exists an element P of BK-model-Plane such that BKtoT2(P) lies between RP3toT2(R_1) and RP3toT2(R_2). The theorem is a consequence of (59) and (58).

(61) Let us consider points P, Q, R of TarskiEuclid2Space. Suppose Q lies between P and R and $\hat{P}$, $\hat{R} \in$ the inside of circle(0,0,1). Then $\hat{Q} \in$ the inside of circle(0,0,1).

Let us consider a non point at ∞ element P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$.

(62) If $P \in$ the absolute, then RP3toREAL2(P) $\in$ circle$(0, 0, 1)$.

(63) If $P \in$ the BK-model, then RP3toREAL2(P) $\in$ the inside of circle(0,0,1). The theorem is a consequence of (26).

(64) Let us consider a non point at ∞ point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and an element Q of the BK-model. If $P = Q$, then RP3toREAL2(P) = BK-to-REAL2(Q). The theorem is a consequence of (26).

(65) Let us consider non point at ∞ elements P, Q, R_1, R_2 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P \neq Q$ and $P \in$ the BK-model and R_1, $R_2 \in$ the absolute and RP3toT2(Q) lies between RP3toT2(P) and RP3toT2(R_1) and RP3toT2(Q) lies between RP3toT2(P) and RP3toT2(R_2). Then $R_1 = R_2$. The theorem is a consequence of (60), (59), (62), (64), (8), and (61).

(66) Let us consider non point at ∞ elements P, Q, P_1, P_2 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P \neq Q$ and P, $Q \in$ the BK-model and P_1, $P_2 \in$ the absolute and $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear. Then

(i) RP3toT2(P) lies between RP3toT2(Q) and RP3toT2(P_1), or

(ii) RP3toT2(P) lies between RP3toT2(Q) and RP3toT2(P_2).

The theorem is a consequence of (55), (53), and (65).

Let us consider elements P, Q of the BK-model. Now we state the propositions:

(67) Suppose $P \neq Q$. Then there exists an element R of the absolute such that for every non point at ∞ elements p, q, r of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $p = P$ and $q = Q$ and $r = R$ holds RP3toT2(p) lies between RP3toT2(q) and RP3toT2(r). The theorem is a consequence of (47) and (66).

(68) Suppose $P \neq Q$. Then there exists an element R of the absolute such that for every non point at ∞ elements p, q, r of the projective space over

$\mathcal{E}^3_{\mathrm{T}}$ such that $p = P$ and $q = Q$ and $r = R$ holds RP3toT2(q) lies between RP3toT2(p) and RP3toT2(r). The theorem is a consequence of (67).

(69) The direction of $[1, 0, 1]$ is an element of the absolute.

(70) Let us consider points a, b of BK-model-Plane. Then $\overline{aa} \cong \overline{bb}$. The theorem is a consequence of (69).

(71) Every element of the BK-model is a non point at ∞ element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The theorem is a consequence of (47).

(72) Every element of the absolute is a non point at ∞ element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The theorem is a consequence of (47).

(73) Let us consider an element P of the BK-model, and a non point at ∞ element P' of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. If $P = P'$, then RP3toREAL2(P') = BK-to-REAL2(P). The theorem is a consequence of (26).

(74) Let us consider points a, q, b, c of BK-model-Plane. Then there exists a point x of BK-model-Plane such that

(i) a lies between q and x, and

(ii) $\overline{ax} \cong \overline{bc}$.

The theorem is a consequence of (71), (68), (72), (12), (70), (48), and (73).

(75) Let us consider points P, Q of BK-model-Plane.
If BKtoT2(P) = BKtoT2(Q), then $P = Q$.

(76) Let us consider real numbers a, b, r, and elements P, Q, R of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P, $R \in$ the inside of circle(a,b,r). Then $\mathcal{L}(P, R) \subseteq$ the inside of circle(a,b,r).

2. The Axiom of Segment Construction

Now we state the proposition:

(77) BK-model-Plane satisfies the axiom of segment construction.

3. The Axiom of Betweenness Identity

Now we state the proposition:

(78) BK-model-Plane satisfies the axiom of betweenness identity. The theorem is a consequence of (12) and (75).

4. The Axiom of Pasch

Now we state the proposition:

(79) BK-model-Plane satisfies the axiom of Pasch. The theorem is a consequence of (12), (8), (25), and (10).

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