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A Simple Example for Linear Partial Differential Equations and Its Solution Using the Method of Separation of Variables

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Summary. In this article, we formalized in Mizar [4], [1] simple partial differential equations. In the first section, we formalized partial differentiability and partial derivative. The next section contains the method of separation of variables for one-dimensional wave equation. In the last section, we formalized the superposition principle. We referred to [6], [3], [5] and [9] in this formalization.

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1. Preliminaries

From now on m, n denote non zero elements of $\mathbb{N}$, i, j, k denote elements of $\mathbb{N}$, Z denotes a subset of $\mathcal{R}^2$, c denotes a real number, I denotes a non empty finite sequence of elements of $\mathbb{N}$, and d_1, d_2 denote elements of $\mathbb{R}$.

Now we state the proposition:

(1) Let us consider a non zero element m of $\mathbb{N}$, a subset X of $\mathcal{R}^m$, a non empty finite sequence I of elements of $\mathbb{N}$, and a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose f is partially differentiable on X w.r.t. I. Then $\mathrm{dom}(f{\restriction}^I X) = X$.

Let us note that $\Omega_{\mathbb{R}}$ is open and $\Omega_{\mathcal{R}^2}$ is open.

Now we state the proposition:

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(2) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a subset Z of $\mathbb{R}$, and a real number x_0. Suppose Z is open and $x_0 \in Z$. Then

(i) f is differentiable in x_0 iff $f{\restriction}Z$ is differentiable in x_0, and

(ii) if f is differentiable in x_0, then $f'(x_0) = (f{\restriction}Z)'(x_0)$.

PROOF: f is differentiable in x_0 iff $f{\restriction}Z$ is differentiable in x_0. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a subset X of $\mathbb{R}$. Now we state the propositions:

(3) If X is open and $X \subseteq \operatorname{dom} f$, then f is differentiable on X iff $f{\restriction}X$ is differentiable on X. The theorem is a consequence of (2).

(4) If X is open and $X \subseteq \operatorname{dom} f$ and f is differentiable on X, then $(f{\restriction}X)'_{\restriction X} = f'_{\restriction X}$. The theorem is a consequence of (3) and (2).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a subset Z of $\mathbb{R}$. Now we state the propositions:

(5) If $Z \subseteq \operatorname{dom} f$ and Z is open and f is differentiable 1 times on Z, then f is differentiable on Z and $(f'(Z))(1) = f'_{\restriction Z}$. The theorem is a consequence of (3) and (4).

(6) Suppose $Z \subseteq \operatorname{dom} f$ and Z is open and f is differentiable 2 times on Z. Then

(i) f is differentiable on Z, and

(ii) $(f'(Z))(1) = f'_{\restriction Z}$, and

(iii) $f'_{\restriction Z}$ is differentiable on Z, and

(iv) $(f'(Z))(2) = (f'_{\restriction Z})'_{\restriction Z}$.

The theorem is a consequence of (5).

(7) Let us consider subsets X, T of $\mathbb{R}$, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a partial function g from $\mathbb{R}$ to $\mathbb{R}$. Suppose $X \subseteq \operatorname{dom} f$ and $T \subseteq \operatorname{dom} g$. Then there exists a partial function u from $\mathcal{R}^2$ to $\mathbb{R}$ such that

(i) $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in X$ and $t \in T\}$, and

(ii) for every real numbers x, t such that $x \in X$ and $t \in T$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$.

PROOF: Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exist real numbers x, t such that $x \in X$ and $t \in T$ and $\$_1 = \langle x, t\rangle$ and $\$_2 = f_{/x} \cdot (g_{/t})$. For every objects z, w_1, w_2 such that $z \in \mathcal{R}^2$ and $\mathcal{Q}[z, w_1]$ and $\mathcal{Q}[z, w_2]$ holds $w_1 = w_2$. Consider u being a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ such that for every object z, $z \in \operatorname{dom} u$ iff $z \in \mathcal{R}^2$ and there exists an object w such that $\mathcal{Q}[z, w]$ and for every object z such that $z \in \operatorname{dom} u$ holds $\mathcal{Q}[z, u(z)]$. For every object z,

$z \in \operatorname{dom} u$ iff $z \in \{\langle x, t\rangle$, where x, t are real numbers : $x \in X$ and $t \in T\}$. Consider x_1, t_1 being real numbers such that $x_1 \in X$ and $t_1 \in T$ and $\langle x, t\rangle = \langle x_1, t_1\rangle$ and $u(\langle x, t\rangle) = f_{/x_1} \cdot (g_{/t_1})$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a partial function g from $\mathbb{R}$ to $\mathbb{R}$, a partial function u from $\mathcal{R}^2$ to $\mathbb{R}$, real numbers x_0, t_0, and an element z of $\mathcal{R}^2$. Now we state the propositions:

(8) Suppose $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g\}$ and for every real numbers x, t such that $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$ and $z = \langle x_0, t_0\rangle$ and $x_0 \in \operatorname{dom} f$ and $t_0 \in \operatorname{dom} g$. Then

(i) $u \cdot (\operatorname{reproj}(1, z)) = g_{/t_0} \cdot f$, and

(ii) $u \cdot (\operatorname{reproj}(2, z)) = f_{/x_0} \cdot g$.

Proof: For every object s, $s \in \operatorname{dom}(u \cdot (\operatorname{reproj}(1, z)))$ iff $s \in \operatorname{dom} f$. For every object s, $s \in \operatorname{dom}(u \cdot (\operatorname{reproj}(2, z)))$ iff $s \in \operatorname{dom} g$. For every object s such that $s \in \operatorname{dom}(u \cdot (\operatorname{reproj}(1, z)))$ holds $(u \cdot (\operatorname{reproj}(1, z)))(s) = (g_{/t_0} \cdot f)(s)$. For every object s such that $s \in \operatorname{dom}(u \cdot (\operatorname{reproj}(2, z)))$ holds $(u \cdot (\operatorname{reproj}(2, z)))(s) = (f_{/x_0} \cdot g)(s)$ by [7, (14)]. □

(9) Suppose $x_0 \in \operatorname{dom} f$ and $t_0 \in \operatorname{dom} g$ and $z = \langle x_0, t_0\rangle$ and $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g\}$ and f is differentiable in x_0 and for every real numbers x, t such that $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$. Then

(i) u is partially differentiable in z w.r.t. 1, and

(ii) $\operatorname{partdiff}(u, z, 1) = f'(x_0) \cdot (g_{/t_0})$.

The theorem is a consequence of (8).

(10) Suppose $x_0 \in \operatorname{dom} f$ and $t_0 \in \operatorname{dom} g$ and $z = \langle x_0, t_0\rangle$ and $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g\}$ and g is differentiable in t_0 and for every real numbers x, t such that $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$. Then

(i) u is partially differentiable in z w.r.t. 2, and

(ii) $\operatorname{partdiff}(u, z, 2) = f_{/x_0} \cdot (g'(t_0))$.

The theorem is a consequence of (8).

Let us consider subsets X, T of $\mathbb{R}$, a subset Z of $\mathcal{R}^2$, a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a partial function g from $\mathbb{R}$ to $\mathbb{R}$, and a partial function u from $\mathcal{R}^2$ to $\mathbb{R}$. Now we state the propositions:

(11) Suppose $X \subseteq \operatorname{dom} f$ and $T \subseteq \operatorname{dom} g$ and X is open and T is open and Z is open and $Z = \{\langle x, t\rangle$, where x, t are real numbers : $x \in X$ and $t \in T\}$ and $\operatorname{dom} u = \{\langle x, t\rangle$, where x, t are real numbers : $x \in \operatorname{dom} f$ and

$t \in \operatorname{dom} g\}$ and f is differentiable on X and g is differentiable on T and for every real numbers x, t such that $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$. Then

(i) u is partially differentiable on Z w.r.t. $\langle 1\rangle$, and

(ii) for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction^{\langle 1\rangle}}Z)_{/\langle x,t\rangle} = f'(x) \cdot (g_{/t})$, and

(iii) u is partially differentiable on Z w.r.t. $\langle 2\rangle$, and

(iv) for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction^{\langle 2\rangle}}Z)_{/\langle x,t\rangle} = f_{/x} \cdot (g'(t))$.

Proof: $Z \subseteq \operatorname{dom} u$. For every element z of $\mathcal{R}^2$ such that $z \in Z$ holds u is partially differentiable in z w.r.t. 1. For every real numbers x, t and for every element z of $\mathcal{R}^2$ such that $x \in X$ and $t \in T$ and $z = \langle x,t\rangle$ holds $\operatorname{partdiff}(u,z,1) = f'(x) \cdot (g_{/t})$. For every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction^{\langle 1\rangle}}Z)_{/\langle x,t\rangle} = f'(x) \cdot (g_{/t})$. For every element z of $\mathcal{R}^2$ such that $z \in Z$ holds u is partially differentiable in z w.r.t. 2. For every real numbers x, t and for every element z of $\mathcal{R}^2$ such that $x \in X$ and $t \in T$ and $z = \langle x,t\rangle$ holds $\operatorname{partdiff}(u,z,2) = f_{/x} \cdot (g'(t))$. $\square$

(12) Suppose $X \subseteq \operatorname{dom} f$ and $T \subseteq \operatorname{dom} g$ and X is open and T is open and Z is open and $Z = \{\langle x,t\rangle$, where x,t are real numbers : $x \in X$ and $t \in T\}$ and $\operatorname{dom} u = \{\langle x,t\rangle$, where x,t are real numbers : $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g\}$ and f is differentiable 2 times on X and g is differentiable 2 times on T and for every real numbers x, t such that $x \in \operatorname{dom} f$ and $t \in \operatorname{dom} g$ holds $u_{/\langle x,t\rangle} = f_{/x} \cdot (g_{/t})$. Then

(i) u is partially differentiable on Z w.r.t. $\langle 1\rangle \frown \langle 1\rangle$, and

(ii) for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction^{\langle 1\rangle \frown \langle 1\rangle}}Z)_{/\langle x,t\rangle} = (f'(X))(2)_{/x} \cdot (g_{/t})$, and

(iii) u is partially differentiable on Z w.r.t. $\langle 2\rangle \frown \langle 2\rangle$, and

(iv) for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction^{\langle 2\rangle \frown \langle 2\rangle}}Z)_{/\langle x,t\rangle} = f_{/x} \cdot ((g'(T))(2)_{/t})$.

Proof: u is partially differentiable on Z w.r.t. $\langle 1\rangle$ and for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction^{\langle 1\rangle}}Z)_{/\langle x,t\rangle} = f'(x) \cdot (g_{/t})$ and u is partially differentiable on Z w.r.t. $\langle 2\rangle$ and for every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u{\restriction^{\langle 2\rangle}}Z)_{/\langle x,t\rangle} = f_{/x} \cdot (g'(t))$. u is partially differentiable on Z w.r.t. 1. For every real numbers x, t such that $x \in \operatorname{dom}(f'_{\restriction X})$ and $t \in \operatorname{dom}(g{\restriction}T)$ holds $(u{\restriction^{\langle 1\rangle}}Z)_{/\langle x,t\rangle} = (f'_{\restriction X})_{/x} \cdot ((g{\restriction}T)_{/t})$. $u{\restriction^{\langle 1\rangle}}Z$ is partially differentiable on Z w.r.t. $\langle 1\rangle$ and for every real numbers x, t such that $x \in X$ and $t \in T$ holds $((u{\restriction^{\langle 1\rangle}}Z){\restriction^{\langle 1\rangle}}Z)_{/\langle x,t\rangle} =$

$(f'_{\restriction X})'(x) \cdot ((g \restriction T)_{/t})$. For every real numbers x, t such that $x \in X$ and $t \in T$ holds $(u \restriction^{\langle 1 \rangle \frown \langle 1 \rangle} Z)_{/\langle x,t \rangle} = (f'(X))(2)_{/x} \cdot (g_{/t})$. u is partially differentiable on Z w.r.t. 2. For every real numbers x, t such that $x \in \operatorname{dom}(f \restriction X)$ and $t \in \operatorname{dom}(g'_{\restriction T})$ holds $(u \restriction^{\langle 2 \rangle} Z)_{/\langle x,t \rangle} = (f \restriction X)_{/x} \cdot ((g'_{\restriction T})_{/t})$. $u \restriction^{\langle 2 \rangle} Z$ is partially differentiable on Z w.r.t. $\langle 2 \rangle$ and for every real numbers x, t such that $x \in X$ and $t \in T$ holds $((u \restriction^{\langle 2 \rangle} Z) \restriction^{\langle 2 \rangle} Z)_{/\langle x,t \rangle} = (f \restriction X)_{/x} \cdot ((g'_{\restriction T})'(t))$. □

(13) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$, a partial function u from $\mathcal{R}^2$ to $\mathbb{R}$, and a real number c. Suppose f is differentiable 2 times on $\Omega_{\mathbb{R}}$ and g is differentiable 2 times on $\Omega_{\mathbb{R}}$ and $\operatorname{dom} u = \Omega_{\mathcal{R}^2}$ and for every real numbers x, t, $u_{/\langle x,t \rangle} = f_{/x} \cdot (g_{/t})$ and for every real numbers x, t, $f_{/x} \cdot ((g'(\Omega_{\mathbb{R}}))(2)_{/t}) = c^{\mathbf{2}} \cdot ((f'(\Omega_{\mathbb{R}}))(2)_{/x}) \cdot (g_{/t})$. Then

(i) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1 \rangle \frown \langle 1 \rangle$, and

(ii) for every real numbers x, t such that x, $t \in \Omega_{\mathbb{R}}$ holds $(u \restriction^{\langle 1 \rangle \frown \langle 1 \rangle} \Omega_{\mathcal{R}^2})_{/\langle x,t \rangle} = (f'(\Omega_{\mathbb{R}}))(2)_{/x} \cdot (g_{/t})$, and

(iii) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2 \rangle \frown \langle 2 \rangle$, and

(iv) for every real numbers x, t such that x, $t \in \Omega_{\mathbb{R}}$ holds $(u \restriction^{\langle 2 \rangle \frown \langle 2 \rangle} \Omega_{\mathcal{R}^2})_{/\langle x,t \rangle} = f_{/x} \cdot ((g'(\Omega_{\mathbb{R}}))(2)_{/t})$, and

(v) for every real numbers x, t, $(u \restriction^{\langle 2 \rangle \frown \langle 2 \rangle} \Omega_{\mathcal{R}^2})_{/\langle x,t \rangle} = c^{\mathbf{2}} \cdot ((u \restriction^{\langle 1 \rangle \frown \langle 1 \rangle} \Omega_{\mathcal{R}^2})_{/\langle x,t \rangle})$.

The theorem is a consequence of (12).

(14) Let us consider real numbers A, B, e, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = A \cdot (\text{the function cos})(e \cdot x) + B \cdot (\text{the function sin})(e \cdot x)$. Then

(i) f is differentiable on $\Omega_{\mathbb{R}}$, and

(ii) for every real number x, $(f'_{\restriction \Omega_{\mathbb{R}}})(x) = -e \cdot (A \cdot (\text{the function sin})(e \cdot x) - B \cdot (\text{the function cos})(e \cdot x))$.

PROOF: Reconsider $f_1 = A \cdot (\text{the function cos}) \cdot (e \cdot \mathrm{id}_{\Omega_{\mathbb{R}}})$, $f_2 = B \cdot (\text{the function sin}) \cdot (e \cdot \mathrm{id}_{\Omega_{\mathbb{R}}})$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. Reconsider $Z = \Omega_{\mathbb{R}}$ as an open subset of $\mathbb{R}$. Reconsider $E = e \cdot \mathrm{id}_{\Omega_{\mathbb{R}}}$ as a function from $\mathbb{R}$ into $\mathbb{R}$. For every real number x such that $x \in Z$ holds $E(x) = e \cdot x$. For every object x such that $x \in \operatorname{dom} f$ holds $f(x) = f_1(x) + f_2(x)$. For every real number x, $(f'_{\restriction \Omega_{\mathbb{R}}})(x) = -e \cdot (A \cdot (\text{the function sin})(e \cdot x) - B \cdot (\text{the function cos})(e \cdot x))$. □

2. The Method of Separation of Variables for One-dimensional Wave Equation

Now we state the propositions:

(15) Let us consider real numbers A, B, e, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = A \cdot (\text{the function cos})(e \cdot x) + B \cdot (\text{the function sin})(e \cdot x)$. Then

(i) f is differentiable 2 times on $\Omega_{\mathbb{R}}$, and

(ii) for every real number x, $(f'_{\restriction \Omega_{\mathbb{R}}})(x) = -e \cdot (A \cdot (\text{the function sin})(e \cdot x) - B \cdot (\text{the function cos})(e \cdot x))$ and $((f'_{\restriction \Omega_{\mathbb{R}}})'_{\restriction \Omega_{\mathbb{R}}})(x) = -e^{\mathbf{2}} \cdot (A \cdot (\text{the function cos})(e \cdot x) + B \cdot (\text{the function sin})(e \cdot x))$ and $(f'(\Omega_{\mathbb{R}}))(2)_{/x} + e^{\mathbf{2}} \cdot (f_{/x}) = 0$.

Proof: f is differentiable on $\Omega_{\mathbb{R}}$ and for every real number x, $(f'_{\restriction \Omega_{\mathbb{R}}})(x) = -e \cdot (A \cdot (\text{the function sin})(e \cdot x) - B \cdot (\text{the function cos})(e \cdot x))$. For every real number x, $(f'_{\restriction \Omega_{\mathbb{R}}})(x) = e \cdot B \cdot (\text{the function cos})(e \cdot x) + (-e \cdot A) \cdot (\text{the function sin})(e \cdot x)$. For every natural number i such that $i \leqslant 2 - 1$ holds $(f'(\Omega_{\mathbb{R}}))(i)$ is differentiable on $\Omega_{\mathbb{R}}$. $\square$

(16) Let us consider real numbers A, B, e. Then there exists a function f from $\mathbb{R}$ into $\mathbb{R}$ such that for every real number x, $f(x) = A \cdot (\text{the function cos})(e \cdot x) + B \cdot (\text{the function sin})(e \cdot x)$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a real number t such that $\$_1 = t$ and $\$_2 = A \cdot (\text{the function cos})(e \cdot t) + B \cdot (\text{the function sin})(e \cdot t)$. For every object x such that $x \in \mathbb{R}$ there exists an object y such that $y \in \mathbb{R}$ and $\mathcal{P}[x, y]$. Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every object x such that $x \in \mathbb{R}$ holds $\mathcal{P}[x, f(x)]$. $\square$

(17) Let us consider real numbers A, B, C, d, c, e, and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = A \cdot (\text{the function cos})(e \cdot x) + B \cdot (\text{the function sin})(e \cdot x)$ and for every real number t, $g(t) = C \cdot (\text{the function cos})(e \cdot c \cdot t) + d \cdot (\text{the function sin})(e \cdot c \cdot t)$. Let us consider real numbers x, t. Then $f_{/x} \cdot ((g'(\Omega_{\mathbb{R}}))(2)_{/t}) = c^{\mathbf{2}} \cdot ((f'(\Omega_{\mathbb{R}}))(2)_{/x}) \cdot (g_{/t})$. The theorem is a consequence of (15).

(18) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$, and a function u from $\mathcal{R}^2$ into $\mathbb{R}$. Suppose f is differentiable 2 times on $\Omega_{\mathbb{R}}$ and g is differentiable 2 times on $\Omega_{\mathbb{R}}$ and for every real numbers x, t, $f_{/x} \cdot ((g'(\Omega_{\mathbb{R}}))(2)_{/t}) = c^{\mathbf{2}} \cdot ((f'(\Omega_{\mathbb{R}}))(2)_{/x}) \cdot (g_{/t})$ and for every real numbers x, t, $u_{/\langle x,t \rangle} = f_{/x} \cdot (g_{/t})$. Then

(i) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1 \rangle$, and

(ii) for every real numbers x, t, $(u {\restriction^{\langle 1 \rangle}} \Omega_{\mathcal{R}^2})_{/\langle x,t \rangle} = f'(x) \cdot (g_{/t})$, and

(iii) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle$, and

(iv) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = f_{/x} \cdot (g'(t))$, and

(v) f is differentiable 2 times on $\Omega_{\mathbb{R}}$, and

(vi) g is differentiable 2 times on $\Omega_{\mathbb{R}}$, and

(vii) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle \frown \langle 1\rangle$, and

(viii) for every real numbers x, t, $(u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = (f'(\Omega_{\mathbb{R}}))(2)_{/x} \cdot (g_{/t})$, and

(ix) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle \frown \langle 2\rangle$, and

(x) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = f_{/x} \cdot ((g'(\Omega_{\mathbb{R}}))(2)_{/t})$, and

(xi) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle})$.

The theorem is a consequence of (11) and (13).

(19) Let us consider real numbers A, B, C, d, e, c, and a function u from $\mathcal{R}^2$ into $\mathbb{R}$. Suppose for every real numbers x, t, $u_{/\langle x,t\rangle} = (A \cdot (\text{the function } \cos)(e \cdot x) + B \cdot (\text{the function } \sin)(e \cdot x)) \cdot (C \cdot (\text{the function } \cos)(e \cdot c \cdot t) + d \cdot (\text{the function } \sin)(e \cdot c \cdot t))$. Then

(i) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle$, and

(ii) for every real numbers x, t, $(u{\restriction}^{\langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = (-A \cdot e \cdot (\text{the function } \sin)(e \cdot x) + B \cdot e \cdot (\text{the function } \cos)(e \cdot x)) \cdot (C \cdot (\text{the function } \cos)(e \cdot c \cdot t) + d \cdot (\text{the function } \sin)(e \cdot c \cdot t))$, and

(iii) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle$, and

(iv) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = (A \cdot (\text{the function } \cos)(e \cdot x) + B \cdot (\text{the function } \sin)(e \cdot x)) \cdot (-C \cdot (e \cdot c) \cdot (\text{the function } \sin)(e \cdot c \cdot t) + d \cdot (e \cdot c) \cdot (\text{the function } \cos)(e \cdot c \cdot t))$, and

(v) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle \frown \langle 1\rangle$, and

(vi) for every real numbers x, t, $(u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -e^{\mathbf{2}} \cdot (A \cdot (\text{the function } \cos)(e \cdot x) + B \cdot (\text{the function } \sin)(e \cdot x)) \cdot (C \cdot (\text{the function } \cos)(e \cdot c \cdot t) + d \cdot (\text{the function } \sin)(e \cdot c \cdot t))$ and u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle \frown \langle 2\rangle$ and for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -(e \cdot c)^{\mathbf{2}} \cdot (A \cdot (\text{the function } \cos)(e \cdot x) + B \cdot (\text{the function } \sin)(e \cdot x)) \cdot (C \cdot (\text{the function } \cos)(e \cdot c \cdot t) + d \cdot (\text{the function } \sin)(e \cdot c \cdot t))$, and

(vii) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle})$.

The theorem is a consequence of (16), (15), (17), (18), and (6).

(20) Let us consider a real number c. Then there exists a partial function u from $\mathcal{R}^2$ to $\mathbb{R}$ such that

(i) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle \frown \langle 1\rangle$ and partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle \frown \langle 2\rangle$, and

(ii) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle})$.

The theorem is a consequence of (16), (7), (15), (17), and (18).

3. The Superposition Principle

Now we state the propositions:

(21) Let us consider real numbers C, d, c, a natural number n, and a function u from $\mathcal{R}^2$ into $\mathbb{R}$. Suppose for every real numbers x, t, $u_{/\langle x,t\rangle} =$ (the function sin)$(n\cdot\pi\cdot x)\cdot(C\cdot$(the function cos)$(n\cdot\pi\cdot c\cdot t)+d\cdot$(the function sin)$(n\cdot\pi\cdot c\cdot t))$. Then

(i) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle$, and

(ii) for every real numbers x, t, $(u{\restriction}^{\langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = n \cdot \pi \cdot$ (the function cos)$(n \cdot \pi \cdot x) \cdot (C \cdot$ (the function cos)$(n \cdot \pi \cdot c \cdot t) + d \cdot$ (the function sin)$(n \cdot \pi \cdot c \cdot t))$, and

(iii) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle$, and

(iv) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} =$ (the function sin)$(n \cdot \pi \cdot x) \cdot (-C \cdot (n \cdot \pi \cdot c) \cdot$ (the function sin)$(n \cdot \pi \cdot c \cdot t) + d \cdot (n \cdot \pi \cdot c) \cdot$ (the function cos)$(n \cdot \pi \cdot c \cdot t))$, and

(v) u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 1\rangle \frown \langle 1\rangle$, and

(vi) for every real numbers x, t, $(u{\restriction}^{\langle 1\rangle \frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -(n \cdot \pi)^{\mathbf{2}} \cdot$ (the function sin)$(n \cdot \pi \cdot x) \cdot (C \cdot$ (the function cos)$(n \cdot \pi \cdot c \cdot t) + d \cdot$ (the function sin)$(n \cdot \pi \cdot c \cdot t))$ and u is partially differentiable on $\Omega_{\mathcal{R}^2}$ w.r.t. $\langle 2\rangle \frown \langle 2\rangle$ and for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle \frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -(n \cdot \pi \cdot c)^{\mathbf{2}} \cdot$ (the function sin)$(n \cdot \pi \cdot x) \cdot (C \cdot$ (the function cos)$(n \cdot \pi \cdot c \cdot t) + d \cdot$ (the function sin)$(n \cdot \pi \cdot c \cdot t))$, and

(vii) for every real number t, $u_{/\langle 0,t\rangle} = 0$ and $u_{/\langle 1,t\rangle} = 0$, and

(viii) for every real numbers x, t, $(u{\restriction}^{\langle 2\rangle ^\frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} =$
$c^{\mathbf{2}} \cdot ((u{\restriction}^{\langle 1\rangle ^\frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle})$.

PROOF: Set $e = n \cdot \pi$. For every real numbers x, t, $(u{\restriction}^{\langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = e \cdot (\text{the function cos})(e \cdot x) \cdot (C \cdot (\text{the function cos})(e \cdot c \cdot t) + d \cdot (\text{the function sin})(e \cdot c \cdot t))$. For every real numbers x, t, $(u{\restriction}^{\langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = (\text{the function sin})(e \cdot x) \cdot (-C \cdot (e \cdot c) \cdot (\text{the function sin})(e \cdot c \cdot t) + d \cdot (e \cdot c) \cdot (\text{the function cos})(e \cdot c \cdot t))$. For every real numbers x, t, $(u{\restriction}^{\langle 1\rangle ^\frown \langle 1\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -e^{\mathbf{2}} \cdot (\text{the function sin})(e \cdot x) \cdot (C \cdot (\text{the function cos})(e \cdot c \cdot t) + d \cdot (\text{the function sin})(e \cdot c \cdot t))$. For every real numbers x, t, $(u{\restriction}^{\langle 2\rangle ^\frown \langle 2\rangle}\Omega_{\mathcal{R}^2})_{/\langle x,t\rangle} = -(e \cdot c)^{\mathbf{2}} \cdot (\text{the function sin})(e \cdot x) \cdot (C \cdot (\text{the function cos})(e \cdot c \cdot t) + d \cdot (\text{the function sin})(e \cdot c \cdot t))$. For every real number t, $u_{/\langle 0,t\rangle} = 0$ and $u_{/\langle 1,t\rangle} = 0$ by [8, (30)]. $\square$

(22) Let us consider partial functions u, v from $\mathcal{R}^2$ to $\mathbb{R}$, a subset Z of $\mathcal{R}^2$, and a real number c. Suppose Z is open and $Z \subseteq \operatorname{dom} u$ and $Z \subseteq \operatorname{dom} v$ and u is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$ and for every real numbers x, t such that $\langle x,t\rangle \in Z$ holds $(u{\restriction}^{\langle 2\rangle ^\frown \langle 2\rangle}Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u{\restriction}^{\langle 1\rangle ^\frown \langle 1\rangle}Z)_{/\langle x,t\rangle})$ and v is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$ and for every real numbers x, t such that $\langle x,t\rangle \in Z$ holds $(v{\restriction}^{\langle 2\rangle ^\frown \langle 2\rangle}Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((v{\restriction}^{\langle 1\rangle ^\frown \langle 1\rangle}Z)_{/\langle x,t\rangle})$. Then

(i) $Z \subseteq \operatorname{dom}(u + v)$, and

(ii) $u + v$ is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$, and

(iii) for every real numbers x, t such that $\langle x,t\rangle \in Z$ holds
$(u + v{\restriction}^{\langle 2\rangle ^\frown \langle 2\rangle}Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u + v{\restriction}^{\langle 1\rangle ^\frown \langle 1\rangle}Z)_{/\langle x,t\rangle})$.

PROOF: For every real numbers x, t such that $\langle x,t\rangle \in Z$ holds $(u + v{\restriction}^{\langle 2\rangle ^\frown \langle 2\rangle}Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u + v{\restriction}^{\langle 1\rangle ^\frown \langle 1\rangle}Z)_{/\langle x,t\rangle})$ by (1), [2, (75)]. $\square$

(23) Let us consider a sequence u of partial functions from $\mathcal{R}^2$ into $\mathbb{R}$, a subset Z of $\mathcal{R}^2$, and a real number c. Suppose Z is open and for every natural number i, $Z \subseteq \operatorname{dom}(u(i))$ and $\operatorname{dom}(u(i)) = \operatorname{dom}(u(0))$ and $u(i)$ is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$ and for every real numbers x, t such that $\langle x,t\rangle \in Z$ holds $(u(i){\restriction}^{\langle 2\rangle ^\frown \langle 2\rangle}Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((u(i){\restriction}^{\langle 1\rangle ^\frown \langle 1\rangle}Z)_{/\langle x,t\rangle})$. Let us consider a natural number i. Then

(i) $Z \subseteq \operatorname{dom}(((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa \in \mathbb{N}})(i))$, and

(ii) $((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa \in \mathbb{N}})(i)$ is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$, and

(iii) for every real numbers x, t such that $\langle x, t\rangle \in Z$ holds $(((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(i){\restriction}^{\langle 2\rangle ^\frown \langle 2\rangle} Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(i){\restriction}^{\langle 1\rangle ^\frown \langle 1\rangle} Z)_{/\langle x,t\rangle})$.

PROOF: Define $\mathcal{X}$[natural number] $\equiv Z \subseteq \mathrm{dom}(((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(\$_1))$ and $((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(\$_1)$ is partially differentiable on Z w.r.t. $\langle 1\rangle ^\frown \langle 1\rangle$ and partially differentiable on Z w.r.t. $\langle 2\rangle ^\frown \langle 2\rangle$ and for every real numbers x, t such that $\langle x, t\rangle \in Z$ holds $(((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(\$_1){\restriction}^{\langle 2\rangle ^\frown \langle 2\rangle} Z)_{/\langle x,t\rangle} = c^{\mathbf{2}} \cdot ((((\sum_{\alpha=0}^{\kappa} u(\alpha))_{\kappa\in\mathbb{N}})(\$_1){\restriction}^{\langle 1\rangle ^\frown \langle 1\rangle} Z)_{/\langle x,t\rangle})$. For every natural number i such that $\mathcal{X}[i]$ holds $\mathcal{X}[i+1]$. For every natural number n, $\mathcal{X}[n]$. □

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