Klein-Beltrami model. Part IV

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Summary. Timothy Makarios (with Isabelle/HOL) and John Harrison (with HOL-Light) shown that “the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski’s axioms except his Euclidean axiom” [2],[3],[4, 5].

With the Mizar system [1] we use some ideas taken from Tim Makarios’s MSc thesis [10] to formalize some definitions and lemmas necessary for the verification of the independence of the parallel postulate. In this article, which is the continuation of [8], we prove that our constructed model satisfies the axioms of segment construction, the axiom of betweenness identity, and the axiom of Pasch due to Tarski, as formalized in [11] and related Mizar articles.

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1. Preliminaries

Let us consider real numbers $a$, $b$. Now we state the propositions:

(1) If $a \neq b$, then $1 - \frac{a}{a-b} = -\frac{b}{a-b}$.

(2) If $0 < a \cdot b$, then $0 < \frac{a}{b}$.

Now we state the propositions:

(3) Let us consider real numbers $a$, $b$, $c$. Suppose $0 \leq a \leq 1$ and $0 < b \cdot c$.

Then $0 \leq \frac{a-c}{(1-a)b+a-c} \leq 1$.


Let us consider real numbers $a, b, c$. Suppose $(1 - a) \cdot b + a \cdot c \neq 0$. Then
\[
1 - \frac{a \cdot c}{b + a \cdot c} = \frac{(1-a) \cdot b}{(1-a) \cdot b + a \cdot c}.
\]

Let us consider real numbers $a, b, c, d$. If $b \neq 0$, then $\frac{a \cdot b \cdot d}{b} = \frac{a \cdot d}{c}$.

Let us consider an element $u$ of $E^3_1$. Then $u = [u(1), u(2), u(3)]$.

Let us consider an element $P$ of the BK-model. Then $\text{BK-to-REAL2}(P) \in \text{TarskiEuclid2Space}$.

Let $P$ be a point of BK-model-Plane. The functor $\text{BKtoT2}(P)$ yielding a point of TarskiEuclid2Space is defined by

(Def. 1) there exists an element $p$ of the BK-model such that $P = p$ and $it = \text{BK-to-REAL2}(p)$.

Let $P$ be a point of TarskiEuclid2Space. Assume $\hat{P} \in$ the inside of circle(0,0,1). The functor $\text{T2toBK}(P)$ yielding a point of BK-model-Plane is defined by

(Def. 2) there exists a non zero element $u$ of $E^3_1$ such that $it = \text{direction of } u$ and $(u)_3 = 1$ and $\hat{P} = [(u)_1, (u)_2]$.

Let us consider a point $P$ of BK-model-Plane. Now we state the propositions:

- $\text{BKTo}\hat{T2}(P) \in \text{the inside of circle}(0,0,1)$.
- $\text{T2toBK}(\text{BKToT2}(P)) = P$.
- Let us consider a point $P$ of TarskiEuclid2Space. Suppose $\hat{P}$ is an element of the inside of circle(0,0,1). Then $\text{BKToT2}(\text{T2toBK}(P)) = P$.
- Let us consider a point $P$ of BK-model-Plane, and an element $p$ of the BK-model. Suppose $P = p$. Then
  - (i) $\text{BKToT2}(P) = \text{BK-to-REAL2}(p)$, and
  - (ii) $\text{BKToT2}(P) = \text{BK-to-REAL2}(p)$.
- Let us consider points $P, Q, R$ of BK-model-Plane, and points $P_2, Q_2, R_2$ of TarskiEuclid2Space. Suppose $P_2 = \text{BKToT2}(P)$ and $Q_2 = \text{BKToT2}(Q)$ and $R_2 = \text{BKToT2}(R)$. Then $Q$ lies between $P$ and $R$ if and only if $Q_2$ lies between $P_2$ and $R_2$. The theorem is a consequence of (11).
- Let us consider elements $P, Q$ of $E^2_1$. If $P \neq Q$, then $P(1) \neq Q(1)$ or $P(2) \neq Q(2)$.
- Let us consider real numbers $a, b$, and elements $P, Q$ of $E^2_1$. If $P \neq Q$ and $(1 - a) \cdot P + a \cdot Q = (1 - b) \cdot P + b \cdot Q$, then $a = b$. The theorem is a consequence of (13).
- Let us consider points $P, Q$ of BK-model-Plane. If $\text{BKTo}\hat{T2}(P) = \text{BKTo}\hat{T2}(Q)$, then $P = Q$. The theorem is a consequence of (11).

Let $P, Q, R$ be points of BK-model-Plane. Assume $Q$ lies between $P$ and $R$ and $P \neq R$. The functor $\text{length}(P, Q, R)$ yielding a real number is defined by
(Def. 3) \( 0 \leq it \leq 1 \) and \( \text{BKto}\tilde{T}2(Q) = (1-it) \cdot (\text{BKto}\tilde{T}2(P)) + it \cdot (\text{BKto}\tilde{T}2(R)) \).

Let us consider points \( P, Q \) of BK-model-Plane. Now we state the propositions:

(16) (i) \( P \) lies between \( P \) and \( Q \), and
(ii) \( Q \) lies between \( P \) and \( Q \).

The theorem is a consequence of (12).

(17) If \( P \neq Q \), then length\( (P,P,Q) = 0 \) and length\( (P,Q,Q) = 1 \). The theorem is a consequence of (16).

(18) Let us consider a square matrix \( N \) over \( \mathbb{R}_F \) of dimension 3. Suppose \( N = \langle \langle 2,0,-1 \rangle, \langle 0,\sqrt{3},0 \rangle, \langle 1,0,-2 \rangle \rangle \). Then

(i) \( \text{Det} N = (-3) \cdot \sqrt{3} \), and
(ii) \( N \) is invertible.

(19) Let us consider elements \( a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32}, b_{33}, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \) of \( \mathbb{R}_F \), and square matrices \( A, B \) over \( \mathbb{R}_F \) of dimension 3.

Suppose \( A = \langle \langle a_{11}, a_{12}, a_{13} \rangle, \langle a_{21}, a_{22}, a_{23} \rangle, \langle a_{31}, a_{32}, a_{33} \rangle \rangle \) and \( B = \langle \langle b_{11}, b_{12}, b_{13} \rangle, \langle b_{21}, b_{22}, b_{23} \rangle, \langle b_{31}, b_{32}, b_{33} \rangle \rangle \) and \( a_1 = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} \) and \( a_2 = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} \) and \( a_3 = a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \) and \( a_4 = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} \).

Suppose \( a_5 = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} \) and \( a_6 = a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \) and \( a_7 = a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} \) and \( a_8 = a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} \) and \( a_9 = a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \).

Then \( A \cdot B = \langle \langle a_1, a_2, a_3 \rangle, \langle a_4, a_5, a_6 \rangle, \langle a_7, a_8, a_9 \rangle \rangle \).

Let us consider square matrices \( N_1, N_2 \) over \( \mathbb{R}_F \) of dimension 3. Now we state the propositions:

(20) Suppose \( N_1 = \langle \langle 2,0,-1 \rangle, \langle 0,\sqrt{3},0 \rangle, \langle 1,0,-2 \rangle \rangle \) and \( N_2 = \langle \langle \frac{2}{3},0,-\frac{1}{3} \rangle, \langle 0,\frac{1}{\sqrt{3}},0 \rangle, \langle \frac{1}{3},0,-\frac{2}{3} \rangle \rangle \). Then \( N_1 \cdot N_2 = \langle \langle 1,0,0 \rangle, \langle 0,1,0 \rangle, \langle 0,0,1 \rangle \rangle \). The theorem is a consequence of (19).

(21) Suppose \( N_2 = \langle \langle 2,0,-1 \rangle, \langle 0,\sqrt{3},0 \rangle, \langle 1,0,-2 \rangle \rangle \) and \( N_1 = \langle \langle \frac{2}{3},0,-\frac{1}{3} \rangle, \langle 0,\frac{1}{\sqrt{3}},0 \rangle, \langle \frac{1}{3},0,-\frac{2}{3} \rangle \rangle \). Then \( N_1 \cdot N_2 = \langle \langle 1,0,0 \rangle, \langle 0,1,0 \rangle, \langle 0,0,1 \rangle \rangle \). The theorem is a consequence of (19).

(22) Suppose \( N_1 = \langle \langle 2,0,-1 \rangle, \langle 0,\sqrt{3},0 \rangle, \langle 1,0,-2 \rangle \rangle \) and \( N_2 = \langle \langle \frac{2}{3},0,-\frac{1}{3} \rangle, \langle 0,\frac{1}{\sqrt{3}},0 \rangle, \langle \frac{1}{3},0,-\frac{2}{3} \rangle \rangle \). Then \( N_1 \) is inverse of \( N_2 \). The theorem is a consequence of (20) and (21).

Let us consider an invertible square matrix \( N \) over \( \mathbb{R}_F \) of dimension 3. Now we state the propositions:
(23) Suppose $N = \langle(\frac{2}{3}, 0, -\frac{1}{3}), (0, \frac{1}{\sqrt{3}}, 0), (\frac{1}{3}, 0, -\frac{2}{3})\rangle$. Then (the homography of $N$)°(the absolute) $\subseteq$ the absolute.

**Proof:** (The homography of $N$)°(the absolute) $\subseteq$ the absolute by [7 (89)], [9 (7)]. □

(24) Suppose $N = \langle(2, 0, -1), (0, \sqrt{3}, 0), (1, 0, -2)\rangle$. Then (the homography of $N$)°(the absolute) = the absolute.

**Proof:** (The homography of $N$)°(the absolute) $\subseteq$ the absolute.
The absolute $\subseteq$ (the homography of $N$)°(the absolute) by [6 (19)], (22), (23). □

(25) Let us consider real numbers $a$, $b$, $r$, and elements $P$, $Q$, $R$ of $E^2_F$. Suppose $Q \in \mathcal{L}(P, R)$ and $P$, $R \in$ the inside of circle($a$, $b$, $r$). Then $Q \in$ the inside of circle($a$, $b$, $r$).

(26) Let us consider non zero elements $u$, $v$ of $E^2_F$. Suppose the direction of $u$ = the direction of $v$ and $u(3) \neq 0$ and $u(3) = v(3)$. Then $u = v$.

(27) Let us consider an element $R$ of the projective space over $E^3_T$, elements $P$, $Q$ of the BK-model, non zero elements $u$, $v$, $w$ of $E^3_T$, and a real number $r$. Suppose $0 \leq r \leq 1$ and $P =$ the direction of $u$ and $Q =$ the direction of $v$ and $R =$ the direction of $w$ and $u(3) = 1$ and $v(3) = 1$ and $w = r \cdot u + (1 - r) \cdot v$. Then $R$ is an element of the BK-model.

**Proof:** Consider $u_2$ being a non zero element of $E^3_T$ such that the direction of $u_2 = P$ and $u_2(3) = 1$ and BK-to-REAL2($P$) = $[u_2(1), u_2(2)]$. $u = u_2$.

Reconsider $r_4 = [u_2(1), u_2(2)]$ as an element of $E^2_T$. Consider $v_2$ being a non zero element of $E^3_T$ such that the direction of $v_2 = Q$ and $v_2(3) = 1$ and BK-to-REAL2($Q$) = $[v_2(1), v_2(2)]$. $v = v_2$.

Reconsider $r_6 = [v_2(1), v_2(2)]$ as an element of $E^2_T$. Reconsider $r_8 = [w(1), w(2)]$ as an element of $E^2_T$. $r_8 = r \cdot r_4 + (1 - r) \cdot r_6$.

Consider $R_3$ being an element of $E^2_T$ such that $R_3 = r_8$ and REAL2-to-BK($r_8$) = the direction of $[(R_3)1, (R_3)2, 1]$. □

(28) Let us consider an invertible square matrix $N$ over $\mathbb{R}_F$ of dimension 3, elements $n_{11}$, $n_{12}$, $n_{13}$, $n_{21}$, $n_{22}$, $n_{23}$, $n_{31}$, $n_{32}$, $n_{33}$ of $\mathbb{R}_F$, points $P$, $Q$ of the projective space over $E^3_T$, and non zero elements $u$, $v$ of $E^3_T$. Suppose $N = \langle\langle n_{11}, n_{12}, n_{13}\rangle, \langle n_{21}, n_{22}, n_{23}\rangle, \langle n_{31}, n_{32}, n_{33}\rangle\rangle$ and $P =$ the direction of $u$ and $Q =$ the direction of $v$ and $Q =$ (the homography of $N$)°($P$) and $u(3) = 1$. Then there exists a non zero real number $a$ such that

(i) $v(1) = a \cdot (n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13})$, and

(ii) $v(2) = a \cdot (n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23})$, and

(iii) $v(3) = a \cdot (n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33})$.

(29) Let us consider an invertible square matrix $N$ over $\mathbb{R}_F$ of dimension 3, elements $n_{11}$, $n_{12}$, $n_{13}$, $n_{21}$, $n_{22}$, $n_{23}$, $n_{31}$, $n_{32}$, $n_{33}$ of $\mathbb{R}_F$, an element
Let us consider an invertible square matrix $N$ over $\mathbb{F}_T$ of dimension 3, an element $h$ of the subgroup of $K$-isometries, elements $n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}$ of $\mathbb{F}_F$, an element $P$ of the absolute, and a non zero element $u$ of $\mathcal{E}^3_T$. Suppose $h = \text{the homography of } N$ and $N = \langle \langle n_{11}, n_{12}, n_{13} \rangle, \langle n_{21}, n_{22}, n_{23} \rangle, \langle n_{31}, n_{32}, n_{33} \rangle \rangle$ and $P = \text{the direction of } u$ and $Q = \text{the direction of } v$ and $Q = (\text{the homography of } N)(P)$ and $u(3) = 1$ and $v(3) = 1$. Then

(i) $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} = 0$, and

(ii) $v(1) = \frac{n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}}{n_{31} \cdot u(1) - n_{32} \cdot u(2) + n_{33}}$, and

(iii) $v(2) = \frac{n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}}{n_{31} \cdot u(1) - n_{32} \cdot u(2) + n_{33}}$.

The theorem is a consequence of (28).

Let us consider an invertible square matrix $N$ over $\mathbb{F}_F$ of dimension 3, an element $h$ of the subgroup of $K$-isometries, elements $n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}$ of $\mathbb{F}_F$, an element $P$ of the BK-model, and a non zero element $u$ of $\mathcal{E}^3_T$. Suppose $h = \text{the homography of } N$ and $N = \langle \langle n_{11}, n_{12}, n_{13} \rangle, \langle n_{21}, n_{22}, n_{23} \rangle, \langle n_{31}, n_{32}, n_{33} \rangle \rangle$ and $P = \text{the direction of } u$ and $u(3) = 1$. Then $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} = 0$. The theorem is a consequence of (29).

Let us consider an invertible square matrix $N$ over $\mathbb{F}_F$ of dimension 3, an element $h$ of the subgroup of $K$-isometries, elements $n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}$ of $\mathbb{F}_F$, an element $P$ of the absolute, and a non zero element $u$ of $\mathcal{E}^3_T$. Suppose $h = \text{the homography of } N$ and $N = \langle \langle n_{11}, n_{12}, n_{13} \rangle, \langle n_{21}, n_{22}, n_{23} \rangle, \langle n_{31}, n_{32}, n_{33} \rangle \rangle$ and $P = \text{the direction of } u$ and $u(3) = 1$. Then $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} = 0$. The theorem is a consequence of (31).

Let us consider an invertible square matrix $N$ over $\mathbb{F}_F$ of dimension 3, an element $h$ of the subgroup of $K$-isometries, elements $n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}$ of $\mathbb{F}_F$, an element $P$ of the BK-model, and a non zero element $u$ of $\mathcal{E}^3_T$. Suppose $h = \text{the homography of } N$ and $N = \langle \langle n_{11}, n_{12}, n_{13} \rangle, \langle n_{21}, n_{22}, n_{23} \rangle, \langle n_{31}, n_{32}, n_{33} \rangle \rangle$ and $P = \text{the direction of } u$ and $u(3) = 1$. Then $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \neq 0$. The theorem is a consequence of (32).

Let us consider an invertible square matrix $N$ over $\mathbb{F}_F$ of dimension 3, an element $h$ of the subgroup of $K$-isometries, elements $n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}$ of $\mathbb{F}_F$, an element $P$ of the BK-model, and a non zero element $u$ of $\mathcal{E}^3_T$. Suppose $h = \text{the homography of } N$ and $N = \langle \langle n_{11}, n_{12}, n_{13} \rangle, \langle n_{21}, n_{22}, n_{23} \rangle, \langle n_{31}, n_{32}, n_{33} \rangle \rangle$ and $P = \text{the direction of } u$ and $u(3) = 1$. Then $n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \neq 0$. The theorem is a consequence of (33).
of \( u \) and \( u(3) = 1 \). Then (the homography of \( N \))(\( P \)) = the direction of 

\[
\begin{bmatrix}
    n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13} & n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23} & n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \cdot 1
\end{bmatrix}.
\]

The theorem is a consequence of (29).

(34) \ Let us consider an invertible square matrix \( N \) over \( \mathbb{R}_F \) of dimension 3, an element \( h \) of the subgroup of \( K \)-isometries, elements \( n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} \) of \( \mathbb{R}_F \), an element \( P \) of the absolute, and a non zero element \( u \) of \( \mathcal{E}_T^3 \). Suppose \( h = \) the homography of \( N \) and \( N = \langle \langle n_{11}, n_{12}, n_{13} \rangle, \langle n_{21}, n_{22}, n_{23} \rangle, \langle n_{31}, n_{32}, n_{33} \rangle \rangle \) and \( P = \) the direction of \( u \) and \( u(3) = 1 \). Then (the homography of \( N \))(\( P \)) = the direction of 

\[
\begin{bmatrix}
    n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13} & n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23} & n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33} \cdot 1
\end{bmatrix}.
\]

The theorem is a consequence of (31).

(35) \ Let us consider a subset \( A \) of \( \mathcal{E}_T^3 \), a convex, non empty subset \( B \) of \( \mathcal{E}_T^2 \), a real number \( r \), and an element \( x \) of \( \mathcal{E}_T^3 \). Suppose \( A = \{ x, \ \text{where} \ x \ \text{is} \ \text{an element of} \ \mathcal{E}_T^3 : [(x)_1, (x)_2] \in B \) and \((x)_3 = r\). Then \( A \) is non empty and convex.

(36) \ Let us consider elements \( n_1, n_2, n_3 \) of \( \mathbb{R}_F \), and elements \( n, u \) of \( \mathcal{E}_T^3 \). Suppose \( n = \langle n_1, n_2, n_3 \rangle \) and \( u(3) = 1 \). Then \(|(n, u)| = n_1 \cdot u(1) + n_2 \cdot u(2) + n_3 \).

(37) \ Let us consider a convex, non empty subset \( A \) of \( \mathcal{E}_T^3 \), and elements \( n, u, v \) of \( \mathcal{E}_T^3 \). Suppose for every element \( w \) of \( \mathcal{E}_T^3 \) such that \( w \in A \) holds \(|(n, w)| \neq 0 \) and \( u, v \in A \). Then \( 0 < |(n, u)| \cdot |(n, v)| \).

**Proof:** Set \( x = |(n, u)| \). Set \( y = |(n, v)| \). Reconsider \( l = \frac{x}{x-y} \) as a non zero real number. Reconsider \( w = l \cdot v + (1-l) \cdot u \) as an element of \( \mathcal{E}_T^3 \). \( x \neq y \). \( 1 - l = -\frac{y}{x-y} \). \(|(n, w)| = 0 \). \( \square \)

Let us consider elements \( n_{31}, n_{32}, n_{33} \) of \( \mathbb{R}_F \) and elements \( u, v \) of \( \mathcal{E}_T^2 \). Now we state the propositions:

(38) \ Suppose \( u, v \in \) the inside of circle(0,0,1) and for every element \( w \) of \( \mathcal{E}_T^2 \) such that \( w \in \) the inside of circle(0,0,1) holds \( n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33} \neq 0 \). Then \( 0 < (n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}) \cdot (n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}) \). The theorem is a consequence of (35), (36), and (37).

(39) \ Suppose \( u \in \) the inside of circle(0,0,1) and \( v \in \) circle(0,0,1) and for every element \( w \) of \( \mathcal{E}_T^2 \) such that \( w \in \) the closed inside of circle(0,0,1) holds \( n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33} \neq 0 \). Then \( 0 < (n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}) \cdot (n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}) \). The theorem is a consequence of (35), (36), and (37).

(40) \ Let us consider real numbers \( l, r \), elements \( u, v, w \) of \( \mathcal{E}_T^3 \), and real numbers \( n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}, m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9 \).
Suppose $m_3 \neq 0$ and $m_6 \neq 0$ and $m_9 \neq 0$ and $r = \frac{l \cdot m_6}{(1-l) \cdot m_3 + l \cdot m_6}$ and $(1-l) \cdot m_3 + l \cdot m_6 \neq 0$ and $w = (1-l) \cdot u + l \cdot v$ and $m_1 = n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}$ and $m_2 = n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}$ and $m_3 = n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}$ and $m_4 = n_{11} \cdot v(1) + n_{12} \cdot v(2) + n_{13}$.

Suppose $m_5 = n_{21} \cdot v(1) + n_{22} \cdot v(2) + n_{23}$ and $m_6 = n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}$ and $m_7 = n_{11} \cdot w(1) + n_{12} \cdot w(2) + n_{13}$ and $m_8 = n_{21} \cdot w(1) + n_{22} \cdot w(2) + n_{23}$ and $m_9 = n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33}$.

Then $(1-r) \cdot [m_1, m_2, 1] + r \cdot [m_4, m_5, 1] = [m_7, m_8, 1]$. The theorem is a consequence of (4) and (5).

(41) Let us consider an invertible square matrix $N$ over $\mathbb{R}_F$ of dimension 3, an element $h$ of the subgroup of $K$-isometries, elements $n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}$ of $\mathbb{R}_F$, and an element $P$ of the BK-model. Suppose $h$ is the homography of $N$ and $N = \langle \langle n_{11}, n_{12}, n_{13}, \langle n_{21}, n_{22}, n_{23}, \langle n_{31}, n_{32}, n_{33} \rangle \rangle \rangle$. Then (the homography of $N$)$(P)$ is the direction of $\frac{[n_{11} \cdot (BK-to-REAL2(P))_1 + n_{12} \cdot (BK-to-REAL2(P))_2 + n_{13} \cdot (BK-to-REAL2(P))_2 + n_{33}, \langle n_{31} \cdot (BK-to-REAL2(P))_1 + n_{32} \cdot (BK-to-REAL2(P))_2 + n_{33}, n_{21} \cdot (BK-to-REAL2(P))_1 + n_{22} \cdot (BK-to-REAL2(P))_2 + n_{23}, n_{31} \cdot (BK-to-REAL2(P))_1 + n_{32} \cdot (BK-to-REAL2(P))_2 + n_{33} \rangle}{m_7, m_8, 1}$. The theorem is a consequence of (33).

(42) Let us consider an element $h$ of the subgroup of $K$-isometries, an invertible square matrix $N$ over $\mathbb{R}_F$ of dimension 3, elements $n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}$ of $\mathbb{R}_F$, and an element $u_2$ of $E^2_T$. Suppose $h$ is the homography of $N$ and $N = \langle \langle n_{11}, n_{12}, n_{13}, \langle n_{21}, n_{22}, n_{23}, \langle n_{31}, n_{32}, n_{33} \rangle \rangle \rangle$ and $u_2 \in$ the inside of circle$(0,0,1)$. Then $n_{31} \cdot u_2(1) + n_{32} \cdot u_2(2) + n_{33} \neq 0$. The theorem is a consequence of (30).

(43) Let us consider a positive real number $r$, and an element $u$ of $E^2_T$. If $u \in$ circle$(0,0,r)$, then $u$ is not zero.

(44) Let us consider an element $h$ of the subgroup of $K$-isometries, an invertible square matrix $N$ over $\mathbb{R}_F$ of dimension 3, elements $n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}$ of $\mathbb{R}_F$, and an element $u_2$ of $E^2_T$. Suppose $h$ is the homography of $N$ and $N = \langle \langle n_{11}, n_{12}, n_{13}, \langle n_{21}, n_{22}, n_{23}, \langle n_{31}, n_{32}, n_{33} \rangle \rangle \rangle$ and $u_2 \in$ the closed inside of circle$(0,0,1)$. Then $n_{31} \cdot u_2(1) + n_{32} \cdot u_2(2) + n_{33} \neq 0$. The theorem is a consequence of (30), (43), and (32).

(45) Let us consider real numbers $a, b, c, d, e, f, r$. Suppose $(1-r) \cdot [a, b, 1] + r \cdot [c, d, 1] = [e, f, 1]$. Then $(1-r) \cdot [a, b] + r \cdot [c, d] = [e, f]$.

(46) Let us consider points $P, Q, R, P', Q', R'$ of BK-model-Plane, elements $p, q, r, p', q', r'$ of the BK-model, an element $h$ of the subgroup of $K$-isometries, and an invertible square matrix $N$ over $\mathbb{R}_F$ of dimension 3. Suppose $h$ is the homography of $N$ and $Q$ lies between $P$ and $R$ and $P = p$ and $Q = q$ and $R = r$ and $p' = (\text{the homography of } N)(p)$ and $q' = (\text{the homography of } N)(q)$ and $r' = (\text{the homography of } N)(r)$ and
\[ P' = p' \text{ and } Q' = q' \text{ and } R' = r'. \] Then \( Q' \) lies between \( P' \) and \( R' \).

**Proof:** Consider \( n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33} \) being elements of \( \mathbb{R}_F \) such that \( N = \langle \langle n_{11}, n_{12}, n_{13} \rangle, \langle n_{21}, n_{22}, n_{23} \rangle, \langle n_{31}, n_{32}, n_{33} \rangle \rangle \). Consider \( u \) being a non zero element of \( E_F^3 \) such that the direction of \( u = p \) and \( u(3) = 1 \) and \( \text{BK-to-REAL2}(p) = [u(1), u(2)] \). Consider \( v \) being a non zero element of \( E_F^3 \) such that the direction of \( v = r \) and \( v(3) = 1 \) and \( \text{BK-to-REAL2}(r) = [v(1), v(2)] \). Consider \( w \) being a non zero element of \( E_F^3 \) such that the direction of \( w = q \) and \( w(3) = 1 \) and \( \text{BK-to-REAL2}(q) = [w(1), w(2)] \).

Reconsider \( m_1 = n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}, m_2 = n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}, m_3 = n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}, m_4 = n_{11} \cdot v(1) + n_{12} \cdot v(2) + n_{13}, m_5 = n_{21} \cdot v(1) + n_{22} \cdot v(2) + n_{23}, m_6 = n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}, m_7 = n_{11} \cdot w(1) + n_{12} \cdot w(2) + n_{13}, m_8 = n_{21} \cdot w(1) + n_{22} \cdot w(2) + n_{23}, m_9 = n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33} \) as a real number. \( \text{BKtoT2}(P) = \text{BK-to-REAL2}(p) \) and \( \text{BKtoT2}(P) = \text{BK-to-REAL2}(p) \) and \( \text{BKtoT2}(Q) = \text{BK-to-REAL2}(q) \) and \( \text{BKtoT2}(R) = \text{BK-to-REAL2}(r) \) and \( \text{BKtoT2}(R) = \text{BK-to-REAL2}(r) \). Consider \( l \) being a real number such that \( 0 \leq l \leq 1 \) and \( \text{BKtoT2}(Q) = (1 - l) \cdot \text{BKtoT2}(P) + l \cdot \text{BKtoT2}(R) \).

Set \( r = \frac{l \cdot m_8}{(1 - l) \cdot m_5 + l \cdot m_6}, \quad (1 - r) \cdot \left[ \frac{m_1}{m_5}, \frac{m_2}{m_6}, 1 \right] = \left[ \frac{m_2}{m_9} \right] = \left[ \frac{m_3}{m_9}, \frac{m_4}{m_9}, 1 \right] \), \( 0 \leq r \leq 1 \). \( \text{BKtoT2}(P') = \text{BK-to-REAL2}(p') \) and \( \text{BKtoT2}(P') = \text{BK-to-REAL2}(p') \) and \( \text{BKtoT2}(Q') = \text{BK-to-REAL2}(q') \) and \( \text{BKtoT2}(Q') = \text{BK-to-REAL2}(q') \) and \( \text{BKtoT2}(R') = \text{BK-to-REAL2}(r') \) and \( \text{BKtoT2}(R') = \text{BK-to-REAL2}(r') \). \( \square \)

Let \( P \) be a point of the projective space over \( E_T^3 \). We say that \( P \) is point at \( \infty \) if and only if

(Def. 4) there exists a non zero element \( u \) of \( E_T^3 \) such that \( P = \text{the direction of} \ u \) and \( (u)_3 = 0 \).

Now we state the proposition:

(47) Let us consider a point \( P \) of the projective space over \( E_T^3 \). Suppose there exists a non zero element \( u \) of \( E_T^3 \) such that \( P = \text{the direction of} \ u \) and \( (u)_3 \neq 0 \). Then \( P \) is not point at \( \infty \).

Note that there exists a point of the projective space over \( E_T^3 \) which is point at \( \infty \) and there exists a point of the projective space over \( E_T^3 \) which is non point at \( \infty \).

Let \( P \) be a non point at \( \infty \) point of the projective space over \( E_T^3 \). The functor \( \text{RP3toREAL2}(P) \) yielding an element of \( \mathcal{R}^2 \) is defined by

(Def. 5) there exists a non zero element \( u \) of \( E_T^3 \) such that \( P = \text{the direction of} \ u \) and \( (u)_3 = 1 \) and \( it = [(u)_1, (u)_2] \).
The functor $\text{RP3toT2}(P)$ yielding a point of TarskiEuclid2Space is defined by the term

(Def. 6) $\text{RP3toREAL2}(P)$.

Now we state the propositions:

(48) Let us consider non point at $\infty$ elements $P, Q, R, P', Q', R'$ of the projective space over $E^3_T$, an element $h$ of the subgroup of $K$-isometries, and an invertible square matrix $N$ over $\mathbb{R}_F$ of dimension 3.

Suppose $h$ = the homography of $N$ and $P, Q \in \text{the BK-model and } R \in \text{the absolute and } P' = (\text{the homography of } N)(P) \text{ and } Q' = (\text{the homography of } N)(Q) \text{ and } R' = (\text{the homography of } N)(R) \text{ and } \text{RP3toT2}(Q) \text{ lies between } \text{RP3toT2}(P) \text{ and } \text{RP3toT2}(R)$.

Then $\text{RP3toT2}(Q')$ lies between $\text{RP3toT2}(P')$ and $\text{RP3toT2}(R')$.

**Proof:** Consider $n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}, n_{31}, n_{32}, n_{33}$ being elements of $\mathbb{R}_F$ such that $N = \{\langle n_{11}, n_{12}, n_{13}, \rangle, \langle n_{21}, n_{22}, n_{23}, \rangle, \langle n_{31}, n_{32}, n_{33}, \rangle\}$. Consider $u$ being a non zero element of $E^3_T$ such that $P = \text{the direction of } u$ and $(u)_3 = 1$ and $\text{RP3toREAL2}(P) = [(u)_1, (u)_2]$. Consider $v$ being a non zero element of $E^3_T$ such that $R = \text{the direction of } v$ and $(v)_3 = 1$ and $\text{RP3toREAL2}(R) = [(v)_1, (v)_2]$. Consider $w$ being a non zero element of $E^3_T$ such that $Q = \text{the direction of } w$ and $(w)_3 = 1$ and $\text{RP3toREAL2}(Q) = [(w)_1, (w)_2]$.

Reconsider $m_1 = n_{11} \cdot u(1) + n_{12} \cdot u(2) + n_{13}, m_2 = n_{21} \cdot u(1) + n_{22} \cdot u(2) + n_{23}, m_3 = n_{31} \cdot u(1) + n_{32} \cdot u(2) + n_{33}, m_4 = n_{11} \cdot v(1) + n_{12} \cdot v(2) + n_{13}, m_5 = n_{21} \cdot v(1) + n_{22} \cdot v(2) + n_{23}, m_6 = n_{31} \cdot v(1) + n_{32} \cdot v(2) + n_{33}, m_7 = n_{11} \cdot w(1) + n_{12} \cdot w(2) + n_{13}, m_8 = n_{21} \cdot w(1) + n_{22} \cdot w(2) + n_{23}, m_9 = n_{31} \cdot w(1) + n_{32} \cdot w(2) + n_{33}$ as a real number.

Consider $l$ being a real number such that $0 \leq l \leq 1$ and $\text{RP3toT2}(Q) = (1-l) \cdot \text{RP3toT2}(P) + l \cdot \text{RP3toT2}(R)$. Set $r = \frac{l \cdot m_6}{(1-l) \cdot m_3 + l \cdot m_6} \cdot (1-r) \cdot \frac{m_2}{m_3}, \frac{m_4}{m_6}, \frac{m_5}{m_6}, 1 = \frac{m_7}{m_9}, \frac{m_8}{m_9}, 1$. $0 \leq r \leq 1$.

(49) Let us consider real numbers $a, b, c$, and elements $u, v, w$ of $E^3_T$. Suppose $a \neq 0$ and $a + b + c = 0$ and $a \cdot u + b \cdot v + c \cdot w = 0_{E^3_T}$. Then $u \in \text{Line}(v, w)$.

(50) Let us consider non point at $\infty$ points $P, Q, R$ of the projective space over $E^3_T$, and non zero elements $u, v, w$ of $E^3_T$. Suppose $P$ = the direction of $u$ and $Q$ = the direction of $v$ and $R$ = the direction of $w$ and $(u)_3 = 1$ and $(v)_3 = 1$ and $(w)_3 = 1$. Then $P, Q$ and $R$ are collinear if and only if $u, v$ and $w$ are collinear. The theorem is a consequence of (49).

(51) Let us consider elements $u, v, w$ of $E^3_T$. Suppose $u \in \mathcal{L}(v, w)$. Then $[(u)_1, (u)_2] \in \mathcal{L}([(v)_1, (v)_2], [(w)_1, (w)_2])$.

(52) Let us consider elements $u, v, w$ of $E^3_T$. Suppose $u \in \mathcal{L}(v, w)$. Then $[(u)_1, (u)_2, 1] \in \mathcal{L}([(v)_1, (v)_2, 1], [(w)_1, (w)_2, 1])$. 
Let us consider non point at $\infty$ elements $P$, $Q$, $R$ of the projective space over $\mathcal{E}_T^3$. Then $P$, $Q$ and $R$ are collinear if and only if $\text{RP3toT2}(P)$, $\text{RP3toT2}(Q)$ and $\text{RP3toT2}(R)$ are collinear. The theorem is a consequence of (50), (51), and (52).

Let us consider non point at $\infty$ elements $P$, $Q$, $P_1$ of the projective space over $\mathcal{E}_T^3$. Suppose $P$, $Q$ $\in$ the BK-model and $P_1$ $\in$ the absolute. Then $\text{RP3toT2}(P_1)$ does not lie between $\text{RP3toT2}(Q)$ and $\text{RP3toT2}(P)$. The theorem is a consequence of (52) and (27).

The functor $\text{Dir001}$ yielding a non point at $\infty$ element of the projective space over $\mathcal{E}_T^3$ is defined by the term

(Def. 7) \[ \text{the direction of } [0,0,1]. \]

The functor $\text{Dir101}$ yielding a non point at $\infty$ element of the projective space over $\mathcal{E}_T^3$ is defined by the term

(Def. 8) \[ \text{the direction of } [1,0,1]. \]

Now we state the propositions:

Let us consider non point at $\infty$ elements $P$, $Q$ of the projective space over $\mathcal{E}_T^3$. Suppose $P$, $Q$ $\in$ the absolute. Then $\text{RP3toT2}(\text{Dir001}) \text{ RP3toT2}(P) \cong \text{RP3toT2}(\text{Dir001}) \text{ RP3toT2}(Q)$.

Let us consider non point at $\infty$ elements $P$, $Q$, $R$ of the projective space over $\mathcal{E}_T^3$, and non zero elements $u$, $v$, $w$ of $\mathcal{E}_T^3$. Suppose $P$, $Q$ $\in$ the absolute and $P \neq Q$ and $P$ $=$ the direction of $u$ and $Q$ $=$ the direction of $v$ and $R$ $=$ the direction of $w$ and $(u)_3 = 1$ and $(v)_3 = 1$ and $w = \frac{(u)_1 + (v)_1}{2}, \frac{(u)_2 + (v)_2}{2}, 1$. Then $R$ $\in$ the BK-model.

Proof: Reconsider $u' = [u(1), u(2)], v' = [v(1), v(2)]$ as an element of $\mathcal{E}_T^2$. $u' \neq v'$. Reconsider $r_8 = [(w)_1, (w)_2]$ as an element of the inside of circle(0,0,1). Consider $R_3$ being an element of $\mathcal{E}_T^2$ such that $R_3 = r_8$ and $\text{REAL2-to-BK}(r_8) = \text{the direction of } [(R_3)_1, (R_3)_2, 1]$. □

Let us consider points $R_1$, $R_2$ of TarskiEuclid2Space. Suppose $\hat{R}_1$, $\hat{R}_2$ $\in$ circle(0,0,1) and $R_1 \neq R_2$. Then there exists an element $P$ of BK-model-Plane such that BKtoT2($P$) lies between $R_1$ and $R_2$. The theorem is a consequence of (47), (57), and (26).

Let us consider non point at $\infty$ elements $P$, $Q$ of the projective space
over $\mathcal{E}_T^3$. If $\text{RP}3\text{to}T2(P) = \text{RP}3\text{to}T2(Q)$, then $P = Q$.

(60) Let us consider non point at $\infty$ elements $R_1, R_2$ of the projective space over $\mathcal{E}_T^3$. Suppose $R_1, R_2 \in$ the absolute and $R_1 \neq R_2$. Then there exists an element $P$ of BK-model-Plane such that $\text{BK}\text{to}T2(P)$ lies between $\text{RP}3\text{to}T2(R_1)$ and $\text{RP}3\text{to}T2(R_2)$. The theorem is a consequence of (59) and (58).

(61) Let us consider points $P, Q, R$ of TarskiEuclid2Space. Suppose $Q$ lies between $P$ and $R$ and $\hat{P}, \hat{R} \in$ the inside of circle$(0,0,1)$. Then $Q \in$ the inside of circle$(0,0,1)$.

Let us consider a non point at $\infty$ element $P$ of the projective space over $\mathcal{E}_T^3$.

(62) If $P \in$ the absolute, then $\text{RP}3\text{to}REAL2(P) \in$ circle$(0,0,1)$.

(63) If $P \in$ the BK-model, then $\text{RP}3\text{to}REAL2(P) \in$ the inside of circle$(0,0,1)$. The theorem is a consequence of (26).

(64) Let us consider a non point at $\infty$ point $P$ of the projective space over $\mathcal{E}_T^3$, and an element $Q$ of the BK-model. If $P = Q$, then $\text{RP}3\text{to}REAL2(P) = \text{BK-to-REAL2}(Q)$. The theorem is a consequence of (26).

(65) Let us consider non point at $\infty$ elements $P, Q, R_1, R_2$ of the projective space over $\mathcal{E}_T^3$. Suppose $P \neq Q$ and $P \in$ the BK-model and $R_1, R_2 \in$ the absolute and $\text{RP}3\text{to}T2(Q)$ lies between $\text{RP}3\text{to}T2(P)$ and $\text{RP}3\text{to}T2(R_1)$ and $\text{RP}3\text{to}T2(Q)$ lies between $\text{RP}3\text{to}T2(P)$ and $\text{RP}3\text{to}T2(R_2)$. Then $R_1 = R_2$. The theorem is a consequence of (60), (59), (62), (64), (8), and (61).

(66) Let us consider non point at $\infty$ elements $P, Q, P_1, P_2$ of the projective space over $\mathcal{E}_T^3$. Suppose $P \neq Q$ and $P, Q \in$ the BK-model and $P_1, P_2 \in$ the absolute and $P_1 \neq P_2$ and $P, Q$ and $P_1$ are collinear and $P, Q$ and $P_2$ are collinear. Then

(i) $\text{RP}3\text{to}T2(P)$ lies between $\text{RP}3\text{to}T2(Q)$ and $\text{RP}3\text{to}T2(P_1)$, or

(ii) $\text{RP}3\text{to}T2(P)$ lies between $\text{RP}3\text{to}T2(Q)$ and $\text{RP}3\text{to}T2(P_2)$.

The theorem is a consequence of (55), (53), and (65).

Let us consider elements $P, Q$ of the BK-model. Now we state the propositions:

(67) Suppose $P \neq Q$. Then there exists an element $R$ of the absolute such that for every non point at $\infty$ elements $p, q, r$ of the projective space over $\mathcal{E}_T^3$ such that $p = P$ and $q = Q$ and $r = R$ holds $\text{RP}3\text{to}T2(p)$ lies between $\text{RP}3\text{to}T2(q)$ and $\text{RP}3\text{to}T2(r)$. The theorem is a consequence of (47) and (66).

(68) Suppose $P \neq Q$. Then there exists an element $R$ of the absolute such that for every non point at $\infty$ elements $p, q, r$ of the projective space over...
such that \( p = P \) and \( q = Q \) and \( r = R \) holds \( \text{RP3toT2}(q) \) lies between \( \text{RP3toT2}(p) \) and \( \text{RP3toT2}(r) \). The theorem is a consequence of (67).

(69) The direction of \([1, 0, 1]\) is an element of the absolute.

(70) Let us consider points \( a, b \) of BK-model-Plane. Then \( \overline{aa} \cong \overline{bb} \). The theorem is a consequence of (69).

(71) Every element of the BK-model is a non point at \( \infty \) element of the projective space over \( \mathcal{E}_T^3 \). The theorem is a consequence of (47).

(72) Every element of the absolute is a non point at \( \infty \) element of the projective space over \( \mathcal{E}_T^3 \). The theorem is a consequence of (47).

(73) Let us consider an element \( P \) of the BK-model, and a non point at \( \infty \) element \( P' \) of the projective space over \( \mathcal{E}_T^3 \). If \( P = P' \), then \( \text{RP3toREAL2}(P') = \text{BKtoREAL2}(P) \). The theorem is a consequence of (26).

(74) Let us consider points \( a, q, b, c \) of BK-model-Plane. Then there exists a point \( x \) of BK-model-Plane such that

(i) \( a \) lies between \( q \) and \( x \), and

(ii) \( \overline{ax} \cong \overline{bc} \).

The theorem is a consequence of (71), (68), (12), (70), (48), and (73).

(75) Let us consider points \( P, Q \) of BK-model-Plane. If \( \text{BKtoT2}(P) = \text{BKtoT2}(Q) \), then \( P = Q \).

(76) Let us consider real numbers \( a, b, r \), and elements \( P, Q, R \) of \( \mathcal{E}_T^3 \). Suppose \( P, R \in \text{the inside of circle}(a,b,r) \). Then \( \mathcal{L}(P,R) \subseteq \text{the inside of circle}(a,b,r) \).

2. THE AXIOM OF SEGMENT CONSTRUCTION

Now we state the proposition:

(77) BK-model-Plane satisfies the axiom of segment construction.

3. THE AXIOM OF BETWEENNESS IDENTITY

Now we state the proposition:

(78) BK-model-Plane satisfies the axiom of betweenness identity. The theorem is a consequence of (12) and (75).
4. The Axiom of Pasch

Now we state the proposition:

(79) BK-model-Plane satisfies the axiom of Pasch. The theorem is a consequence of (12), (8), (25), and (10).

References


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