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Concatenation of Finite Sequences

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Summary. The coexistence of "classical" *finite sequences* [1] and their zero-based equivalents *finite 0-sequences* [6] in Mizar has been regarded as a disadvantage. However the suggested replacement of the former type with the latter [5] has not yet been implemented, despite of several advantages of this form, such as the identity of length and domain operators [4]. On the other hand the number of theorems formalized using *finite sequence* notation is much larger then of those based on *finite 0-sequences*, so such translation would require quite an effort.

The paper addresses this problem with another solution, using the Mizar system [3], [2]. Instead of removing one notation it is possible to introduce operators which would concatenate sequences of various types, and in this way allow utilization of the whole range of formalized theorems. While the operation could replace existing `FS2XFS`, `XFS2FS` commands (by using empty sequences as initial elements) its universal notation (independent on sequences that are concatenated to the initial object) allows to "forget" about the type of sequences that are concatenated on further positions, and thus simplify the proofs.

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1. Preliminaries

Let a be a real number and b be a non negative real number. One can check that $a -' (a + b)$ is zero.

One can check that $a + b -' a$ reduces to b.

Let n, m be natural numbers. We identify $n \cap m$ with $\min(m, n)$. We identify $\min(m, n)$ with $n \cap m$. We identify $\max(m, n)$ with $n \cup m$. Let n, m be non negative real numbers. Observe that $\min(n + m, n)$ reduces to n and $\max(n + m, n)$ reduces to $n + m$.

Now we state the propositions:

(1) Let us consider a binary relation f, and natural numbers n, m. Then $(f{\restriction}(n + m)){\restriction}n = f{\restriction}n$.

(2) Let us consider a function f, a natural number n, and a non zero natural number m. Then $(f{\restriction}(n + m))(n) = f(n)$.

Let D be a non empty set, x be a sequence of D, and n be a natural number. Let us note that $\mathrm{dom}(x{\restriction}n)$ reduces to n. Observe that $x{\restriction}n$ is finite and transfinite sequence-like and $x{\restriction}n$ is n-element.

2. Complex-Valued Sequences

Now we state the proposition:

(3) Let us consider complex-valued functions f, g, and a set X. Then $(f \cdot g){\restriction}X = (f{\restriction}X) \cdot (g{\restriction}X)$.

Proof: For every object x such that $x \in \mathrm{dom}((f \cdot g){\restriction}X)$ holds $((f \cdot g){\restriction}X)(x) = ((f{\restriction}X) \cdot (g{\restriction}X))(x)$. $\square$

Let D be a non empty set and f, g be sequences of D. Let us note that $f{+}{\cdot}g$ is transfinite sequence-like.

Let f be a constant complex sequence and n be a natural number. Let us note that $f \uparrow n$ is constant and there exists a complex sequence which is empty yielding and there exists a sequence of real numbers which is empty yielding and every complex sequence which is empty yielding is also natural-valued and there exists a complex sequence which is constant and real-valued.

Now we state the proposition:

(4) Let us consider a sequence s of real numbers, and a natural number n. Then $((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})(n) = \sum(s{\restriction}\mathbb{Z}_{n+1})$.

Let c be a complex number. The functor $\{c\}_{n \in \mathbb{N}}$ yielding a complex sequence is defined by the term

(Def. 1) $\mathbb{N} \longmapsto c$.

Let n be a natural number. One can check that $(\{c\}_{n \in \mathbb{N}})(n)$ reduces to c.

Now we state the proposition:

(5) Let us consider complex-valued functions f, g, and a set X. Then $(f + g){\restriction}X = f{\restriction}X + g{\restriction}X$.

Proof: For every object x such that $x \in \mathrm{dom}((f + g){\restriction}X)$ holds $((f + g){\restriction}X)(x) = (f{\restriction}X + g{\restriction}X)(x)$. $\square$

Let f be a 1-element finite sequence. One can verify that $\langle f(1)\rangle$ reduces to f.

Let f be a 2-element finite sequence. Let us note that $\langle f(1), f(2)\rangle$ reduces to f.

Let f be a 3-element finite sequence. Let us note that $\langle f(1), f(2), f(3)\rangle$ reduces to f.

Now we state the propositions:

(6) Let us consider a complex-valued finite sequence f. Then $\sum f = f(1) + \sum f_{\downharpoonleft 1}$.

(7) Let us consider a non empty, complex-valued finite sequence f. Then $\prod f = f(1) \cdot (\prod f_{\downharpoonleft 1})$.

(8) Let us consider a natural number n, a non zero natural number m, and an $(n+m)$-element finite sequence f. Then $f \restriction (n+1) = (f \restriction n) ^\frown \langle f(n+1)\rangle$.

(9) Let us consider a complex-valued finite sequence f, and a natural number n. Then $\prod f = \prod (f \restriction n) \cdot \prod f_{\downharpoonleft n}$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \prod f = (\prod (f \restriction \$_1)) \cdot (\prod f_{\downharpoonleft \$_1})$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [8, (35)], (7). For every natural number x, $\mathcal{P}[x]$. $\square$

(10) Let us consider complex-valued finite sequences f, g. Then $\prod (f ^\frown g) = (\prod f) \cdot (\prod g)$. The theorem is a consequence of (9).

3. On Product and Sum of Complex Sequences

Let s be a complex sequence. The partial product of s yielding a complex sequence is defined by

(Def. 2) $it(0) = s(0)$ and for every natural number n, $it(n+1) = it(n) \cdot s(n+1)$.

Now we state the propositions:

(11) Let us consider a complex sequence f, and a natural number n. Suppose $f(n) = 0$. Then (the partial product of f)$(n) = 0$.

(12) Let us consider a complex sequence f, and natural numbers n, m. Suppose $f(n) = 0$. Then (the partial product of f)$(n + m) = 0$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ (the partial product of f)$(n+\$_1) = 0$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number x, $\mathcal{P}[x]$. $\square$

Let c be a complex number and n be a non zero natural number. Observe that the functor c^n is defined by the term

(Def. 3) (the partial product of $\{c\}_{n \in \mathbb{N}}$)$(n - 1)$.

Now we state the proposition:

(13) Let us consider a natural number n. Then (the partial product of $\{0_{\mathbb{C}}\}_{n\in\mathbb{N}})(n) = 0$. The theorem is a consequence of (12).

Let k be a natural number. Let us note that (the partial product of $\{0\}_{n\in\mathbb{N}})(k)$ reduces to 0.

One can verify that every complex sequence which is empty yielding is also absolutely summable and every sequence of real numbers which is empty yielding is also absolutely summable.

Observe that $(\sum_{\alpha=0}^{\kappa}(\mathbb{N} \longmapsto 0)(\alpha))_{\kappa\in\mathbb{N}}$ reduces to $\mathbb{N} \longmapsto 0$ and the partial product of $\{0\}_{n\in\mathbb{N}}$ reduces to $\{0\}_{n\in\mathbb{N}}$. One can verify that every complex sequence is transfinite sequence-like and there exists a sequence of $\mathbb{C}$ which is summable.

Let s_1 be an empty yielding complex sequence. One can check that $\sum s_1$ is zero.

Let s_1 be an empty yielding sequence of real numbers. Let us note that $\sum s_1$ is zero.

4. Finite 0-sequences

Let c be a complex number. Observe that $\langle c\rangle$ is complex-valued.

One can verify that $\sum\langle c\rangle$ reduces to c.

Let n be a natural number. One can verify that there exists a natural-valued finite 0-sequence which is n-element.

Let k be an object. One can check that $n \longmapsto k$ is n-element and there exists a finite 0-sequence which is n-element.

Let f be an n-element finite 0-sequence. Let us note that $f{\restriction}n$ reduces to f.

Let n, m be natural numbers. One can check that $f{\restriction}(n+m)$ reduces to f.

Let f be a 1-element finite 0-sequence. Let us note that $\langle f(0)\rangle$ reduces to f.

Let f be a 2-element finite 0-sequence. Let us note that $\langle f(0), f(1)\rangle$ reduces to f.

Let f be a 3-element finite 0-sequence. One can verify that $\langle f(0), f(1), f(2)\rangle$ reduces to f.

Now we state the propositions:

(14) Let us consider natural numbers n, k. If $k \in \mathbb{Z}_{n+1}$, then $n-k$ is a natural number.

(15) Let us consider complex numbers a, b, and natural numbers n, k. Suppose $k \in \mathbb{Z}_{n+1}$. Then there exists an object c and there exists a natural number l such that $l = n - k$ and $c = a^l \cdot (b^k)$. The theorem is a consequence of (14).

5. Shifting Sequences

Let f be a complex-valued finite 0-sequence and s_1 be a complex sequence. The functor $f \frown s_1$ yielding a complex sequence is defined by the term

(Def. 4) $f \cup \mathrm{Shift}(s_1, \mathrm{len}\, f)$.

Let f be a function. The functor $s_1 \frown f$ yielding a sequence of $\mathbb{C}$ is defined by the term

(Def. 5) s_1.

Now we state the propositions:

(16) Let us consider an object x. Then x is a real-valued complex sequence if and only if x is a sequence of real numbers.

(17) Let us consider a sequence r_1 of real numbers, and a complex sequence c_1. Suppose $c_1 = r_1$. Then the partial product of $r_1 =$ the partial product of c_1.

Let f be a complex-valued finite 0-sequence and s_1 be a sequence of real numbers. The functor $f \frown s_1$ yielding a complex sequence is defined by the term

(Def. 6) $f \cup \mathrm{Shift}(s_1, \mathrm{len}\, f)$.

Now we state the proposition:

(18) Let us consider a sequence r_1 of real numbers. Then $\langle\rangle_{\mathbb{R}} \frown r_1$ is a real-valued complex sequence.

Let f be a sequence of real numbers and g be a function. The functor $f \frown g$ yielding a real-valued sequence of $\mathbb{C}$ is defined by the term

(Def. 7) f.

Let f be a complex-valued finite 0-sequence and s_1 be a complex sequence. Let us observe that $(f \frown s_1) \restriction \mathrm{dom}\, f$ reduces to f.

Let s_1 be a sequence of real numbers. Let us note that $(f \frown s_1) \restriction \mathrm{dom}\, f$ reduces to f.

Now we state the propositions:

(19) Let us consider a complex-valued finite 0-sequence f, and a natural number x. Then $(f \frown \{0\}_{n \in \mathbb{N}})(x) = f(x)$.

(20) Let us consider a sequence f of real numbers. Then $f \frown f$ is a real-valued complex sequence.

Let f be a real-valued complex sequence. Note that $\Im(f)$ is empty yielding. One can check that $\Re(f)$ reduces to f.

Let us observe that there exists a sequence of real numbers which is empty yielding and every sequence of real numbers is transfinite sequence-like.

Let r be a real number. Let us note that $\Re(r \cdot (i))$ is zero.

One can check that $\Im(r \cdot (i))$ reduces to r.

Let f be a complex-valued finite 0-sequence. Let us note that $\Re(f)$ is real-valued, finite, and transfinite sequence-like and $\Im(f)$ is real-valued, finite, and transfinite sequence-like and $\Re(f)$ is $(\operatorname{len} f)$-element and $\Im(f)$ is $(\operatorname{len} f)$-element.

Let f be a complex-valued finite sequence. Note that $\Re(f)$ is real-valued and finite sequence-like and $\Im(f)$ is real-valued and finite sequence-like.

Let f be a complex-valued function. Let us observe that $\Re(\Re(f))$ reduces to $\Re(f)$ and $\Re(\Im(f))$ reduces to $\Im(f)$. Let us note that $\Im(\Re(f))$ is empty yielding and $\Im(\Im(f))$ is empty yielding.

One can check that $\Re(\Re(f)+i\cdot\Im(f))$ reduces to $\Re(f)$ and $\Im(\Re(f)+i\cdot\Im(f))$ reduces to $\Im(f)$ and $\Re(f)+i\cdot\Im(f)$ reduces to f.

Let n be a natural number. One can check that there exists a finite function which is n-element.

Let f be a finite, complex-valued transfinite sequence. Note that $\operatorname{Shift}(f,n)$ is finite and $\operatorname{Shift}(f,n)$ is $(\operatorname{len} f)$-element and $\{0\}_{n\in\mathbb{N}}$ is empty yielding.

6. Converting Complex 0-sequences into Ordinary Ones

Let f be a complex-valued finite 0-sequence. The functor $\operatorname{Sequel} f$ yielding a complex sequence is defined by the term

(Def. 8) $(\mathbb{N} \longmapsto 0){+\cdot}f$.

Now we state the propositions:

(21) Let us consider a complex-valued finite 0-sequence f, and a natural number x. Then $(\operatorname{Sequel} f)(x) = f(x)$.

(22) Let us consider a complex-valued finite 0-sequence f. Then $\operatorname{Sequel} f = f \frown \{0\}_{n\in\mathbb{N}}$.
PROOF: $\operatorname{dom}(\operatorname{Sequel} f) = \operatorname{dom}(f \frown \{0\}_{n\in\mathbb{N}})$. For every natural number x, $(\operatorname{Sequel} f)(x) = (f \frown \{0\}_{n\in\mathbb{N}})(x)$. $\square$

(23) Let us consider a complex sequence s_1. Then $s_1 = \Re(s_1) + i\cdot\Im(s_1)$.

Let us consider a complex-valued finite 0-sequence f. Now we state the propositions:

(24) $\Re(\operatorname{Sequel} f) = \operatorname{Sequel}\Re(f)$. The theorem is a consequence of (21).

(25) $\Im(\operatorname{Sequel} f) = \operatorname{Sequel}\Im(f)$. The theorem is a consequence of (21).

Now we state the propositions:

(26) Let us consider a complex number c. Then $0\cdot(\mathbb{N} \longmapsto c) = \mathbb{N} \longmapsto 0$.

(27) Let us consider a complex sequence s_1, and a natural number x. Suppose for every natural number k such that $k \geqslant x$ holds $s_1(k) = 0$. Then s_1 is summable.

(28) Let us consider a sequence s_1 of real numbers, and a natural number x. Suppose for every natural number k such that $k \geqslant x$ holds $s_1(k) = 0$. Then s_1 is summable.

Let f be a complex-valued finite 0-sequence. One can check that Sequel f is summable.

7. PROPERTIES OF CONCATENATION

Let f be a finite 0-sequence and g be a finite sequence. The functor $f \frown g$ yielding a finite 0-sequence is defined by

(Def. 9) $\operatorname{dom} it = \operatorname{len} f + \operatorname{len} g$ and for every natural number k such that $k \in \operatorname{dom} f$ holds $it(k) = f(k)$ and for every natural number k such that $k \in \operatorname{dom} g$ holds $it(\operatorname{len} f + k - 1) = g(k)$.

Let f be a finite sequence and g be a finite 0-sequence. The functor $f \frown g$ yielding a finite sequence is defined by

(Def. 10) $\operatorname{dom} it = \operatorname{Seg}(\operatorname{len} f + \operatorname{len} g)$ and for every natural number k such that $k \in \operatorname{dom} f$ holds $it(k) = f(k)$ and for every natural number k such that $k \in \operatorname{dom} g$ holds $it(\operatorname{len} f + k + 1) = g(k)$.

Now we state the proposition:

(29) Let us consider a finite 0-sequence f, and a finite sequence g. Then

(i) $\operatorname{len}(f \frown g) = \operatorname{len} f + \operatorname{len} g$, and

(ii) $\operatorname{len}(g \frown f) = \operatorname{len} f + \operatorname{len} g$.

Let n, m be natural numbers, f be an n-element finite 0-sequence, and g be an m-element finite sequence. Let us note that $f \frown g$ is $(n+m)$-element and $g \frown f$ is $(n+m)$-element.

Now we state the propositions:

(30) Let us consider a finite 0-sequence f, a finite sequence g, and a natural number x. Then $x \in \operatorname{dom}(f \frown g)$ if and only if $x \in \operatorname{dom} f$ or $x + 1 - \operatorname{len} f \in \operatorname{dom} g$.

PROOF: If $x \in \operatorname{dom}(f \frown g)$, then $x \in \operatorname{dom} f$ or $x + 1 - \operatorname{len} f \in \operatorname{dom} g$. If $x \in \operatorname{dom} f$ or $x + 1 - \operatorname{len} f \in \operatorname{dom} g$, then $x \in \operatorname{dom}(f \frown g)$. □

(31) Let us consider a finite sequence f, a finite 0-sequence g, and a natural number x. Then $x \in \operatorname{dom}(f \frown g)$ if and only if $x \in \operatorname{dom} f$ or $x - (\operatorname{len} f + 1) \in \operatorname{dom} g$.

PROOF: If $x \in \operatorname{dom}(f \frown g)$, then $x \in \operatorname{dom} f$ or $x - (\operatorname{len} f + 1) \in \operatorname{dom} g$. □

(32) Let us consider a finite sequence f, and a finite 0-sequence g. Then

(i) $\operatorname{rng}(f \frown g) = \operatorname{rng} f \cup \operatorname{rng} g$, and

(ii) $\operatorname{rng}(g \frown f) = \operatorname{rng} f \cup \operatorname{rng} g$.

PROOF: $\operatorname{rng}(f \frown g) \subseteq \operatorname{rng} f \cup \operatorname{rng} g$. $\operatorname{rng} f \cup \operatorname{rng} g \subseteq \operatorname{rng}(f \frown g)$. $\operatorname{rng}(g \frown f) \subseteq \operatorname{rng} f \cup \operatorname{rng} g$. $\operatorname{rng} f \cup \operatorname{rng} g \subseteq \operatorname{rng}(g \frown f)$. □

(33) Let us consider a non empty finite 0-sequence f, and a finite sequence g. Then $\operatorname{dom}(f \cup \operatorname{Shift}(g, \operatorname{len} f - 1)) = \mathbb{Z}_{\operatorname{len} f + \operatorname{len} g}$.
PROOF: For every object x, $x \in \operatorname{dom}(f \cup \operatorname{Shift}(g, \operatorname{len} f - 1))$ iff $x \in \mathbb{Z}_{\operatorname{len} f + \operatorname{len} g}$. □

(34) Let us consider a finite sequence f, and a finite 0-sequence g. Then $\operatorname{dom}(f \cup \operatorname{Shift}(g, \operatorname{len} f + 1)) = \operatorname{Seg}(\operatorname{len} f + \operatorname{len} g)$.
PROOF: For every object x, $x \in \operatorname{dom}(f \cup \operatorname{Shift}(g, \operatorname{len} f + 1))$ iff $x \in \operatorname{Seg}(\operatorname{len} f + \operatorname{len} g)$. □

Let f be a complex-valued finite sequence. One can verify that $\langle\rangle_{\mathbb{C}} \frown f$ is complex-valued.

Let f be a complex-valued finite 0-sequence. Let us note that $\varepsilon_{\mathbb{C}} \frown f$ is complex-valued.

Let f be a finite 0-sequence and g be a finite sequence. One can verify that $(f \frown g) \restriction \operatorname{len} f$ reduces to f and $(g \frown f) \restriction \operatorname{len} g$ reduces to g.

Now we state the propositions:

(35) Let us consider a set D, a finite 0-sequence f, and a finite sequence g of elements of D. Then $(f \frown g)_{\lfloor \operatorname{len} f} = \operatorname{FS2XFS}(g)$.
PROOF: For every natural number i such that $i \in \operatorname{dom}((f \frown g)_{\lfloor \operatorname{len} f})$ holds $((f \frown g)_{\lfloor \operatorname{len} f})(i) = (\operatorname{FS2XFS}(g))(i)$. □

(36) Every finite 0-sequence is a finite 0-sequence of $\operatorname{rng} f \cup \{1\}$.

(37) Let us consider a set D, a finite sequence f, and a finite 0-sequence g of D. Then $(f \frown g)_{\lfloor \operatorname{len} f} = \operatorname{XFS2FS}(g)$.
PROOF: $\operatorname{len} f \leqslant \operatorname{len}(f \frown g)$. For every natural number i such that $i \in \operatorname{dom}((f \frown g)_{\lfloor \operatorname{len} f})$ holds $((f \frown g)_{\lfloor \operatorname{len} f})(i) = (\operatorname{XFS2FS}(g))(i)$. □

Let D be a set and f be a finite 0-sequence of D. One can verify that the functor $\operatorname{XFS2FS}(f)$ is defined by the term

(Def. 11) $\varepsilon_D \frown f$.

Now we state the proposition:

(38) Let us consider a set D, and a finite 0-sequence f of D.
Then $\operatorname{dom}(\operatorname{Shift}(f, 1)) = \operatorname{Seg} \operatorname{len} f$.
PROOF: For every object x such that $x \in \operatorname{Seg} \operatorname{len} f$ holds $x \in \operatorname{dom}(\operatorname{Shift}(f, 1))$. For every object x such that $x \in \operatorname{dom}(\operatorname{Shift}(f, 1))$ holds $x \in \operatorname{Seg} \operatorname{len} f$ by [7, (106)]. □

Let D be a set and f be a finite 0-sequence of D. One can verify that the functor $\operatorname{XFS2FS}(f)$ is defined by the term

(Def. 12) $\mathrm{Shift}(f, 1)$.

Let f be a finite sequence of elements of D. One can check that the functor $\mathrm{FS2XFS}(f)$ is defined by the term

(Def. 13) $\langle\rangle_D \mathbin{^\frown} f$.

Now we state the propositions:

(39) Let us consider a set D, and finite 0-sequences f, g of D. Then $f \mathbin{^\frown} g = f \mathbin{^\frown} \mathrm{XFS2FS}(g)$.

PROOF: For every natural number k such that $k \in \mathrm{dom}(f \mathbin{^\frown} g)$ holds $(f \mathbin{^\frown} g)(k) = (f \mathbin{^\frown} \mathrm{XFS2FS}(g))(k)$. □

(40) Let us consider a set D, and finite sequences f, g of elements of D. Then $f \mathbin{^\frown} g = f \mathbin{^\frown} \mathrm{FS2XFS}(g)$.

PROOF: For every natural number k such that $k \in \mathrm{dom}(f \mathbin{^\frown} g)$ holds $(f \mathbin{^\frown} g)(k) = (f \mathbin{^\frown} \mathrm{FS2XFS}(g))(k)$. □

Let f be a finite 0-sequence of $\mathbb{R}$. Let us observe that $\mathrm{Sequel}\, f \restriction \mathrm{dom}\, f$ reduces to f. One can check that $\mathrm{Shift}(f, 1)$ is finite sequence-like and $\mathrm{Sequel}\, f \uparrow \mathrm{dom}\, f$ is empty yielding.

Now we state the propositions:

(41) Let us consider a set D, a finite sequence f of elements of D, and a finite 0-sequence g of D. Then $f \mathbin{^\frown} g = f \mathbin{^\frown} \mathrm{XFS2FS}(g)$. The theorem is a consequence of (40).

(42) Let us consider a set D, a finite 0-sequence f of D, and a finite sequence g of elements of D. Then $f \mathbin{^\frown} g = f \mathbin{^\frown} \mathrm{FS2XFS}(g)$. The theorem is a consequence of (39).

(43) Let us consider a set D, and finite sequences f, g of elements of D. Then $\mathrm{FS2XFS}(f \mathbin{^\frown} g) = \mathrm{FS2XFS}(f) \mathbin{^\frown} \mathrm{FS2XFS}(g)$.

PROOF: For every natural number x such that $x \in \mathrm{dom}(\mathrm{FS2XFS}(f \mathbin{^\frown} g))$ holds $(\mathrm{FS2XFS}(f \mathbin{^\frown} g))(x) = (\mathrm{FS2XFS}(f) \mathbin{^\frown} \mathrm{FS2XFS}(g))(x)$. □

Let D be a set, f be a finite sequence of elements of D, and g be a finite 0-sequence of D. Note that the functor $f \mathbin{^\frown} g$ yields a finite sequence of elements of D. Now we state the propositions:

(44) Let us consider a set D, a finite sequence f of elements of D, and a finite 0-sequence g of D. Then $\mathrm{FS2XFS}(f \mathbin{^\frown} g) = \mathrm{FS2XFS}(f) \mathbin{^\frown} g$. The theorem is a consequence of (43) and (40).

(45) Let us consider a set D, and finite 0-sequences f, g of D. Then $\mathrm{XFS2FS}(f \mathbin{^\frown} g) = \mathrm{XFS2FS}(f) \mathbin{^\frown} \mathrm{XFS2FS}(g)$.

PROOF: For every natural number x such that $x \in \mathrm{dom}(\mathrm{XFS2FS}(f \mathbin{^\frown} g))$ holds $(\mathrm{XFS2FS}(f \mathbin{^\frown} g))(x) = (\mathrm{XFS2FS}(f) \mathbin{^\frown} \mathrm{XFS2FS}(g))(x)$. □

Let D be a set, f be a finite 0-sequence of D, and g be a finite sequence of elements of D. One can check that the functor $f \frown g$ yields a finite 0-sequence of D. Now we state the propositions:

(46) Let us consider a set D, a finite 0-sequence f of D, and a finite sequence g of elements of D. Then $\mathrm{XFS2FS}(f \frown g) = \mathrm{XFS2FS}(f) \frown g$. The theorem is a consequence of (45) and (39).

(47) Let us consider a set D, finite 0-sequences f, g of D, and a finite sequence h of elements of D. Then

(i) $(f \frown g) \frown h = f \frown (g \frown h)$, and

(ii) $(f \frown h) \frown g = f \frown (h \frown g)$, and

(iii) $(h \frown f) \frown g = h \frown (f \frown g)$.

The theorem is a consequence of (42), (39), (43), (41), and (45).

8. Sum of Finite 0-sequences

Now we state the proposition:

(48) Let us consider a complex-valued finite 0-sequence f. Then $\sum(f{\restriction}1) = f(0)$.

Let n, m be natural numbers and f be an $(n+m)$-element finite 0-sequence. One can verify that $f{\restriction}n$ is n-element. Let n be a natural number and p be an n-element, complex-valued finite 0-sequence. Let us observe that $-p$ is n-element and p^{-1} is n-element and $p^{\mathbf{2}}$ is n-element and $|p|$ is n-element and $\mathrm{Rev}(p)$ is n-element.

Let m be a natural number and q be an $(n+m)$-element, complex-valued finite 0-sequence. Let us observe that $\mathrm{dom}\, p \cap \mathrm{dom}\, q$ reduces to $\mathrm{dom}\, p$. Note that $p+q$ is n-element and $p-q$ is n-element and $p \cdot q$ is n-element and p/q is n-element. Let p, q be n-element, complex-valued finite 0-sequences. Note that $p+q$ is n-element and $p-q$ is n-element and $p \cdot q$ is n-element and p/q is n-element. Now we state the propositions:

(49) Let us consider a natural number n, and n-element, complex-valued finite 0-sequences f_1, f_2. Then $\sum(f_1 + f_2) = \sum f_1 + \sum f_2$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every $\$_1$-element, complex-valued finite 0-sequences f_1, f_2, $\sum(f_1 + f_2) = \sum f_1 + \sum f_2$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(50) Let us consider a complex number c. Then $\mathrm{XFS2FS}(\langle c \rangle) = \langle c \rangle$.
Proof: For every natural number k such that $k \in \mathrm{dom}\langle c \rangle$ holds $(\mathrm{XFS2FS}(\langle c \rangle))(k) = \langle c \rangle(k)$. $\square$

(51) Let us consider a finite 0-sequence f of $\mathbb{R}$. Then $\sum \mathrm{XFS2FS}(f) = \sum f$. The theorem is a consequence of (16).

(52) Let us consider a complex-valued finite 0-sequence f. Then $\sum f = \sum \Re(f) + (i) \cdot (\sum \Im(f))$. The theorem is a consequence of (49).

(53) Let us consider a complex-valued transfinite sequence f, and a natural number n. Then

(i) $\Re(\mathrm{Shift}(f, n)) = \mathrm{Shift}(\Re(f), n)$, and

(ii) $\Im(\mathrm{Shift}(f, n)) = \mathrm{Shift}(\Im(f), n)$.

Let us consider a complex-valued finite 0-sequence f.

(54) (i) $\mathrm{XFS2FS}(\Re(f)) = \Re(\mathrm{XFS2FS}(f))$, and

(ii) $\mathrm{XFS2FS}(\Im(f)) = \Im(\mathrm{XFS2FS}(f))$.

(55) $\sum \mathrm{XFS2FS}(f) = \sum f$. The theorem is a consequence of (52), (51), and (53).

(56) Let us consider a finite sequence f of elements of $\mathbb{C}$. Then $\sum \mathrm{FS2XFS}(f) = \sum f$. The theorem is a consequence of (55).

(57) Let us consider a real-valued finite 0-sequence f. Then $\sum f = \sum \mathrm{Sequel}\, f$.

Note that there exists a real-valued complex sequence which is summable.

Let f be a summable complex sequence. The functors: $\Re(f)$ and $\Im(f)$ yield summable, real-valued complex sequences. Now we state the propositions:

(58) Let us consider a complex-valued finite 0-sequence f. Then $\sum f = \sum \mathrm{Sequel}\, f$. The theorem is a consequence of (57), (24), (25), and (52).

(59) Let us consider a finite 0-sequence f of $\mathbb{C}$, and a finite sequence g of elements of $\mathbb{C}$. Then

(i) $\sum(f \frown g) = \sum f + \sum g$, and

(ii) $\sum(g \frown f) = \sum g + \sum f$.

The theorem is a consequence of (39), (56), (40), and (55).

9. Product of Finite 0-sequences

Let f be a finite 0-sequence. The functor $\prod f$ yielding an element of $\mathbb{C}$ is defined by the term

(Def. 14) $\cdot_{\mathbb{C}} \odot f$.

Now we state the proposition:

(60) Let us consider an empty finite 0-sequence f. Then $\prod f = 1$.

Let c be a complex number. One can check that $\prod \langle c \rangle$ reduces to c.

(61) Let us consider a natural number n, and a complex-valued finite 0-sequence f. Suppose $n \in \operatorname{dom} f$. Then $\prod(f \restriction n) \cdot f(n) = \prod(f \restriction (n+1))$.

(62) Let us consider a natural number n, and a complex sequence f. Then $\prod(f \restriction n) \cdot f(n) = \prod(f \restriction (n+1))$. The theorem is a consequence of (61).

(63) Let us consider a non empty, complex-valued finite 0-sequence f. Then $\prod(f \restriction 1) = f(0)$.

(64) Let us consider a natural number n, and n-element, complex-valued finite 0-sequences f_1, f_2. Then $\prod(f_1 \cdot f_2) = (\prod f_1) \cdot (\prod f_2)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every $\$_1$-element, complex-valued finite 0-sequences f_1, f_2, $\prod(f_1 \cdot f_2) = (\prod f_1) \cdot (\prod f_2)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(65) Let us consider a complex sequence f, and a natural number n. Then $\prod(f \restriction (n+1)) = (\text{the partial product of } f)(n)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \prod(f \restriction (\$_1+1)) = (\text{the partial product of } f)(\$_1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number x, $\mathcal{P}[x]$. $\square$

(66) Let us consider a complex-valued finite 0-sequence f.
Then $\prod \mathrm{XFS2FS}(f) = \prod f$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \prod \mathrm{XFS2FS}(f \restriction \$_1) = \prod(f \restriction \$_1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number x, $\mathcal{P}[x]$. $\square$

(67) Let us consider a finite sequence f of elements of $\mathbb{C}$. Then $\prod \mathrm{FS2XFS}(f) = \prod f$. The theorem is a consequence of (66).

(68) Let us consider a finite 0-sequence f of $\mathbb{C}$, and a finite sequence g of elements of $\mathbb{C}$. Then

(i) $\prod(f \frown g) = (\prod f) \cdot (\prod g)$, and

(ii) $\prod(g \frown f) = (\prod g) \cdot (\prod f)$.

The theorem is a consequence of (66), (46), (10), and (40).

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