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Some Remarks about Product Spaces

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Summary. This article covers some technical aspects about the product topology which are usually not given much of a thought in mathematics and standard literature like [7] and [6], not even by Bourbaki in [4].

Let $\{\mathcal{T}_i\}_{i \in I}$ be a family of topological spaces. The prebasis of the product space $\mathcal{T} = \prod_{i \in I} \mathcal{T}_i$ is defined in [5] as the set of all $\pi_i^{-1}(V)$ with $i \in I$ and V open in $\mathcal{T}_i$. Here it is shown that the basis generated by this prebasis consists exactly of the sets $\prod_{i \in I} V_i$ with V_i open in $\mathcal{T}_i$ and for all but finitely many $i \in I$ holds $V_i = \mathcal{T}_i$. Given $I = \{a\}$ we have $\mathcal{T} \cong \mathcal{T}_a$, given $I = \{a, b\}$ with $a \neq b$ we have $\mathcal{T} \cong \mathcal{T}_a \times \mathcal{T}_b$. Given another family of topological spaces $\{\mathcal{S}_i\}_{i \in I}$ such that $\mathcal{S}_i \cong \mathcal{T}_i$ for all $i \in I$, we have $\mathcal{S} = \prod_{i \in I} \mathcal{S}_i \cong \mathcal{T}$. If instead S_i is a subspace of T_i for each $i \in I$, then $\mathcal{S}$ is a subspace of $\mathcal{T}$.

These results are obvious for mathematicians, but formally proven here by means of the Mizar system [3], [2].

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1. Preliminaries

Now we state the propositions:

(1) Let us consider a one-to-one function f, and an object y. Suppose $\operatorname{rng} f = \{y\}$. Then $\operatorname{dom} f = \{(f^{-1})(y)\}$.
PROOF: Consider x_0 being an object such that $x_0 \in \operatorname{dom} f$ and $f(x_0) = y$. For every object x, $x \in \operatorname{dom} f$ iff $x = (f^{-1})(y)$. $\square$

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(2) Let us consider a one-to-one function f, and objects y_1, y_2. Suppose $\operatorname{rng} f = \{y_1, y_2\}$. Then $\operatorname{dom} f = \{(f^{-1})(y_1), (f^{-1})(y_2)\}$.
PROOF: Consider x_1 being an object such that $x_1 \in \operatorname{dom} f$ and $f(x_1) = y_1$. Consider x_2 being an object such that $x_2 \in \operatorname{dom} f$ and $f(x_2) = y_2$. For every object x, $x \in \operatorname{dom} f$ iff $x = (f^{-1})(y_1)$ or $x = (f^{-1})(y_2)$. $\square$

Let X, Y be sets. Note that there exists a function which is empty, X-defined, Y-valued, and one-to-one.

Let T, S be sets, f be a function from T into S, and G be a finite family of subsets of T. Let us note that $f^\circ G$ is finite.

Now we state the propositions:

(3) Let us consider a set A, a family F of subsets of A, and a binary relation R. Then $R^\circ(\bigcap F) \subseteq \bigcap\{R^\circ X$, where X is a subset of $A : X \in F\}$.

(4) Let us consider a set A, a family F of subsets of A, and a one-to-one function f. Then $f^\circ(\bigcap F) = \bigcap\{f^\circ X$, where X is a subset of $A : X \in F\}$.
PROOF: Set $S = \{f^\circ X$, where X is a subset of $A : X \in F\}$. $\bigcap S \subseteq f^\circ(\bigcap F)$. $f^\circ(\bigcap F) \subseteq \bigcap S$. $\square$

(5) Let us consider a set X, a non empty set Y, and a function f from X into Y. Then $\{f^{-1}(\{y\})$, where y is an element of $Y : y \in \operatorname{rng} f\}$ is a partition of X.
PROOF: Set $P = \{f^{-1}(\{y\})$, where y is an element of $Y : y \in \operatorname{rng} f\}$. For every object x, $x \in X$ iff there exists a set A such that $x \in A$ and $A \in P$. For every subset A of X such that $A \in P$ holds $A \neq \emptyset$ and for every subset B of X such that $B \in P$ holds $A = B$ or A misses B. $P \subseteq 2^X$. $\square$

(6) Let us consider a non empty set X, and objects x, y. If $X \longmapsto x = X \longmapsto y$, then $x = y$.

(7) Let us consider an object i, and a many sorted set J indexed by $\{i\}$. Then $J = \{i\} \longmapsto J(i)$.
PROOF: For every object x such that $x \in \operatorname{dom} J$ holds $J(x) = (\{i\} \longmapsto J(i))(x)$. $\square$

(8) Let us consider a 2-element set I, and elements i, j of I. If $i \neq j$, then $I = \{i, j\}$.
PROOF: For every object x, $x = i$ or $x = j$ iff $x \in I$. $\square$

(9) Let us consider a 2-element set I, a many sorted set f indexed by I, and elements i, j of I. If $i \neq j$, then $f = [i \longmapsto f(i), j \longmapsto f(j)]$. The theorem is a consequence of (8).

(10) Let us consider objects a, b, c, d. If $a \neq b$, then $[a \longmapsto c, b \longmapsto d] = [b \longmapsto d, a \longmapsto c]$.
PROOF: For every object x such that $x \in \operatorname{dom}[a \longmapsto c, b \longmapsto d]$ holds $[a \longmapsto c, b \longmapsto d](x) = [b \longmapsto d, a \longmapsto c](x)$. $\square$

(11) Let us consider a function f, and objects i, j. If i, $j \in \operatorname{dom} f$, then $f = f+\cdot[i \longmapsto f(i), j \longmapsto f(j)]$.

(12) Let us consider objects x, y, z. Then $x \dot{\longmapsto} y +\cdot (x \dot{\longmapsto} z) = x \dot{\longmapsto} z$.

Let us observe that there exists a function which is non non-empty.

Now we state the propositions:

(13) Let us consider non empty sets X, Y, and an element y of Y. Then $X \longmapsto y \in \prod(X \longmapsto Y)$.

Proof: Set $f = X \longmapsto y$. For every object x such that $x \in \operatorname{dom}(X \longmapsto Y)$ holds $f(x) \in (X \longmapsto Y)(x)$. $\square$

(14) Let us consider a non empty set X, a set Y, and a subset Z of Y. Then $\prod(X \longmapsto Z) \subseteq \prod(X \longmapsto Y)$.

(15) Let us consider a non empty set X, and an object i. Then $\prod(\{i\} \longmapsto X) = \{\{i\} \longmapsto x$, where x is an element of $X\}$.

Proof: Set $S = \{\{i\} \longmapsto x$, where x is an element of $X\}$. For every object z, $z \in \prod(\{i\} \longmapsto X)$ iff $z \in S$. $\square$

(16) Let us consider a non empty set X, and objects i, f. Then $f \in \prod(\{i\} \longmapsto X)$ if and only if there exists an element x of X such that $f = \{i\} \longmapsto x$. The theorem is a consequence of (15).

(17) Let us consider a non empty set X, an object i, and an element x of X. Then $(\operatorname{proj}(\{i\} \longmapsto X, i))(\{i\} \longmapsto x) = x$. The theorem is a consequence of (13).

(18) Let us consider sets X, Y. Then $X \neq \emptyset$ and $Y = \emptyset$ if and only if $\prod(X \longmapsto Y) = \emptyset$.

Let f be an empty function and x be an object. Let us note that $\operatorname{proj}(f, x)$ is trivial.

Now we state the proposition:

(19) Let us consider a trivial function f, and an object x. If $x \in \operatorname{dom} f$, then $\operatorname{proj}(f, x)$ is one-to-one.

Proof: Consider t being an object such that $\operatorname{dom} f = \{t\}$. Set $F = \operatorname{proj}(f, x)$. For every objects y, z such that y, $z \in \operatorname{dom} F$ and $F(y) = F(z)$ holds $y = z$. $\square$

Let x, y be objects. Note that $\operatorname{proj}(x \dot{\longmapsto} y, x)$ is one-to-one.

Let I be a 1-element set, J be a many sorted set indexed by I, and i be an element of I. One can verify that $\operatorname{proj}(J, i)$ is one-to-one.

Now we state the propositions:

(20) Let us consider a non empty set X, a subset Y of X, and an object i. Then $(\operatorname{proj}(\{i\} \longmapsto X, i))^{\circ}(\prod(\{i\} \longmapsto Y)) = Y$. The theorem is a consequence of (16), (13), and (14).

(21) Let us consider non-empty functions f, g, and objects i, x. Suppose $x \in \prod f \cap \prod(f+\cdot g)$. Then $(\mathrm{proj}(f,i))(x) = (\mathrm{proj}(f+\cdot g,i))(x)$.

(22) Let us consider non-empty functions f, g, an object i, and a set A. Suppose $A \subseteq \prod f \cap \prod(f+\cdot g)$. Then $(\mathrm{proj}(f,i))^{\circ}A = (\mathrm{proj}(f+\cdot g,i))^{\circ}A$. The theorem is a consequence of (21).

(23) Let us consider non-empty functions f, g. Suppose $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and for every object i such that $i \in \mathrm{dom}\, g$ holds $g(i) \subseteq f(i)$. Then $\prod(f+\cdot g) \subseteq \prod f$.

Let us consider non-empty functions f, g and an object i. Now we state the propositions:

(24) Suppose $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and for every object i such that $i \in \mathrm{dom}\, g$ holds $g(i) \subseteq f(i)$. Then if $i \in \mathrm{dom}\, f \setminus \mathrm{dom}\, g$, then $(\mathrm{proj}(f,i))^{\circ}(\prod(f+\cdot g)) = f(i)$. The theorem is a consequence of (23) and (22).

(25) Suppose $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and for every object i such that $i \in \mathrm{dom}\, g$ holds $g(i) \subseteq f(i)$. Then if $i \in \mathrm{dom}\, g$, then $(\mathrm{proj}(f,i))^{\circ}(\prod(f+\cdot g)) = g(i)$. The theorem is a consequence of (23) and (22).

(26) Suppose $\mathrm{dom}\, g = \mathrm{dom}\, f$ and for every object i such that $i \in \mathrm{dom}\, g$ holds $g(i) \subseteq f(i)$. Then if $i \in \mathrm{dom}\, g$, then $(\mathrm{proj}(f,i))^{\circ}(\prod g) = g(i)$. The theorem is a consequence of (25).

(27) Let us consider a function f, sets X, Y, and an object i. Suppose $X \subseteq Y$. Then $\prod(f+\cdot(i \dot{\longmapsto} X)) \subseteq \prod(f+\cdot(i \dot{\longmapsto} Y))$.

(28) Let us consider objects i, j, and sets A, B, C, D. Suppose $A \subseteq C$ and $B \subseteq D$. Then $\prod[i \longmapsto A, j \longmapsto B] \subseteq \prod[i \longmapsto C, j \longmapsto D]$. The theorem is a consequence of (14).

(29) Let us consider sets X, Y, and objects f, i, j. Suppose $i \neq j$. Then $f \in \prod[i \longmapsto X, j \longmapsto Y]$ if and only if there exist objects x, y such that $x \in X$ and $y \in Y$ and $f = [i \longmapsto x, j \longmapsto y]$.

PROOF: If $f \in \prod[i \longmapsto X, j \longmapsto Y]$, then there exist objects x, y such that $x \in X$ and $y \in Y$ and $f = [i \longmapsto x, j \longmapsto y]$. Reconsider $g = f$ as a function. For every object z such that $z \in \mathrm{dom}[i \longmapsto X, j \longmapsto Y]$ holds $g(z) \in [i \longmapsto X, j \longmapsto Y](z)$. $\square$

(30) Let us consider a non-empty function f, sets X, Y, objects i, j, x, y, and a function g. Suppose $x \in X$ and $y \in Y$ and $i \neq j$ and $g \in \prod f$. Then $g+\cdot[i \longmapsto x, j \longmapsto y] \in \prod(f+\cdot[i \longmapsto X, j \longmapsto Y])$.

PROOF: For every object z such that $z \in \mathrm{dom}(f+\cdot[i \longmapsto X, j \longmapsto Y])$ holds $(g+\cdot[i \longmapsto x, j \longmapsto y])(z) \in (f+\cdot[i \longmapsto X, j \longmapsto Y])(z)$. $\square$

(31) Let us consider a function f, sets A, B, C, D, and objects i, j. Suppose $A \subseteq C$ and $B \subseteq D$. Then $\prod(f+\cdot[i \longmapsto A, j \longmapsto B]) \subseteq \prod(f+\cdot[i \longmapsto$

$C, j \longmapsto D]$). The theorem is a consequence of (27).

(32) Let us consider a function f, sets A, B, and objects i, j. Suppose i, $j \in \operatorname{dom} f$ and $A \subseteq f(i)$ and $B \subseteq f(j)$. Then $\prod(f {+\cdot} [i \longmapsto A, j \longmapsto B]) \subseteq \prod f$. The theorem is a consequence of (11) and (31).

(33) Let us consider a set I, and many sorted sets f, g indexed by I. Then $\prod f \cap \prod g = \prod(f \cap g)$.
Proof: For every object x, $x \in \prod f \cap \prod g$ iff there exists a function h such that $h = x$ and $\operatorname{dom} h = \operatorname{dom}(f \cap g)$ and for every object y such that $y \in \operatorname{dom}(f \cap g)$ holds $h(y) \in (f \cap g)(y)$. □

(34) Let us consider a 2-element set I, a many sorted set f indexed by I, elements i, j of I, and an object x. Suppose $i \neq j$. Then

(i) $f +\cdot (i, x) = [i \longmapsto x, j \longmapsto f(j)]$, and

(ii) $f +\cdot (j, x) = [i \longmapsto f(i), j \longmapsto x]$.

The theorem is a consequence of (10).

Let us consider a non-empty function f, a set X, and an object i. Now we state the propositions:

(35) If $i \in \operatorname{dom} f$, then $f +\cdot (i, X)$ is non-empty iff X is not empty.
Proof: For every object x such that $x \in \operatorname{dom}(f +\cdot (i, X))$ holds $(f +\cdot (i, X))(x)$ is not empty. □

(36) If $i \in \operatorname{dom} f$, then $\prod(f +\cdot (i, X)) = \emptyset$ iff X is empty. The theorem is a consequence of (35).

(37) Let us consider a non-empty function f, a set X, objects i, x, and a function g. Suppose $i \in \operatorname{dom} f$ and $x \in X$ and $g \in \prod f$. Then $g +\cdot (i, x) \in \prod(f +\cdot (i, X))$.
Proof: For every object y such that $y \in \operatorname{dom}(f +\cdot (i, X))$ holds $(g +\cdot (i, x))(y) \in (f +\cdot (i, X))(y)$. □

(38) Let us consider a function f, sets X, Y, and an object i. Suppose $i \in \operatorname{dom} f$ and $X \subseteq Y$. Then $\prod(f +\cdot (i, X)) \subseteq \prod(f +\cdot (i, Y))$. The theorem is a consequence of (27).

(39) Let us consider a function f, a set X, and an object i. Suppose $i \in \operatorname{dom} f$ and $X \subseteq f(i)$. Then $\prod(f +\cdot (i, X)) \subseteq \prod f$. The theorem is a consequence of (38).

(40) Let us consider a non-empty function f, non empty sets X, Y, and objects i, j. Suppose $i, j \in \operatorname{dom} f$ and ($X \not\subseteq f(i)$ or $f(j) \not\subseteq Y$) and $\prod(f {+\cdot} (i, X)) \subseteq \prod(f +\cdot (j, Y))$. Then

(i) $i = j$, and

(ii) $X \subseteq Y$.

PROOF: $f +\cdot (i, X)$ is non-empty and $f +\cdot (j, Y)$ is non-empty. $i = j$. Set $g =$ the element of $\prod f$. $g +\cdot (i, x) \in \prod(f +\cdot (i, X))$. $\square$

(41) Let us consider a non-empty function f, a set X, and an object i. Suppose $i \in \operatorname{dom} f$ and $\prod(f +\cdot (i, X)) \subseteq \prod f$. Then $X \subseteq f(i)$. The theorem is a consequence of (37).

(42) Let us consider a non-empty function f, non empty sets X, Y, and objects i, j. Suppose $i, j \in \operatorname{dom} f$ and ($X \neq f(i)$ or $Y \neq f(j)$) and $\prod(f +\cdot (i, X)) = \prod(f +\cdot (j, Y))$. Then

(i) $i = j$, and

(ii) $X = Y$.

PROOF: $f +\cdot (i, X)$ is non-empty and $f +\cdot (j, Y)$ is non-empty. $i = j$. $\square$

(43) Let us consider a non-empty function f, a set X, and an object i. Suppose $i \in \operatorname{dom} f$ and $X \subseteq f(i)$. Then $(\operatorname{proj}(f, i))^\circ(\prod(f +\cdot (i, X))) = X$. The theorem is a consequence of (25).

(44) Let us consider objects x, y, z. Then $x \dot{\longmapsto} y +\cdot (x, z) = x \dot{\longmapsto} z$. The theorem is a consequence of (12).

Let I be a non empty set and J be a 1-sorted yielding, nonempty many sorted set indexed by I. Let us observe that the support of J is non-empty.

2. Remarks about Product Spaces

Now we state the propositions:

(45) Let us consider topological spaces T, S, and a function f from T into S. Then f is open if and only if there exists a basis B of T such that for every subset V of T such that $V \in B$ holds $f^\circ V$ is open.

(46) Let us consider non empty topological spaces T_1, T_2, S_1, S_2, a function f from T_1 into S_1, and a function g from T_2 into S_2. If f is open and g is open, then $f \times g$ is open.

PROOF: There exists a basis B of $T_1 \times T_2$ such that for every subset P of $T_1 \times T_2$ such that $P \in B$ holds $(f \times g)^\circ P$ is open. $\square$

Let us consider non empty topological spaces S, T and a function f from S into T. Now we state the propositions:

(47) If f is bijective and there exists a basis K of S and there exists a basis L of T such that $f^\circ K = L$, then f is a homeomorphism.

PROOF: For every subset W of T such that $W \in L$ holds $f^{-1}(W)$ is open. For every subset V of S such that $V \in K$ holds $f^\circ V$ is open. f is open. $\square$

(48) If f is bijective and there exists a prebasis K of S and there exists a prebasis L of T such that $f^{\circ}K = L$, then f is a homeomorphism.
PROOF: Reconsider $K_0 = \mathrm{FinMeetCl}(K)$ as a basis of S. Reconsider $L_0 = \mathrm{FinMeetCl}(L)$ as a basis of T. For every subset W of T, $W \in L_0$ iff there exists a subset V of S such that $V \in K_0$ and $f^{\circ}V = W$. □

Let us consider topological spaces S, T. Now we state the propositions:

(49) If there exists a basis K of S and there exists a basis L of T such that $K = L \Cap \{\Omega_S\}$, then S is a subspace of T.
PROOF: For every subset A of S, $A \in$ the topology of S iff there exists a subset B of T such that $B \in$ the topology of T and $A = B \cap \Omega_S$. Consider B being a subset of T such that $B \in$ the topology of T and the carrier of $S = B \cap \Omega_S$. □

(50) Suppose $\Omega_S \subseteq \Omega_T$ and there exists a prebasis K of S and there exists a prebasis L of T such that $K = L \Cap \{\Omega_S\}$. Then S is a subspace of T.
PROOF: Reconsider $K_0 = \mathrm{FinMeetCl}(K)$ as a basis of S. Reconsider $L_0 = \mathrm{FinMeetCl}(L)$ as a basis of T. For every object x, $x \in K_0$ iff $x \in L_0 \Cap \{\Omega_S\}$. □

(51) If there exists a prebasis K of S and there exists a prebasis L of T such that $\Omega_S \in K$ and $K = L \Cap \{\Omega_S\}$, then S is a subspace of T. The theorem is a consequence of (50).

(52) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, and an element i of I. Then $\mathrm{rng}\,\mathrm{proj}(J, i) =$ the carrier of $J(i)$.

Let X be a set and T be a topological structure. Observe that $X \longmapsto T$ is topological structure yielding.

Let F be a binary relation. We say that F is topological space yielding if and only if

(Def. 1) for every object x such that $x \in \mathrm{rng}\,F$ holds x is a topological space.

Note that every binary relation which is topological space yielding is also topological structure yielding and every function which is topological space yielding is also 1-sorted yielding.

Let X be a set and T be a topological space. One can verify that $X \longmapsto T$ is topological space yielding.

Let I be a set. One can verify that there exists a many sorted set indexed by I which is topological space yielding and nonempty.

Let I be a non empty set, J be a topological space yielding, nonempty many sorted set indexed by I, and i be an element of I. Let us note that the functor $J(i)$ yields a non empty topological space. Let f be a function. The functor $\mathrm{ProjMap}\,f$ yielding a many sorted function indexed by $\mathrm{dom}\,f$ is defined by

(Def. 2) for every object x such that $x \in \operatorname{dom} f$ holds $it(x) = \operatorname{proj}(f, x)$.

Let f be an empty function. One can verify that ProjMap f is empty.

Let f be a non-empty function. Note that ProjMap f is non-empty.

Let f be a non non-empty function. Let us note that ProjMap f is empty yielding.

Let I be a non empty set and J be a topological structure yielding, nonempty many sorted set indexed by I. The functor ProjMap J yielding a many sorted set indexed by I is defined by the term

(Def. 3) ProjMap(the support of J).

Observe that ProjMap J is function yielding, non empty, and non-empty.

Now we state the proposition:

(53) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, and an element i of I. Then $(\operatorname{ProjMap} J)(i) = \operatorname{proj}(J, i)$.

Let I be a non empty set, J be a topological structure yielding, nonempty many sorted set indexed by I, and f be a one-to-one, I-valued function. The functor ProdBasSel(J, f) yielding a many sorted set indexed by rng f is defined by the term

(Def. 4) $(\operatorname{ProjMap} J) \circ (I\text{-indexing}\, f^{-1}) \restriction \operatorname{rng} f$.

Let f be an empty, one-to-one, I-valued function. Note that ProdBasSel(J, f) is empty.

Now we state the propositions:

(54) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, a one-to-one, I-valued function f, and an element i of I. Suppose $i \in \operatorname{rng} f$. Then $(\operatorname{ProdBasSel}(J, f))(i) = (\operatorname{proj}(J, i))^{\circ}(f^{-1})(i)$. The theorem is a consequence of (53).

(55) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, and a one-to-one, I-valued function f. Suppose f^{-1} is non-empty and $\operatorname{dom} f \subseteq 2^{\prod \alpha}$. Then ProdBasSel$(J, f)$ is non-empty, where α is the support of J. The theorem is a consequence of (54).

(56) Let us consider a non empty set I, and a topological space yielding, nonempty many sorted set J indexed by I. Then $\emptyset \in$ the product prebasis for J. The theorem is a consequence of (36).

(57) Let us consider a non empty set I, a topological structure yielding, nonempty many sorted set J indexed by I, and a non empty subset P of $\prod$(the support of J). Suppose $P \in$ the product prebasis for J. Then there exists an element i of I such that

(i) $(\mathrm{proj}(J,i))^{\circ}P$ is open, and

(ii) for every element j of I such that $j \neq i$ holds $(\mathrm{proj}(J,j))^{\circ}P = \Omega_{J(j)}$.

Proof: Consider i being a set, T being a topological structure, V being a subset of T such that $i \in I$ and V is open and $T = J(i)$ and $P = \prod((\text{the support of } J) +\cdot (i, V))$. $\mathrm{rng}\,\mathrm{proj}(J,i) =$ the carrier of $J(i)$. For every object x, $x \in (\mathrm{proj}(J,j))^{\circ}P$ iff $x \in \Omega_{J(j)}$ by [1, (30), (32)], [9, (8)], [8, (7)]. □

(58) Let us consider a non empty set I, a topological space yielding, non-empty many sorted set J indexed by I, and a non empty subset P of $\prod(\text{the support of } J)$. Suppose $P \in$ the product prebasis for J. Then

(i) for every element j of I, $(\mathrm{proj}(J,j))^{\circ}P$ is open, and

(ii) there exists an element i of I such that for every element j of I such that $j \neq i$ holds $(\mathrm{proj}(J,j))^{\circ}P = \Omega_{J(j)}$.

The theorem is a consequence of (57).

(59) Let us consider a non empty set I, a topological structure yielding, non-empty many sorted set J indexed by I, a one-to-one, I-valued function f, and a family X of subsets of $\prod(\text{the support of } J)$. Suppose $X \subseteq$ the product prebasis for J and $\mathrm{dom}\, f = X$ and f^{-1} is non-empty and for every subset A of $\prod(\text{the support of } J)$ such that $A \in X$ holds $(\mathrm{proj}(J, f_{/A}))^{\circ}A$ is open. Let us consider an element i of I. Then

(i) if $i \notin \mathrm{rng}\, f$, then $(\mathrm{proj}(J,i))^{\circ}(\prod((\text{the support of } J) +\cdot \mathrm{ProdBasSel}(J,f))) = \Omega_{J(i)}$, and

(ii) if $i \in \mathrm{rng}\, f$, then $(\mathrm{proj}(J,i))^{\circ}(\prod((\text{the support of } J) +\cdot \mathrm{ProdBasSel}(J,f)))$ is open.

Proof: Set $g = \mathrm{ProdBasSel}(J,f)$. Set $P = \prod((\text{the support of } J) +\cdot g)$. g is non-empty. If $i \notin \mathrm{rng}\, f$, then $(\mathrm{proj}(J,i))^{\circ}P = \Omega_{J(i)}$. □

(60) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, a one-to-one, I-valued function f, and a family X of subsets of $\prod(\text{the support of } J)$. Suppose $X \subseteq$ the product prebasis for J and $\mathrm{dom}\, f = X$ and f^{-1} is non-empty and for every subset A of $\prod(\text{the support of } J)$ such that $A \in X$ holds $(\mathrm{proj}(J, f_{/A}))^{\circ}A$ is open. Let us consider an element i of I. Then

(i) $(\mathrm{proj}(J,i))^{\circ}(\prod((\text{the support of } J) +\cdot \mathrm{ProdBasSel}(J,f)))$ is open, and

(ii) if $i \notin \mathrm{rng}\, f$, then $(\mathrm{proj}(J,i))^{\circ}(\prod((\text{the support of } J) +\cdot \mathrm{ProdBasSel}(J,f))) = \Omega_{J(i)}$.

The theorem is a consequence of (59).

(61) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, and a subset P of $\prod$(the support of J). Then $P \in$ FinMeetCl(the product prebasis for J) if and only if there exists a family X of subsets of $\prod$(the support of J) and there exists a one-to-one, I-valued function f such that $X \subseteq$ the product prebasis for J and X is finite and $P = \text{Intersect}(X)$ and $\text{dom}\, f = X$ and $P = \prod$((the support of J)$+\cdot$ ProdBasSel(J, f)).

Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, and a non empty subset P of $\prod$(the support of J). Now we state the propositions:

(62) Suppose $P \in$ FinMeetCl(the product prebasis for J). Then there exists a family X of subsets of $\prod$(the support of J) and there exists a one-to-one, I-valued function f such that $X \subseteq$ the product prebasis for J and X is finite and $P = \text{Intersect}(X)$ and $\text{dom}\, f = X$ and for every element i of I, $(\text{proj}(J, i))^{\circ}P$ is open and if $i \notin \text{rng}\, f$, then $(\text{proj}(J, i))^{\circ}P = \Omega_{J(i)}$.
PROOF: Consider X being a family of subsets of $\prod$(the support of J), f being a one-to-one, I-valued function such that $X \subseteq$ the product prebasis for J and X is finite and $P = \text{Intersect}(X)$ and $\text{dom}\, f = X$ and $P = \prod$((the support of J)$+\cdot$ ProdBasSel(J, f)). f^{-1} is non-empty. □

(63) Suppose $P \in$ FinMeetCl(the product prebasis for J). Then there exists a finite subset I_0 of I such that for every element i of I, $(\text{proj}(J, i))^{\circ}P$ is open and if $i \notin I_0$, then $(\text{proj}(J, i))^{\circ}P = \Omega_{J(i)}$. The theorem is a consequence of (62).

(64) Let us consider a 1-element set I, a topological structure yielding, nonempty many sorted set J indexed by I, an element i of I, and a subset P of $\prod$(the support of J). Then $P \in$ the product prebasis for J if and only if there exists a subset V of $J(i)$ such that V is open and $P = \prod(\{i\} \longmapsto V)$. The theorem is a consequence of (7) and (44).

(65) Let us consider a 1-element set I, and a topological space yielding, nonempty many sorted set J indexed by I. Then the topology of $\prod J =$ the product prebasis for J.

(66) Let us consider a 1-element set I, a topological space yielding, nonempty many sorted set J indexed by I, an element i of I, and a subset P of $\prod J$. Then P is open if and only if there exists a subset V of $J(i)$ such that V is open and $P = \prod(\{i\} \longmapsto V)$. The theorem is a consequence of (65) and (64).

Let I be a non empty set, J be a topological structure yielding, nonempty many sorted set indexed by I, and i be an element of I. Note that $\text{proj}(J, i)$ is continuous and onto.

Let J be a topological space yielding, nonempty many sorted set indexed by I. Note that $\mathrm{proj}(J, i)$ is open.

Let us consider a 1-element set I, a topological space yielding, nonempty many sorted set J indexed by I, and an element i of I. Now we state the propositions:

(67) $\mathrm{proj}(J, i)$ is a homeomorphism. The theorem is a consequence of (7).

(68) $\prod J$ and $J(i)$ are homeomorphic. The theorem is a consequence of (67).

Let us consider a 2-element set I, a topological space yielding, nonempty many sorted set J indexed by I, elements i, j of I, and a subset P of $\prod$(the support of J). Now we state the propositions:

(69) Suppose $i \neq j$. Then $P \in$ the product prebasis for J if and only if there exists a subset V of $J(i)$ such that V is open and $P = \prod[i \longmapsto V, j \longmapsto \Omega_{J(j)}]$ or there exists a subset W of $J(j)$ such that W is open and $P = \prod[i \longmapsto \Omega_{J(i)}, j \longmapsto W]$. The theorem is a consequence of (34).

(70) Suppose $i \neq j$. Then $P \in$ FinMeetCl(the product prebasis for J) if and only if there exists a subset V of $J(i)$ and there exists a subset W of $J(j)$ such that V is open and W is open and $P = \prod[i \longmapsto V, j \longmapsto W]$.

Proof: There exists a family Y of subsets of $\prod$(the support of J) such that $Y \subseteq$ the product prebasis for J and Y is finite and $P = \mathrm{Intersect}(Y)$. $\square$

(71) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, and elements i, j of I. Then $\langle \mathrm{proj}(J, i), \mathrm{proj}(J, j) \rangle$ is a function from $\prod J$ into $J(i) \times J(j)$.

(72) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, a subset P of $\prod$(the support of J), and elements i, j of I. Suppose $i \neq j$ and there exists a many sorted set F indexed by I such that $P = \prod F$ and for every element k of I, $F(k) \subseteq$ (the support of $J)(k)$. Then $\langle \mathrm{proj}(J, i), \mathrm{proj}(J, j) \rangle^{\circ} P = (\mathrm{proj}(J, i))^{\circ} P \times (\mathrm{proj}(J, j))^{\circ} P$. The theorem is a consequence of (26), (30), and (11).

(73) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, elements i, j of I, and a function f from $\prod J$ into $J(i) \times J(j)$. Suppose $i \neq j$ and $f = \langle \mathrm{proj}(J, i), \mathrm{proj}(J, j) \rangle$. Then f is onto and open.

Proof: For every element k of I, $(\mathrm{proj}(J, k))^{\circ}(\Omega_{\prod \alpha}) =$ the carrier of $J(k)$, where α is the support of J. There exists a basis B of $\prod J$ such that for every subset P of $\prod J$ such that $P \in B$ holds $f^{\circ} P$ is open. $\square$

(74) Let us consider a 2-element set I, a topological space yielding, nonempty many sorted set J indexed by I, elements i, j of I, and a function f from

$\prod J$ into $J(i) \times J(j)$. Suppose $i \neq j$ and $f = \langle \mathrm{proj}(J, i), \mathrm{proj}(J, j) \rangle$. Then f is a homeomorphism.

PROOF: f is onto and open. For every objects x_1, x_2 such that x_1, $x_2 \in \mathrm{dom}\, f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$. □

(75) Let us consider a 2-element set I, a topological space yielding, nonempty many sorted set J indexed by I, and elements i, j of I. If $i \neq j$, then $\prod J$ and $J(i) \times J(j)$ are homeomorphic. The theorem is a consequence of (74).

Let I_1, I_2 be non empty sets, J be a topological space yielding, nonempty many sorted set indexed by I_2, and f be a function from I_1 into I_2. One can check that $J \cdot f$ is topological space yielding and nonempty.

Let J_1 be a topological space yielding, nonempty many sorted set indexed by I_1, J_2 be a topological space yielding, nonempty many sorted set indexed by I_2, and p be a function from I_1 into I_2. Assume p is bijective and for every element i of I_1, $J_1(i)$ and $(J_2 \cdot p)(i)$ are homeomorphic.

A product homeomorphism of J_1, J_2 and p is a function from $\prod J_1$ into $\prod J_2$ defined by

(Def. 5) there exists a many sorted function F indexed by I_1 such that for every element i of I_1, there exists a function f from $J_1(i)$ into $(J_2 \cdot p)(i)$ such that $F(i) = f$ and f is a homeomorphism and for every element g of $\prod J_1$ and for every element i of I_1, $(it(g))(p(i)) = F(i)(g(i))$.

Now we state the proposition:

(76) Let us consider non empty sets I_1, I_2, a topological space yielding, nonempty many sorted set J_1 indexed by I_1, a topological space yielding, nonempty many sorted set J_2 indexed by I_2, a function p from I_1 into I_2, a product homeomorphism H of J_1, J_2 and p, and a many sorted function F indexed by I_1. Suppose p is bijective and for every element i of I_1, there exists a function f from $J_1(i)$ into $(J_2 \cdot p)(i)$ such that $F(i) = f$ and f is a homeomorphism and for every element g of $\prod J_1$ and for every element i of I_1, $(H(g))(p(i)) = F(i)(g(i))$. Let us consider an element i of I_1, and a subset U of $J_1(i)$. Then $H^\circ(\prod((\text{the support of } J_1) +\cdot (i, U))) = \prod((\text{the support of } J_2) +\cdot (p(i), F(i)^\circ U))$.

PROOF: Reconsider $j = p(i)$ as an element of I_2. Consider f being a function from $J_1(i)$ into $(J_2 \cdot p)(i)$ such that $F(i) = f$ and f is a homeomorphism. For every object y, $y \in H^\circ(\prod((\text{the support of } J_1) +\cdot (i, U)))$ iff $y \in \prod((\text{the support of } J_2) +\cdot (j, F(i)^\circ U))$. □

Let us consider non empty sets I_1, I_2, a topological space yielding, nonempty many sorted set J_1 indexed by I_1, a topological space yielding, nonempty many sorted set J_2 indexed by I_2, a function p from I_1 into I_2, and a product homeomorphism H of J_1, J_2 and p. Now we state the propositions:

(77) If p is bijective and for every element i of I_1, $J_1(i)$ and $(J_2 \cdot p)(i)$ are homeomorphic, then H is bijective.
PROOF: Consider F being a many sorted function indexed by I_1 such that for every element i of I_1, there exists a function f from $J_1(i)$ into $(J_2 \cdot p)(i)$ such that $F(i) = f$ and f is a homeomorphism and for every element g of $\prod J_1$ and for every element i of I_1, $(H(g))(p(i)) = F(i)(g(i))$. For every objects x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} H$ and $H(x_1) = H(x_2)$ holds $x_1 = x_2$. Set $i_0 =$ the element of I_1. Consider f_0 being a function from $J_1(i_0)$ into $(J_2 \cdot p)(i_0)$ such that $F(i_0) = f_0$ and f_0 is a homeomorphism. $\square$

(78) If p is bijective and for every element i of I_1, $J_1(i)$ and $(J_2 \cdot p)(i)$ are homeomorphic, then H is a homeomorphism.
PROOF: Consider F being a many sorted function indexed by I_1 such that for every element i of I_1, there exists a function f from $J_1(i)$ into $(J_2 \cdot p)(i)$ such that $F(i) = f$ and f is a homeomorphism and for every element g of $\prod J_1$ and for every element i of I_1, $(H(g))(p(i)) = F(i)(g(i))$. H is bijective. There exists a prebasis K of $\prod J_1$ and there exists a prebasis L of $\prod J_2$ such that $H^\circ K = L$. $\square$

(79) Let us consider non empty sets I_1, I_2, a topological space yielding, nonempty many sorted set J_1 indexed by I_1, a topological space yielding, nonempty many sorted set J_2 indexed by I_2, and a function p from I_1 into I_2. Suppose p is bijective and for every element i of I_1, $J_1(i)$ and $(J_2 \cdot p)(i)$ are homeomorphic. Then $\prod J_1$ and $\prod J_2$ are homeomorphic. The theorem is a consequence of (78).

(80) Let us consider a non empty set I, topological space yielding, nonempty many sorted sets J_1, J_2 indexed by I, and a permutation p of I. Suppose for every element i of I, $J_1(i)$ and $(J_2 \cdot p)(i)$ are homeomorphic. Then $\prod J_1$ and $\prod J_2$ are homeomorphic.

(81) Let us consider a non empty set I, a topological space yielding, nonempty many sorted set J indexed by I, and a permutation p of I. Then $\prod J$ and $\prod J \cdot p$ are homeomorphic. The theorem is a consequence of (79).

(82) Let us consider a non empty set I, and topological space yielding, nonempty many sorted sets J_1, J_2 indexed by I. Suppose for every element i of I, $J_1(i)$ is a subspace of $J_2(i)$. Then $\prod J_1$ is a subspace of $\prod J_2$.
PROOF: There exists a prebasis K_1 of $\prod J_1$ and there exists a prebasis K_2 of $\prod J_2$ such that $\Omega_{\prod J_1} \in K_1$ and $K_1 = K_2 \Cap \{\Omega_{\prod J_1}\}$. $\square$

References

[1] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[3] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.

[4] Nicolas Bourbaki. *Elements de Mathematique*, volume Topologie Generale. HERMANN, troisieme edition, 1960.

[5] Jarosław Gryko. Injective spaces. *Formalized Mathematics*, 7(**1**):57–62, 1998.

[6] John L. Kelley. *General Topology*, volume 27 of *Graduate Texts in Mathematics*. Springer-Verlag, 1955.

[7] James Raymond Munkres. *Topology*. Prentice-Hall, Upper Saddle River, NJ, 2 edition, 2000.

[8] Adam Naumowicz. On the characterization of collineations of the Segre product of strongly connected partial linear spaces. *Formalized Mathematics*, 13(**1**):125–131, 2005.

[9] Bartłomiej Skorulski. The Tichonov Theorem. *Formalized Mathematics*, 9(**2**):373–376, 2001.

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