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Formalizing Two Generalized Approximation Operators

Adam Grabowski
Institute of Informatics
University of Białystok
Poland

Michał Sielwiesiuk
Institute of Informatics
University of Białystok
Poland

Summary. Rough sets, developed by Pawlak [15], are important tool to describe situation of incomplete or partially unknown information. In this article we give the formal characterization of two closely related rough approximations, along the lines proposed in a paper by Gomolińska [2]. We continue the formalization of rough sets in Mizar [1] started in [6].

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0. Introduction

In the paper [9] published in 2010 we discussed some pros and cons of various approaches to rough operators dealing with some of the issues raised by Anna Gomolińska [2]. Even if our chosen formal framework [6] faithfully reflected Pawlak's ideas [15], also possibility of other views for the same topic was quite tempting. Our question was if the Mizar Mathematical Library is ready to do some formal reasoning without much additional work needed to bridge the gap between informal knowledge and its formal countepart present in the repository of automatically verified mathematical knowledge. This expectation is not really that trivial as we noted after an unsatisfactory – at least from our point of view – attempt to formalize Rough Concept Analysis in Mizar [12]. On the other hand, reuse of lattice theory to develop a rough framework [4] according to Järvinen

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[13] or bulding a theory of approximations based on pure set theory in style of [16], [17] showed the usefulness of automated theorem proving methods [3] in order to obtain new results, with possibility of theory merging, taking care of possible duplications [11].

Our main aim was to use the existing implementation of rough sets in Mizar to provide the formal proof of the following theorem (original notation of [2]):

Theorem 4.1 For any sets $x, y \subseteq U$, objects $u, w \in U$, and $i = 0, 1$, it holds that:

1. $f_0^d \leqslant id \leqslant f_0$.
2. $f_1^d \leqslant id \leqslant f_1$.
3. $f_0(x)$ is definable.
4. $\forall_u \in f_1(x).\kappa(I(u), x) > 0$.
5. $\forall_u \in f_1^d(x).\kappa(I(u), x) = 1$.
6. If $\tau(u) = \tau(w)$, then $u \in f_0(x)$ iff $w \in f_0(x)$; and similarly for f_0^d.
7. If $I(u) = I(w)$, then $u \in f_1(x)$ iff $w \in f_1(x)$; and similarly for f_1^d.
8. $f_i(\emptyset) = \emptyset$ and $f_i(U) = U$; and similarly for f_i^d.
9. f_i and f_i^d are monotone.
10. $f_i(x \cup y) = f_i(x) \cup f_i(y)$.
11. $f_i^d(x \cup y) \supseteq f_i^d(x) \cup f_i^d(y)$.
12. $f_i(x \cap y) \subseteq f_i(x) \cap f_i(y)$.
13. $f_i^d(x \cap y) = f_i^d(x) \cap f_i^d(y)$.

With the exception of two subitems (4. and 5.) dealing with κ as rough inclusion operator, we succeeded.

It should be mentioned, that most of the reasoning on the properties of the generalized approximation operator was done under the assumption

$$\forall_{u \in U}\ u \in I(u),$$

which we called `map-reflexive` of the uncertainty mapping I. Another, more general relational aproach was adopted in [8] which is a Mizar counterpart of [17]. There the reflexivity of binary indiscernibility relation was assumed where needed.

Automated math-asistants can offer a new – semiautomated – insight [10] also for quite elementary notions: in Section 4, we introduced more general Mizar functor dealing with arbitrary maps from the universe into its powerset, so that we could obtain most of properties of mappings f_0 and f_1 as straightforward consequences. We kept them both however, to assure full compatibility with [2].

1. Preliminaries: Map-Reflexivity

Let R be a non empty set and I be a function from R into 2^R. We say that I is map-reflexive if and only if

(Def. 1) for every element u of R, $u \in I(u)$.

The functor $\mathrm{singleton}_R$ yielding a function from R into 2^R is defined by

(Def. 2) for every element x of R, $it(x) = \{x\}$.

Let us observe that $\mathrm{singleton}_R$ is map-reflexive.

Now we state the proposition:

(1) Let us consider a non empty relational structure R, and a function I from the carrier of R into 2^α. Suppose I is map-reflexive. Then the carrier of $R = \bigcup(I^\circ(\Omega_R))$, where α is the carrier of R.

From now on f, g denote functions and R denotes a non empty, reflexive relational structure.

Now we state the propositions:

(2) $\mathrm{LAp}(R) \mathrel{\dot\subseteq} \mathrm{id}_{2^\alpha}$, where α is the carrier of R.

Proof: Set $f = \mathrm{LAp}(R)$. Set $g = \mathrm{id}_{2^{(\text{the carrier of } R)}}$. For every set i such that $i \in \mathrm{dom}\, f$ holds $f(i) \subseteq g(i)$ by [7, (35)]. $\square$

(3) $\mathrm{id}_{2^\alpha} \mathrel{\dot\subseteq} \mathrm{UAp}(R)$, where α is the carrier of R.

Proof: Set $f = \mathrm{id}_{2^{(\text{the carrier of } R)}}$. Set $g = \mathrm{UAp}(R)$. For every set i such that $i \in \mathrm{dom}\, f$ holds $f(i) \subseteq g(i)$. $\square$

2. Properties of Flipping Operator f^d

From now on R denotes a non empty relational structure.

Now we state the propositions:

(4) Let us consider a map f of R, and subsets x, y of R. Then $\mathrm{Flip}\,\mathrm{Flip}\, f = f$.

(5) Let us consider maps f, g of R. Then $\mathrm{Flip}\, f \cdot g = (\mathrm{Flip}\, f) \cdot (\mathrm{Flip}\, g)$.

Proof: Set $f_2 = \mathrm{Flip}\, f \cdot g$. Set $f_1 = \mathrm{Flip}\, f$. Set $g_1 = \mathrm{Flip}\, g$. For every subset x of R, $f_2(x) = (f_1 \cdot g_1)(x)$. $\square$

(6) Let us consider a map f of R. Then $f(\emptyset) = \emptyset$ if and only if $(\mathrm{Flip}\, f)(\text{the carrier of } R) = \text{the carrier of } R$.

3. Uncertainty Mappings I and τ

Let R be a non empty relational structure. The functor I_R yielding a function from the carrier of R into $2^{(\text{the carrier of } R)}$ is defined by

(Def. 3) for every element x of R, $it(x) = \mathrm{Coim}((\text{the internal relation of } R), x)$.

Now we state the proposition:

(7) Let us consider elements w, u of R. Then $\langle w, u\rangle \in$ the internal relation of R if and only if $w \in (I_R)(u)$.

Let R be a non empty relational structure. The functor τ_R yielding a function from the carrier of R into $2^{(\text{the carrier of } R)}$ is defined by

(Def. 4) for every element u of R, $it(u) = (\text{the internal relation of } R)^{\circ}u$.

Now we state the propositions:

(8) Let us consider elements u, w of R. Then $u \in (\text{the internal relation of } R)^{\circ}w$ if and only if $w \in \mathrm{Coim}((\text{the internal relation of } R), u)$.
PROOF: If $u \in (\text{the internal relation of } R)^{\circ}w$, then $w \in \mathrm{Coim}((\text{the internal relation of } R), u)$. Consider t being an object such that $\langle w, t\rangle \in$ the internal relation of R and $t \in \{u\}$. $\square$

(9) Let us consider elements w, u of R. Then $\langle w, u\rangle \in$ the internal relation of R if and only if $u \in (\tau_R)(w)$.
PROOF: If $\langle w, u\rangle \in$ the internal relation of R, then $u \in (\tau_R)(w)$. $w \in \mathrm{Coim}((\text{the internal relation of } R), u)$. Consider x being an object such that $\langle w, x\rangle \in$ the internal relation of R and $x \in \{u\}$. $\square$

4. Generalized Approximation Mappings

Let R be a non empty relational structure and f be a function from the carrier of R into $2^{(\text{the carrier of } R)}$. The functor UAp_f yielding a map of R is defined by

(Def. 5) for every subset x of R, $it(x) = \{u$, where u is an element of $R : f(u)$ meets $x\}$.

The functors: $f_0(R)$ and $f_1(R)$ yielding maps of R are defined by terms

(Def. 6) UAp_{τ_R},

(Def. 7) UAp_{I_R},

respectively. Now we state the propositions:

(10) If the internal relation of R is symmetric, then $I_R = \tau_R$.
PROOF: Set $f = I_R$. Set $g = \tau_R$. For every element x of R, $f(x) = g(x)$ by [14, (20)]. $\square$

(11) If the internal relation of R is symmetric, then $f_0(R) = f_1(R)$. The theorem is a consequence of (10).

(12) the internal relation of R is symmetric if and only if for every elements u, v of R such that $u \in (\tau_R)(v)$ holds $v \in (\tau_R)(u)$. The theorem is a consequence of (10), (7), and (9).

(13) $f_0(R) = \mathrm{UAp}(R)$.

(14) Flip $f_0(R) = \mathrm{LAp}(R)$. The theorem is a consequence of (13).

(15) Let us consider an approximation space R, and a subset x of R. Then $(f_0(R))(x)$ is exact. The theorem is a consequence of (13).

5. The Ordering of Approximation Mappings

Now we state the propositions:

(16) If the internal relation of R is total and reflexive, then $\mathrm{id}_{2^\alpha} \dot{\subseteq} f_0(R)$, where α is the carrier of R.

PROOF: Set $f = \mathrm{id}_{2^{(\text{the carrier of } R)}}$. Set $g = f_0(R)$. For every set i such that $i \in \mathrm{dom}\, f$ holds $f(i) \subseteq g(i)$ by [5, (1)], (9). $\square$

(17) If R is reflexive, then Flip $f_0(R) \dot{\subseteq} \mathrm{id}_{2^\alpha}$, where α is the carrier of R. The theorem is a consequence of (14) and (2).

(18) If the internal relation of R is total and reflexive, then $\mathrm{id}_{2^\alpha} \dot{\subseteq} f_1(R)$, where α is the carrier of R.

PROOF: Set $f = \mathrm{id}_{2^{(\text{the carrier of } R)}}$. Set $g = f_1(R)$. For every set i such that $i \in \mathrm{dom}\, f$ holds $f(i) \subseteq g(i)$. $\square$

6. Acting on the Empty Set and the Universe

In the sequel f denotes a function from the carrier of R into $2^{(\text{the carrier of } R)}$.

Now we state the proposition:

(19) $(\mathrm{UAp}_f)(\emptyset) = \emptyset$.

Let us consider R and f. One can check that UAp_f preserves empty set.

(20) $(f_0(R))(\emptyset) = \emptyset$.

(21) $(f_1(R))(\emptyset) = \emptyset$.

Let R be a non empty, reflexive relational structure. Let us observe that the internal relation of R is total and reflexive.

(22) If f is map-reflexive, then $(\mathrm{UAp}_f)(\text{the carrier of } R) = \text{the carrier of } R$.

(23) Suppose the internal relation of R is reflexive and total.
Then $(f_0(R))$(the carrier of R) = the carrier of R.
PROOF: The carrier of $R \subseteq \{u$, where u is an element of $R : (\tau_R)(u)$ meets $\Omega_R\}$. □

(24) Suppose the internal relation of R is reflexive and total.
Then $(f_1(R))$(the carrier of R) = the carrier of R.
PROOF: The carrier of $R \subseteq \{u$, where u is an element of $R : (I_R)(u)$ meets $\Omega_R\}$. □

7. Standard Properties of Approximations

Let us consider elements u, w of R and a subset x of R. Now we state the propositions:

(25) If $f(u) = f(w)$, then $u \in (\mathrm{UAp}_f)(x)$ iff $w \in (\mathrm{UAp}_f)(x)$.

(26) If $(I_R)(u) = (I_R)(w)$, then $u \in (f_1(R))(x)$ iff $w \in (f_1(R))(x)$.

(27) If $(\tau_R)(u) = (\tau_R)(w)$, then $u \in (f_0(R))(x)$ iff $w \in (f_0(R))(x)$.

(28) Let us consider a function f from the carrier of R into 2^α, and a subset x of R. Then $(\mathrm{Flip}(\mathrm{UAp}_f))(x) = \{w$, where w is an element of $R : f(w) \subseteq x\}$, where α is the carrier of R.
PROOF: $(\mathrm{Flip}(\mathrm{UAp}_f))(x) \subseteq \{w$, where w is an element of $R : f(w) \subseteq x\}$. Consider w being an element of R such that $y = w$ and $f(w) \subseteq x$. Reconsider $y_1 = y$ as an element of R. $y_1 \notin (\mathrm{UAp}_f)(x^c)$. □

Let us consider a subset x of R. Now we state the propositions:

(29) $(\mathrm{Flip}\, f_0(R))(x) = \{w$, where w is an element of $R : (\tau_R)(w) \subseteq x\}$.
PROOF: $(\mathrm{Flip}\, f_0(R))(x) \subseteq \{w$, where w is an element of $R : (\tau_R)(w) \subseteq x\}$. Consider w being an element of R such that $y = w$ and $(\tau_R)(w) \subseteq x$. Reconsider $y_1 = y$ as an element of R. $y_1 \notin (f_0(R))(x^c)$. □

(30) $(\mathrm{Flip}\, f_1(R))(x) = \{w$, where w is an element of $R : (I_R)(w) \subseteq x\}$.
PROOF: $(\mathrm{Flip}\, f_1(R))(x) \subseteq \{w$, where w is an element of $R : (I_R)(w) \subseteq x\}$. Consider w being an element of R such that $y = w$ and $(I_R)(w) \subseteq x$. Reconsider $y_1 = y$ as an element of R. $y_1 \notin (f_1(R))(x^c)$. □

Let us consider elements u, w of R and a subset x of R. Now we state the propositions:

(31) If $f(u) = f(w)$, then $u \in (\mathrm{Flip}(\mathrm{UAp}_f))(x)$ iff $w \in (\mathrm{Flip}(\mathrm{UAp}_f))(x)$. The theorem is a consequence of (28).

(32) If $(\tau_R)(u) = (\tau_R)(w)$, then $u \in (\mathrm{Flip}\, f_0(R))(x)$ iff $w \in (\mathrm{Flip}\, f_0(R))(x)$. The theorem is a consequence of (29).

(33) If $(I_R)(u) = (I_R)(w)$, then $u \in (\mathrm{Flip}\, f_1(R))(x)$ iff $w \in (\mathrm{Flip}\, f_1(R))(x)$. The theorem is a consequence of (30).

Let us consider an element w of R. Now we state the propositions:

(34) If R is reflexive, then $w \in (I_R)(w)$. The theorem is a consequence of (7).

(35) If R is reflexive, then $w \in (\tau_R)(w)$. The theorem is a consequence of (9).

Let R be a reflexive, non empty relational structure. One can verify that I_R is map-reflexive and τ_R is map-reflexive.

Now we state the propositions:

(36) If R is reflexive, then $\mathrm{Flip}\, f_1(R) \dot{\subseteq} \mathrm{id}_{2^\alpha}$, where α is the carrier of R. The theorem is a consequence of (34) and (30).

(37) $(f_0(R)) \cdot (f_0(R)) = f_0(R)$ if and only if $(\mathrm{Flip}\, f_0(R)) \cdot (\mathrm{Flip}\, f_0(R)) = \mathrm{Flip}\, f_0(R)$. The theorem is a consequence of (5).

(38) If R is reflexive, then $\bigcup((I_R)^\circ(\Omega_R)) =$ the carrier of R. The theorem is a consequence of (34).

8. Monotonicity of Approximations

Let R be a non empty relational structure. One can verify that $f_0(R)$ is $\subseteq$-monotone and $f_1(R)$ is $\subseteq$-monotone.

Now we state the propositions:

(39) Let us consider a map f of R. Suppose f is $\subseteq$-monotone. Then $\mathrm{Flip}\, f$ is $\subseteq$-monotone.

Proof: Set $g = \mathrm{Flip}\, f$. For every subsets A, B of R such that $A \subseteq B$ holds $g(A) \subseteq g(B)$. $\square$

(40) $\mathrm{Flip}\, f_0(R)$ is $\subseteq$-monotone.

(41) $\mathrm{Flip}\, f_1(R)$ is $\subseteq$-monotone.

9. Distributivity wrt. Set-Theoretic Operations

Now we state the proposition:

(42) Let us consider a function f from the carrier of R into 2^α, and subsets x, y of R. Then $(\mathrm{UAp}_f)(x \cup y) = (\mathrm{UAp}_f)(x) \cup (\mathrm{UAp}_f)(y)$, where α is the carrier of R.

Let us consider subsets x, y of R. Now we state the propositions:

(43) $(f_0(R))(x \cup y) = (f_0(R))(x) \cup (f_0(R))(y)$. The theorem is a consequence of (42).

(44) $(f_1(R))(x \cup y) = (f_1(R))(x) \cup (f_1(R))(y)$. The theorem is a consequence of (42).

(45) Let us consider a function f from the carrier of R into 2^α, and subsets x, y of R. Then $(\mathrm{Flip}(\mathrm{UAp}_f))(x) \cup (\mathrm{Flip}(\mathrm{UAp}_f))(y) \subseteq (\mathrm{Flip}(\mathrm{UAp}_f))(x \cup y)$, where α is the carrier of R. The theorem is a consequence of (28).

Let us consider subsets x, y of R. Now we state the propositions:

(46) $(\mathrm{Flip}\, f_0(R))(x) \cup (\mathrm{Flip}\, f_0(R))(y) \subseteq (\mathrm{Flip}\, f_0(R))(x \cup y)$. The theorem is a consequence of (45).

(47) $(\mathrm{Flip}\, f_1(R))(x) \cup (\mathrm{Flip}\, f_1(R))(y) \subseteq (\mathrm{Flip}\, f_1(R))(x \cup y)$. The theorem is a consequence of (45).

(48) Let us consider a function f from the carrier of R into 2^α, and subsets x, y of R. Then $(\mathrm{UAp}_f)(x \cap y) \subseteq (\mathrm{UAp}_f)(x) \cap (\mathrm{UAp}_f)(y)$, where α is the carrier of R.

Let us consider subsets x, y of R. Now we state the propositions:

(49) $(f_0(R))(x \cap y) \subseteq (f_0(R))(x) \cap (f_0(R))(y)$. The theorem is a consequence of (48).

(50) $(f_1(R))(x \cap y) \subseteq (f_1(R))(x) \cap (f_1(R))(y)$. The theorem is a consequence of (48).

(51) Let us consider a function f from the carrier of R into 2^α, and subsets x, y of R. Then $(\mathrm{Flip}(\mathrm{UAp}_f))(x) \cap (\mathrm{Flip}(\mathrm{UAp}_f))(y) = (\mathrm{Flip}(\mathrm{UAp}_f))(x \cap y)$, where α is the carrier of R.

Let us consider subsets x, y of R. Now we state the propositions:

(52) $(\mathrm{Flip}\, f_0(R))(x) \cap (\mathrm{Flip}\, f_0(R))(y) = (\mathrm{Flip}\, f_0(R))(x \cap y)$. The theorem is a consequence of (51).

(53) $(\mathrm{Flip}\, f_1(R))(x) \cap (\mathrm{Flip}\, f_1(R))(y) = (\mathrm{Flip}\, f_1(R))(x \cap y)$. The theorem is a consequence of (51).

References

[1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[2] Anna Gomolińska. A comparative study of some generalized rough approximations. *Fundamenta Informaticae*, 51:103–119, 2002.

[3] Adam Grabowski. Automated discovery of properties of rough sets. *Fundamenta Informaticae*, 128:65–79, 2013. doi:10.3233/FI-2013-933.

[4] Adam Grabowski. Lattice theory for rough sets – a case study with Mizar. *Fundamenta Informaticae*, 147(2–3):223–240, 2016. doi:10.3233/FI-2016-1406.

[5] Adam Grabowski. Formalization of generalized almost distributive lattices. *Formalized Mathematics*, 22(**3**):257–267, 2014. doi:10.2478/forma-2014-0026.

[6] Adam Grabowski. Basic properties of rough sets and rough membership function. *Formalized Mathematics*, 12(**1**):21–28, 2004.

[7] Adam Grabowski. Relational formal characterization of rough sets. *Formalized Mathematics*, 21(**1**):55–64, 2013. doi:10.2478/forma-2013-0006.

[8] Adam Grabowski. Binary relations-based rough sets – an automated approach. *Formalized Mathematics*, 24(**2**):143–155, 2016. doi:10.1515/forma-2016-0011.

[9] Adam Grabowski and Magdalena Jastrzębska. A note on a formal approach to rough operators. In Marcin S. Szczuka and Marzena Kryszkiewicz et al., editors, *Rough Sets and Current Trends in Computing – 7th International Conference, RSCTC 2010, Warsaw, Poland, June 28-30, 2010. Proceedings*, volume 6086 of *Lecture Notes in Computer Science*, pages 307–316. Springer, 2010. doi:10.1007/978-3-642-13529-3_33.

[10] Adam Grabowski and Magdalena Jastrzębska. Rough set theory from a math-assistant perspective. In *Rough Sets and Intelligent Systems Paradigms, International Conference, RSEISP 2007, Warsaw, Poland, June 28–30, 2007, Proceedings*, pages 152–161, 2007. doi:10.1007/978-3-540-73451-2_17.

[11] Adam Grabowski and Christoph Schwarzweller. On duplication in mathematical repositories. In Serge Autexier, Jacques Calmet, David Delahaye, Patrick D. F. Ion, Laurence Rideau, Renaud Rioboo, and Alan P. Sexton, editors, *Intelligent Computer Mathematics, 10th International Conference, AISC 2010, 17th Symposium, Calculemus 2010, and 9th International Conference, MKM 2010, Paris, France, July 5–10, 2010. Proceedings*, volume 6167 of *Lecture Notes in Computer Science*, pages 300–314. Springer, 2010. doi:10.1007/978-3-642-14128-7_26.

[12] Adam Grabowski and Christoph Schwarzweller. Rough Concept Analysis - theory development in the Mizar system. In Asperti, Andrea and Bancerek, Grzegorz and Trybulec, Andrzej, editor, *Mathematical Knowledge Management, Third International Conference, MKM 2004, Bialowieza, Poland, September 19–21, 2004, Proceedings*, volume 3119 of *Lecture Notes in Computer Science*, pages 130–144, 2004. doi:10.1007/978-3-540-27818-4_10. 3rd International Conference on Mathematical Knowledge Management, Bialowieza, Poland, Sep. 19-21, 2004.

[13] Jouni Järvinen. Lattice theory for rough sets. *Transactions of Rough Sets, VI, Lecture Notes in Computer Science*, 4374:400–498, 2007.

[14] Eliza Niewiadomska and Adam Grabowski. Introduction to formal preference spaces. *Formalized Mathematics*, 21(**3**):223–233, 2013. doi:10.2478/forma-2013-0024.

[15] Zdzisław Pawlak. Rough sets. *International Journal of Parallel Programming*, 11:341–356, 1982. doi:10.1007/BF01001956.

[16] Y.Y. Yao. Two views of the theory of rough sets in finite universes. *International Journal of Approximate Reasoning*, 15(4):291–317, 1996. doi:10.1016/S0888-613X(96)00071-0.

[17] William Zhu. Generalized rough sets based on relations. *Information Sciences*, 177: 4997–5011, 2007.

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