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Klein-Beltrami model. Part III

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Summary. Timothy Makarios (with Isabelle/HOL[1]) and John Harrison (with HOL-Light[2]) shown that "the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski's axioms except his Euclidean axiom" [2],[3],[4],[5].

With the Mizar system [1] we use some ideas taken from Tim Makarios's MSc thesis [10] to formalize some definitions (like the absolute) and lemmas necessary for the verification of the independence of the parallel postulate. In this article we prove that our constructed model (we prefer "Beltrami-Klein" name over "Klein-Beltrami", which can be seen in the naming convention for Mizar functors, and even MML identifiers) satisfies the congruence symmetry, the congruence equivalence relation, and the congruence identity axioms formulated by Tarski (and formalized in Mizar as described briefly in [8]).

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1. Preliminaries

Now we state the propositions:

(1) Let us consider real numbers x, y. If $x \cdot y < 0$, then $0 < \frac{x}{x-y} < 1$.

(2) Let us consider a non zero real number a, and real numbers b, r. Suppose $r = \sqrt{a^{\mathbf{2}} + b^{\mathbf{2}}}$. Then

(i) r is positive, and

[1] https://www.isa-afp.org/entries/Tarskis_Geometry.html
[2] https://github.com/jrh13/hol-light/blob/master/100/independence.ml

(ii) $(\frac{a}{r})^{\mathbf{2}} + (\frac{b}{r})^{\mathbf{2}} = 1$.

(3) Let us consider a non zero real number a, and real numbers b, c, d, e. Suppose $a \cdot b = c - d \cdot e$. Then $b^{\mathbf{2}} = \frac{c^{\mathbf{2}}}{a^{\mathbf{2}}} - 2 \cdot \frac{c \cdot d}{a \cdot a} \cdot e + \frac{d^{\mathbf{2}}}{a^{\mathbf{2}}} \cdot e^{\mathbf{2}}$.

Let us consider complex numbers a, b, c. Now we state the propositions:

(4) If $a \neq 0$, then $\frac{a^{\mathbf{2}} \cdot b \cdot c}{a^{\mathbf{2}}} = b \cdot c$.

(5) If $a \neq 0$, then $\frac{2 \cdot a^{\mathbf{2}} \cdot b \cdot c}{a^{\mathbf{2}}} = 2 \cdot b \cdot c$. The theorem is a consequence of (4).

(6) Let us consider a real number a. If $1 < a$, then $\frac{1}{a} - 1 < 0$.

(7) Let us consider real numbers a, b. If $0 < a$ and $1 < b$, then $\frac{a}{b} - a < 0$. The theorem is a consequence of (6).

(8) Let us consider a non zero real number a, and real numbers b, c, d. Suppose $a^{\mathbf{2}} + c^{\mathbf{2}} = b^{\mathbf{2}}$ and $1 < b^{\mathbf{2}}$. Then $\frac{(b^{\mathbf{2}})^{\mathbf{2}}}{a^{\mathbf{2}}} - 2 \cdot \frac{b^{\mathbf{2}} \cdot c}{a \cdot a} \cdot d + \frac{c^{\mathbf{2}}}{a^{\mathbf{2}}} \cdot d^{\mathbf{2}} + d^{\mathbf{2}} \neq 1$. The theorem is a consequence of (5) and (7).

(9) Let us consider real numbers a, b, c. If $a \cdot (-b) = c$ and $a \cdot c = b$, then $c = 0$ and $b = 0$.

(10) Let us consider a positive real number a. Then $\frac{\sqrt{a}}{a} = \frac{1}{\sqrt{a}}$.

2. Planar Lemmas

Let a be a non zero real number and b, c be real numbers. Observe that $[a, b, c]$ is non zero as an element of $\mathcal{E}^3_{\mathrm{T}}$ and $[c, a, b]$ is non zero as an element of $\mathcal{E}^3_{\mathrm{T}}$ and $[b, c, a]$ is non zero as an element of $\mathcal{E}^3_{\mathrm{T}}$.

Let P be an element of the real projective plane. Assume $P \in$ (the absolute)$\cup$ (the BK-model). The functor $\# P$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by

(Def. 1) the direction of $it = P$ and $it(3) = 1$.

Now we state the propositions:

(11) Let us consider an element P of the real projective plane. Then there exists an element Q of the BK-model such that $P \neq Q$.

From now on P denotes an element of the BK-model.

(12) There exist elements P_1, P_2 of the absolute such that

(i) $P_1 \neq P_2$, and

(ii) P_1, P and P_2 are collinear.

The theorem is a consequence of (11).

(13) Let us consider an element Q of the absolute. Then there exists an element R of the BK-model such that

(i) $P \neq R$, and

(ii) P, Q and R are collinear.

(14) Let us consider a line L of Inc-ProjSp(the real projective plane). Suppose $P \in L$. Then there exist elements P_1, P_2 of the absolute such that

(i) $P_1 \neq P_2$, and

(ii) P_1, $P_2 \in L$.

Let N be an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3. The functor Line-homography(N) yielding a function from the lines of Inc-ProjSp(the real projective plane) into the lines of Inc-ProjSp(the real projective plane) is defined by

(Def. 2) for every line x of Inc-ProjSp(the real projective plane), $it(x) =$ {(the homography of N)(P), where P is a point of Inc-ProjSp(the real projective plane) : P lies on x}.

In the sequel N, N_1, N_2 denote invertible square matrices over $\mathbb{R}_{\rm F}$ of dimension 3 and l, l_1, l_2 denote elements of the lines of Inc-ProjSp(the real projective plane). Now we state the propositions:

(15) (Line-homography(N_1))((Line-homography(N_2))(l)) = (Line-homography($N_1 \cdot N_2$))(l).

PROOF: Reconsider $l_2 =$ (Line-homography(N_2))(l) as a line of Inc-ProjSp (the real projective plane). {(the homography of N_1)(P), where P is a point of Inc-ProjSp(the real projective plane) : P lies on l_2} = {(the homography of $N_1 \cdot N_2$)(P), where P is a point of Inc-ProjSp(the real projective plane) : P lies on l} by [9, (3), (4), (5)], [6, (13)]. □

(16) (Line-homography($I^{3\times 3}_{\mathbb{R}_{\rm F}}$))($l$) $= l$.

PROOF: Set X = {(the homography of $I^{3\times 3}_{\mathbb{R}_{\rm F}}$)($P$), where P is a point of Inc-ProjSp(the real projective plane) : P lies on l}. $X \subseteq l$. $l \subseteq X$. □

(17) (i) (Line-homography(N))((Line-homography($N^{\smile}$))(l)) $= l$, and

(ii) (Line-homography($N^{\smile}$))((Line-homography(N))(l)) $= l$.

The theorem is a consequence of (15) and (16).

(18) If (Line-homography(N))(l_1) = (Line-homography(N))(l_2), then $l_1 = l_2$. The theorem is a consequence of (17).

The functor SetLineHom3 yielding a set is defined by the term

(Def. 3) the set of all Line-homography(N) where N is an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3.

Observe that SetLineHom3 is non empty. Let h_1, h_2 be elements of SetLineHom3. The functor $h_1 \circ h_2$ yielding an element of SetLineHom3 is defined by

(Def. 4) there exist invertible square matrices N_1, N_2 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h_1 = \text{Line-homography}(N_1)$ and $h_2 = \text{Line-homography}(N_2)$ and $it = \text{Line-homography}(N_1 \cdot N_2)$.

Now we state the propositions:

(19) Let us consider elements h_1, h_2 of SetLineHom3. Suppose $h_1 = \text{Line-homography}(N_1)$ and $h_2 = \text{Line-homography}(N_2)$.
Then $\text{Line-homography}(N_1 \cdot N_2) = h_1 \circ h_2$. The theorem is a consequence of (15).

(20) Let us consider elements x, y, z of SetLineHom3. Then $(x \circ y) \circ z = x \circ (y \circ z)$. The theorem is a consequence of (19).

The functor BinOpLineHom3 yielding a binary operation on SetLineHom3 is defined by

(Def. 5) for every elements h_1, h_2 of SetLineHom3, $it(h_1, h_2) = h_1 \circ h_2$.

The functor GroupLineHom3 yielding a strict multiplicative magma is defined by the term

(Def. 6) $\langle \text{SetLineHom3}, \text{BinOpLineHom3} \rangle$.

Let us observe that GroupLineHom3 is non empty, associative, and group-like. Now we state the propositions:

(21) $\mathbf{1}_{\text{GroupLineHom3}} = \text{Line-homography}(I^{3\times 3}_{\mathbb{R}_{\mathrm{F}}})$.

(22) Let us consider elements h, g of GroupLineHom3, and invertible square matrices N, N_1 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $h = \text{Line-homography}(N)$ and $g = \text{Line-homography}(N_1)$ and $N_1 = N^{\smile}$. Then $g = h^{-1}$. The theorem is a consequence of (21).

In the sequel P denotes a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ and l denotes a line of Inc-ProjSp(the real projective plane).

(23) If (the homography of N)$(P) \in l$, then $P \in (\text{Line-homography}(N^{\smile}))(l)$.

(24) If $P \in (\text{Line-homography}(N))(l)$, then (the homography of $N^{\smile}$)$(P) \in l$.

(25) $P \in l$ if and only if (the homography of N)$(P) \in (\text{Line-homography}(N))$ (l). The theorem is a consequence of (23) and (17).

(26) Let us consider non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $(u)_{\mathbf{3}} = 1$ and $(v)_{\mathbf{1}} = -(u)_{\mathbf{2}}$ and $(v)_{\mathbf{2}} = (u)_{\mathbf{1}}$ and $(v)_{\mathbf{3}} = 0$ and $(w)_{\mathbf{3}} = 1$ and $\langle |u, v, w| \rangle = 0$. Then $((u)_{\mathbf{1}})^{\mathbf{2}} + ((u)_{\mathbf{2}})^{\mathbf{2}} - (u)_{\mathbf{1}} \cdot (w)_{\mathbf{1}} - (u)_{\mathbf{2}} \cdot (w)_{\mathbf{2}} = 0$.

(27) Let us consider a non zero real number a, and real numbers b, c. Then $a \cdot [\frac{b}{a}, \frac{c}{a}, 1] = [b, c, a]$.

Let us consider non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(28) Suppose $(u)_{\mathbf{1}} \neq 0$ and $(u)_{\mathbf{3}} = 1$ and $(v)_{\mathbf{1}} = -(u)_{\mathbf{2}}$ and $(v)_{\mathbf{2}} = (u)_{\mathbf{1}}$ and $(v)_{\mathbf{3}} = 0$ and $(w)_{\mathbf{3}} = 1$ and $\langle|u, v, w|\rangle = 0$ and $1 < ((u)_{\mathbf{1}})^{\mathbf{2}} + ((u)_{\mathbf{2}})^{\mathbf{2}}$. Then $((w)_{\mathbf{1}})^{\mathbf{2}} + ((w)_{\mathbf{2}})^{\mathbf{2}} \neq 1$. The theorem is a consequence of (26), (2), (3), and (8).

(29) Suppose $(u)_{\mathbf{2}} \neq 0$ and $(u)_{\mathbf{3}} = 1$ and $(v)_{\mathbf{1}} = -(u)_{\mathbf{2}}$ and $(v)_{\mathbf{2}} = (u)_{\mathbf{1}}$ and $(v)_{\mathbf{3}} = 0$ and $(w)_{\mathbf{3}} = 1$ and $\langle|u, v, w|\rangle = 0$ and $1 < ((u)_{\mathbf{1}})^{\mathbf{2}} + ((u)_{\mathbf{2}})^{\mathbf{2}}$. Then $((w)_{\mathbf{1}})^{\mathbf{2}} + ((w)_{\mathbf{2}})^{\mathbf{2}} \neq 1$. The theorem is a consequence of (26), (2), (3), and (8).

(30) Let us consider an element P of the absolute. Then there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that

  (i) $u(3) = 1$, and

  (ii) $P =$ the direction of u.

(31) Let us consider real numbers a, b, c, d, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = [a, b, 1]$ and $v = [c, d, 0]$. Then the direction of $u \neq$ the direction of v.

(32) Let us consider a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u(1)^{\mathbf{2}} + u(2)^{\mathbf{2}} < 1$ and $u(3) = 1$. Then the direction of u is an element of the BK-model.

(33) Let us consider real numbers a, b. Suppose $a^{\mathbf{2}}+b^{\mathbf{2}} \leqslant 1$. Then the direction of $[a, b, 1] \in$ (the BK-model) $\cup$ (the absolute). The theorem is a consequence of (32).

(34) If $P \notin$ (the BK-model) $\cup$ (the absolute), then there exists l such that $P \in l$ and l misses the absolute. The theorem is a consequence of (9), (30), (27), (31), (33), (28), and (29).

(35) Let us consider a point P of the real projective plane, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $h =$ the homography of N. Then P is an element of the absolute if and only if (the homography of N)(P) is an element of the absolute.

Let us consider an element P of the BK-model, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3.

(36) If $h =$ the homography of N, then (the homography of N)(P) is an element of the BK-model.

PROOF: Set $h_1 =$ (the homography of N)(P). h_1 is not an element of the absolute by (35), [7, (1)]. Consider l such that $h_1 \in l$ and l misses the absolute. Reconsider $L =$ (Line-homography($N^{\smile}$))(l) as a line of the real projective plane. Reconsider $L' = L$ as a line of Inc-ProjSp(the real projective plane). Consider P_1, P_2 being elements of the absolute such that

$P_1 \neq P_2$ and $P_1 \in L'$ and $P_2 \in L'$. (The homography of $N)(P_1)$ is an element of the absolute. (The homography of $N)(P_1) \in$ (Line-homography(N)) (L). (The homography of $N)(P_1) \in l$. $\square$

(37) Suppose $h =$ the homography of N. Then there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that

(i) (the homography of $N)(P) =$ the direction of u, and

(ii) $u(3) = 1$.

The theorem is a consequence of (36).

3. The Construction of Beltrami-Klein Model

The functor BK-model-Betweenness yielding a relation between (the BK-model) $\times$ (the BK-model) and the BK-model is defined by

(Def. 7) for every elements a, b, c of the BK-model, $\langle\langle a, b\rangle, c\rangle \in it$ iff BK-to-REAL2(b) $\in \mathcal{L}$(BK-to-REAL2(a), BK-to-REAL2(c)).

The functor BK-model-Equidistance yielding a relation between (the BK-model) $\times$ (the BK-model) and (the BK-model) $\times$ (the BK-model) is defined by

(Def. 8) for every elements a, b, c, d of the BK-model, $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in it$ iff there exists an element h of the subgroup of K-isometries and there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h =$ the homography of N and (the homography of $N)(a) = c$ and (the homography of $N)(b) = d$.

The functor BK-model-Plane yielding a Tarski plane is defined by the term

(Def. 9) $\langle$the BK-model, BK-model-Betweenness, BK-model-Equidistance$\rangle$.

4. Congruence Symmetry

Now we state the proposition:

(38) BK-model-Plane satisfies the axiom of congruence symmetry.

5. Congruence Equivalence Relation

Now we state the proposition:

(39) BK-model-Plane satisfies the axiom of congruence equivalence relation.

6. Congruence Identity

Now we state the proposition:

(40) BK-model-Plane satisfies the axiom of congruence identity.

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