

# Sequences of Prime Reciprocals. Preliminaries

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**Summary.** In the article we formalize some properties needed to prove that sequences of prime reciprocals are divergent. The aim is to show that the series exhibits log-log growth. We introduce some auxiliary notions as harmonic numbers, telescoping series, and prove some standard properties of logarithms and exponents absent in the Mizar Mathematical Library. At the end we proceed with square-free and square-containing parts of a natural number and reciprocals of corresponding products.

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## 0. INTRODUCTION

The aim of this article is to provide preliminaries needed to prove that sequences of reciprocals of prime numbers are divergent. One of the proofs (which we follow) relies on the unique decomposition of natural numbers into the square and its square-free part (similarly to Euler's 1737 original proof [4]). Essentially, it is the proof that the series exhibits log-log growth.

We start with preliminary lemmas, mainly on integrals. Section 2 introduces the notion of  $n$ -th harmonic number,

$$H_n = \sum_{i=0}^n \frac{1}{i},$$

as well as its basic properties.

We proved main steps of the proof that prime harmonic series diverges:

- the lower estimate  $1 + x < \exp(x)$  for the exponential function (13), which holds for all  $x > 0$ ,
- $\int_n^{n+1} \frac{1}{x} dx < \frac{1}{n}$  (21),
- $\ln(n+1) < H_n$  (22),
- the formula for telescoping sum (24).

Although the solution of the Basel problem states [11] that

$$\sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6},$$

we proved rough upper bound, using a telescoping sum, namely, for  $n \geq 2$ ,

$$\sum_{k=1}^n \frac{1}{k^2} < \frac{5}{3}.$$

Evidently however, as in the Mizar Mathematical Library there are at least four strategies for counting the (finite) sum (and the product, similarly): to treat counted objects as

- elements of certain finite subsets;
- values of finite sequences (which is probably most frequent in MML [1], but needs recounting after concatenation or deletion of elements);
- values of the type `Real_Sequence` which are functions from  $\mathbb{N}$  into  $\mathbb{R}$  (this could be generalized into partial functions and is obviously more general than the previous one);
- bags – objects quite well developed during formalization of polynomials, in terms of bags the fundamental theorem of arithmetic is expressed in Mizar;

there is a need to propose unified approach which will be suggested to use, even if the differences between all of them are purely technical. This probably needs some revisions [6] resulting in removing existing duplications [5] of the Mizar repository. Section 6 discusses some of the details, and also introduces a Mizar functor generating bag from a given finite subset of primes. Additionally, we define two sequences of reciprocals needed for proper summing (and multiplying) later on.

The formula we still definitely need to show is

$$\sum_{i=1}^n \frac{1}{i} \leq \prod_{p \leq n} \left(1 + \frac{1}{p}\right) \sum_{k=1}^n \frac{1}{k^2}.$$

The product in the formula above corresponds to the square-free part of  $i$  and the sum corresponds to the square part of  $i$ .

In order to simplify the operations on  $\mathbb{P}_n$  (introduced in Section 4 as the set of all primes less or equal  $n$  for arbitrary  $n \in \mathbb{N}$ ), we conclude our article with selected properties of reciprocals of products of sets of prime numbers.

1. PRELIMINARIES

From now on  $n, i, k, m$  denote natural numbers and  $p$  denotes a prime number.

One can check that there exists a natural number which is non zero, square, and non trivial.

Let  $Z$  be a subset of  $\mathbb{R}$ ,  $f$  be a partial function from  $Z$  to  $\mathbb{R}$ , and  $A$  be a subset of  $\mathbb{R}$ . Let us note that  $f \upharpoonright A$  is  $A$ -defined as a partial function from  $Z$  to  $\mathbb{R}$ .

Let us consider a subset  $Z$  of  $\mathbb{R}$ . Now we state the propositions:

- (1) If  $0 \in Z$ , then  $(\text{id}_Z)^{-1}(\{0\}) = \{0\}$ .
- (2) If  $0 \notin Z$ , then  $(\text{id}_Z)^{-1}(\{0\}) = \emptyset$ .

PROOF:  $(\text{id}_Z)^{-1}(\{0\}) \subseteq \emptyset$ .  $\square$

Let us consider an open subset  $Z$  of  $\mathbb{R}$  and a non empty, closed interval subset  $A$  of  $\mathbb{R}$ . Now we state the propositions:

- (3) If  $0 \notin Z$  and  $A \subseteq Z$ , then  $\frac{1}{\text{id}_Z} \upharpoonright A$  is continuous.
- (4) Suppose  $Z = ]0, +\infty[$  and  $A = [1, n+1]$ . Then  $\int_A \frac{1}{\text{id}_Z}(x)dx = (\text{the function } \ln)(n+1)$ . The theorem is a consequence of (2) and (3).
- (5) Suppose  $Z = ]0, +\infty[$  and  $0 < n$  and  $A = [n, n+1]$ . Then  $\int_A \frac{1}{\text{id}_Z}(x)dx = (\text{the function } \ln)(\frac{n+1}{n})$ . The theorem is a consequence of (2) and (3).
- (6) Let us consider real numbers  $x, r$ . Suppose  $x > 0$  and  $r > 0$ . Then Maclaurin(the function  $\exp, ]-r, r[, x$ ) is positive yielding.

PROOF: Set  $f = \text{Maclaurin}(\text{the function } \exp, ]-r, r[, x)$  by [10, (8)]. For every real number  $r$  such that  $r \in \text{rng } f$  holds  $0 < r$ .  $\square$

- (7) Let us consider a summable sequence  $f$  of real numbers, and a natural number  $n$ . If  $f$  is positive yielding, then  $\sum(f \upharpoonright (n+1)) > 0$ .

PROOF: Set  $L = f \upharpoonright (n+1)$ . For every natural number  $i, 0 \leq L(i)$ . There exists a natural number  $i$  such that  $i \in \text{dom } L$  and  $0 < L(i)$ . Consider  $k$  being a natural number such that  $k \in \text{dom } L$  and  $L(k) > 0$ .  $\square$

## 2. HARMONIC NUMBERS

Let  $n$  be a natural number. The functor  $H_n$  yielding a real number is defined by the term

$$\text{(Def. 1)} \quad \left(\sum_{\alpha=0}^{\kappa} (\text{inv}_{\mathbb{N}})(\alpha)\right)_{\kappa \in \mathbb{N}}(n).$$

Now we state the propositions:

$$(8) \quad H_0 = 0.$$

$$(9) \quad H_{n+1} = H_n + \frac{1}{n+1}.$$

$$(10) \quad H_1 = 1. \text{ The theorem is a consequence of (9) and (8).}$$

$$(11) \quad H_2 = \frac{3}{2}. \text{ The theorem is a consequence of (9) and (10).}$$

## 3. ON EXPONENTS AND LOGARITHMS

Now we state the proposition:

$$(12) \quad (\text{The function } \ln)(1) = 0.$$

Let us consider a real number  $x$ . Now we state the propositions:

$$(13) \quad \text{If } x > 0, \text{ then } (\text{the function } \exp)(x) > x + 1. \text{ The theorem is a consequence of (6) and (7).}$$

$$(14) \quad \text{If } x > 0, \text{ then } (\text{the function } \ln)(x + 1) < x.$$

$$(15) \quad \text{Let us consider a natural number } n. \text{ If } n > 0, \text{ then } (\text{the function } \ln)\left(\frac{n+1}{n}\right) < \frac{1}{n}. \text{ The theorem is a consequence of (14).}$$

$$(16) \quad \text{Let us consider a real number } x. \text{ Then } (\text{the function } \ln)((\text{the function } \exp)(x)) = x.$$

$$(17) \quad \text{Let us consider real numbers } x, y. \text{ Suppose } 0 < x < y. \text{ Then } (\text{the function } \ln)(x) < (\text{the function } \ln)(y).$$

$$(18) \quad \text{Let us consider a non zero natural number } n. \text{ Then } (\text{the function } \ln)(n + 1) > 0. \text{ The theorem is a consequence of (12) and (17).}$$

$$(19) \quad \text{Let us consider real numbers } x, y. \text{ Suppose } 0 < x \text{ and } 0 < y. \text{ Then } (\text{the function } \ln)(x \cdot y) = (\text{the function } \ln)(x) + (\text{the function } \ln)(y).$$

$$(20) \quad \text{Let us consider a real number } x. \text{ Then there exists a non zero natural number } y \text{ such that } x < (\text{the function } \ln)((\text{the function } \ln)(y + 1)). \text{ The theorem is a consequence of (17) and (16).}$$

$$(21) \quad \text{Let us consider a non empty, closed interval subset } A \text{ of } \mathbb{R}, \text{ an open subset } Z \text{ of } \mathbb{R}, \text{ and a non zero natural number } n. \text{ Suppose } Z = ]0, +\infty[ \text{ and } A = [n, n + 1]. \text{ Then } \int_A \frac{1}{\text{id}_Z}(x) dx < \frac{1}{n}. \text{ The theorem is a consequence of (2), (3), and (15).}$$

(22) Let us consider a non zero natural number  $n$ . Then (the function  $\ln$ )( $n + 1$ )  $< H_n$ .

PROOF: Set  $A = [1, n + 1]$ . Reconsider  $Z = ]0, +\infty[$  as an open subset of  $\mathbb{R}$ .  $A \subseteq Z$ .  $\frac{1}{\text{id}_Z} \upharpoonright A$  is continuous. Set  $g = \frac{1}{\text{id}_Z}$ . Define  $\mathcal{P}[\text{natural number}] \equiv$

$\int_1^{\$1+1} g(x)dx < H_{\$1}$ .  $\int_{A_1} g(x)dx < \frac{1}{1}$ .  $\mathcal{P}[1]$ . For every non zero natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$  by [2, (11), (10)], [3, (17)]. For every non zero natural number  $n$ ,  $\mathcal{P}[n]$ .  $\square$

(23) Let us consider natural numbers  $n_1, n_2$ . If  $n_1^2 = n_2^2$ , then  $n_1 = n_2$ .

Let  $n$  be a non trivial natural number. Let us note that  $n^2$  is non trivial.

(24) TELESCOPING SERIES:

Let us consider sequences  $a, b, s$  of real numbers. Suppose for every natural number  $n$ ,  $s(n) = a(n) + b(n)$  and for every natural number  $k$ ,  $b(k) = -a(k + 1)$ . Let us consider a natural number  $n$ . Then  $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n) = a(0) + b(n)$ .

PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\$1) = a(0) + b(\$1)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$ . For every natural number  $n$ ,  $\mathcal{P}[n]$ .  $\square$

(25) Let us consider sequences  $f_1, f_2$  of real numbers, and a non trivial natural number  $n$ . Suppose for every non trivial natural number  $k$  such that  $k \leq n$  holds  $f_1(k) < f_2(k)$ . Then  $\sum_{\kappa=1+1}^n f_1(\kappa) < \sum_{\kappa=1+1}^n f_2(\kappa)$ .

PROOF: Define  $\mathcal{X}[\text{natural number}] \equiv$  if for every non trivial natural number  $k$  such that  $k \leq \$1$  holds  $f_1(k) < f_2(k)$ , then  $(\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}}(\$1) - (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}}(1) < (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}(\$1) - (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}(1)$ . For every non trivial natural number  $n$  such that  $\mathcal{X}[n]$  holds  $\mathcal{X}[n + 1]$ . For every non trivial natural number  $n$ ,  $\mathcal{X}[n]$ .  $\square$

#### 4. SOME SPECIAL SEQUENCES

The functor Rec1-seq1 yielding a sequence of real numbers is defined by

(Def. 2) for every natural number  $n$ ,  $it(n) = \frac{1}{n^2 - \frac{1}{4}}$ .

Now we state the propositions:

(26) Let us consider a natural number  $n$ . Then  $(\text{Rec1-seq1})(n) = \frac{1}{n - \frac{1}{2}} - \frac{1}{n + \frac{1}{2}}$ .

(27)  $\text{Rec1-seq1} = \text{rseq}(0, 1, 1, -\frac{1}{2}) + -\text{rseq}(0, 1, 1, \frac{1}{2})$ .

Let us consider a natural number  $n$ .

(28)  $(\sum_{\alpha=0}^{\kappa} (\text{Rec1-seq1})(\alpha))_{\kappa \in \mathbb{N}}(n) < -2$ . The theorem is a consequence of (24).

(29)  $\sum_{\kappa=1+1}^n \text{Reci-seq1}(\kappa) < \frac{2}{3}$ . The theorem is a consequence of (24).

Note that Basel-seq is summable.

(30) Let us consider a natural number  $n$ . Then  $(\sum_{\alpha=0}^{\kappa} (\text{Reci-seq1})(\alpha))_{\kappa \in \mathbb{N}}(n) = -2 + -\frac{1}{n+\frac{1}{2}}$ . The theorem is a consequence of (24).

Let us consider a non trivial natural number  $n$ .

(31)  $\sum_{\kappa=1+1}^n \text{Basel-seq}(\kappa) < \sum_{\kappa=1+1}^n \text{Reci-seq1}(\kappa)$ .

PROOF: For every non trivial natural number  $k$  such that  $k \leq n$  holds  $(\text{Basel-seq})(k) < (\text{Reci-seq1})(k)$  by [9, (29)].  $\square$

(32)  $\sum_{\kappa=0}^n \text{Basel-seq}(\kappa) < \frac{5}{3}$ . The theorem is a consequence of (31) and (29).

(33)  $(\sum_{\alpha=0}^{\kappa} (\text{Basel-seq})(\alpha))_{\kappa \in \mathbb{N}}(n) < \frac{5}{3}$ . The theorem is a consequence of (32).

The functor Rec-seq2 yielding a sequence of real numbers is defined by

(Def. 3) for every natural number  $n$ ,  $it(n) = 1 + \frac{1}{pr(n)}$ .

Now we state the proposition:

(34)  $\sum \text{Sgm}\{1\} = 1$ .

Let  $n$  be a natural number. The functor  $\mathbb{P}_n$  yielding a subset of  $\mathbb{N}$  is defined by the term

(Def. 4)  $\mathbb{P} \cap \text{Seg } n$ .

One can verify that  $\mathbb{P}_n$  is finite.

Now we state the propositions:

(35) Let us consider natural numbers  $m, n$ . If  $m \leq n$ , then  $\mathbb{P}_m \subseteq \mathbb{P}_n$ .

(36) If  $n + 1$  is not a prime number, then  $\mathbb{P}_{n+1} = \mathbb{P}_n$ .

(37) (i)  $\mathbb{P}_0 = \emptyset$ , and

(ii)  $\mathbb{P}_1 = \emptyset$ .

The theorem is a consequence of (36).

(38) If  $n + 1$  is a prime number, then  $\mathbb{P}_{n+1} = \mathbb{P}_n \cup \{n + 1\}$ .

(39) Let us consider a prime number  $p$ . If  $p > 2$ , then  $p + 1$  is not a prime number.

(40)  $\mathbb{P}_2 = \{2\}$ .

(41)  $n + 1 \notin \mathbb{P}_n$ .

Let  $n$  be a natural number. The functor  $\text{indexp}(n)$  yielding a natural number is defined by the term

(Def. 5)  $\overline{\mathbb{P}_n}$ .

Now we state the proposition:

(42) Let us consider a natural number  $n$ . Then  $\text{indexp}(n) \leq n$ .

5. SQUARE-FREE AND SQUARE-CONTAINING PARTS OF A NATURAL NUMBER

Let us consider a non zero natural number  $n$ . Now we state the propositions:

(43)  $n = (\text{TSqF } n) \cdot (n \text{ div TSqF } n)$ .

(44)  $(\text{SqF } n)^2 \mid n$ .

PROOF: Define  $\mathcal{F}$ (non zero natural number) =  $(\prod \text{SqFactors } \$1)^2$ . Define  $\mathcal{G}$ (non zero natural number) =  $\text{SqFactors } \$1$ . Define  $\mathcal{P}$ [natural number]  $\equiv$  for every non zero natural number  $n$  such that  $\text{support } \mathcal{G}(n) \subseteq \text{Seg } \$1$  holds  $\mathcal{F}(n) \mid n$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$ .  $\mathcal{P}[0]$  by [8, (20)]. For every natural number  $k$ ,  $\mathcal{P}[k]$ .  $\square$

(45) Let us consider a finite-support, natural-valued many sorted set  $m$  indexed by  $\mathbb{P}$ , and a prime number  $p$ . If  $\text{support } m = \{p\}$ , then  $\prod m = m(p)$ .

PROOF: Consider  $f$  being a finite sequence of elements of  $\mathbb{C}$  such that  $\prod m = \prod f$  and  $f = m \cdot \text{CFS}(\text{support } m)$ .  $m \cdot \langle p \rangle = \langle m(p) \rangle$ .  $\square$

(46) Let us consider a non zero natural number  $n$ . Then  $(\text{SqF } n)^2 = \text{TSqF } n$ .

PROOF: Define  $\mathcal{F}$ (non zero natural number) =  $(\prod \text{SqFactors } \$1)^2$ . Define  $\mathcal{G}$ (non zero natural number) =  $\text{SqFactors } \$1$ . Define  $\mathcal{H}$ (non zero natural number) =  $\prod \text{TSqFactors } \$1$ . Define  $\mathcal{P}$ [natural number]  $\equiv$  for every non zero natural number  $n$  such that  $\text{support } \mathcal{G}(n) \subseteq \text{Seg } \$1$  holds  $\mathcal{F}(n) = \mathcal{H}(n)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$ .  $\mathcal{P}[0]$ . For every natural number  $k$ ,  $\mathcal{P}[k]$ .  $\square$

Let  $n$  be a non zero natural number. Note that  $n \text{ div } (\text{SqF } n)^2$  is square-free as a natural number.

The functor  $\text{SquarefreePart}(n)$  yielding a non zero natural number is defined by the term

(Def. 6)  $n \text{ div TSqF } n$ .

Let us observe that  $\text{SquarefreePart}(n)$  is square-free.

Let us consider a non zero natural number  $n$ . Now we state the propositions:

(47)  $n = \text{SquarefreePart}(n) \cdot (\text{SqF } n)^2$ . The theorem is a consequence of (44) and (46).

(48)  $\text{support PFFexp}(n) \subseteq \text{Seg } n$ .

(49)  $\text{support PPF}(n) \subseteq \text{Seg } n$ .

(50)  $\text{Seg SquarefreePart}(n) \subseteq \text{Seg } n$ . The theorem is a consequence of (47).

(51) Let us consider non zero natural numbers  $k, n$ . Then  $k^2 \mid \text{SquarefreePart}(n)$  if and only if  $k = 1$ .

(52) Let us consider non zero natural numbers  $m, n$ . Suppose  $\text{SquarefreePart}(n) = \text{SquarefreePart}(m)$  and  $\text{TSqF } m = \text{TSqF } n$ . Then  $m = n$ . The theorem is a consequence of (47) and (46).

6. GENERATING BAGS FROM SUBSETS OF PRIME NUMBERS

Let  $A$  be a finite subset of  $\mathbb{P}$ . The functor  $A$ -bag yielding a bag of  $\mathbb{P}$  is defined by the term

(Def. 7)  $\text{EmptyBag } \mathbb{P} + \text{id}_A$ .

Let us consider a finite subset  $A$  of  $\mathbb{P}$ . Now we state the propositions:

(53)  $\text{support } A\text{-bag} = A$ .

PROOF: Set  $f = A\text{-bag}$ .  $\text{support } f \subseteq A$ .  $A \subseteq \text{support } f$ .  $\square$

(54) If  $A = \emptyset$ , then  $A\text{-bag} = \text{EmptyBag } \mathbb{P}$ . The theorem is a consequence of (53).

(55) Let us consider a finite subset  $A$  of  $\mathbb{P}$ , and an object  $i$ . If  $i \in \text{support } A\text{-bag}$ , then  $(A\text{-bag})(i) = i$ . The theorem is a consequence of (53).

(56) Let us consider finite subsets  $A, B$  of  $\mathbb{P}$ . If  $A\text{-bag} = B\text{-bag}$ , then  $A = B$ . The theorem is a consequence of (53).

Let  $A$  be a finite subset of  $\mathbb{P}$ . Let us observe that  $A$ -bag is prime-factorization-like and  $\prod A\text{-bag}$  is square-free as a natural number.

Let us consider a non zero natural number  $n$  and an object  $x$ . Now we state the propositions:

(57) If  $x \in 2^{\mathbb{P}^n}$ , then  $x$  is a finite subset of  $\mathbb{P}$ .

(58) If  $x \in 2^{\mathbb{P}^n} \times \text{Seg } n$ , then  $(x)_1$  is a finite subset of  $\mathbb{P}$ .

(59)  $\text{rseq}(0, 1, 1, 0) = \text{inv}_{\mathbb{N}}$ .

(60)  $\text{indexp}(0) = 0$ .

Let us consider a natural number  $n$ .

(61) (The partial product of  $\text{Reci-seq2}$ )( $n$ )  $> 0$ .

(62) (The function  $\text{ln}$ )((the partial product of  $\text{Reci-seq2}$ )( $n$ ))  $\leq (\sum_{\alpha=0}^{\kappa} (\text{inv}_{\mathbb{P}})(\alpha))_{\kappa \in \mathbb{N}}(n)$ .

PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv (\text{the function } \text{ln})((\text{the partial product of } \text{Reci-seq2})(\$1)) \leq (\sum_{\alpha=0}^{\kappa} (\text{inv}_{\mathbb{P}})(\alpha))_{\kappa \in \mathbb{N}}(\$1)$ .  $\mathcal{P}[0]$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$ . For every natural number  $n$ ,  $\mathcal{P}[n]$ .  $\square$

(63) (The function  $\text{ln}$ )((the partial product of  $\text{Reci-seq2}$ )( $\text{indexp}(n)$ ))  $\leq (\sum_{\alpha=0}^{\kappa} (\text{inv}_{\mathbb{P}})(\alpha))_{\kappa \in \mathbb{N}}(n)$ . The theorem is a consequence of (62) and (42).

The functors:  $\text{Reci-Sqf}$  and  $\text{Reci-TSq}$  yielding sequences of real numbers are defined by conditions

(Def. 8)  $\text{Reci-Sqf}(0) = 0$  and for every non zero natural number  $i$ ,  $\text{Reci-Sqf}(i) = \frac{1}{\text{SquarefreePart}(i)}$ ,



(Def. 9)  $\text{Reci-TSq}(0) = 0$  and for every non zero natural number  $i$ ,  $\text{Reci-TSq}(i) = \frac{1}{\text{TSqF}i}$ ,

respectively. Now we state the proposition:

$$(64) \quad \text{rseq}(0, 1, 1, 0) = \text{Reci-Sqf} \cdot \text{Reci-TSq}.$$

From now on  $s, s_1, s_2$  denote sequences of real numbers.

Let us consider a natural number  $n$ .

$$(65) \quad (\text{Reci-Sqf})(n) \geq 0.$$

$$(66) \quad (\text{Reci-TSq})(n) \geq 0.$$

$$(67) \quad (\text{Basel-seq})(n) \geq 0.$$

$$(68) \quad \left( \sum_{\alpha=0}^{\kappa} (\text{rseq}(0, 1, 1, 0))(\alpha) \right)_{\kappa \in \mathbb{N}}(n) \leq \left( \sum_{\alpha=0}^{\kappa} (\text{Reci-Sqf})(\alpha) \right)_{\kappa \in \mathbb{N}}(n) \cdot \left( \sum_{\alpha=0}^{\kappa} (\text{Reci-TSq})(\alpha) \right)_{\kappa \in \mathbb{N}}(n).$$

Let  $n$  be a non zero natural number. The functor  $\text{Compose}(n)$  yielding a function from  $2^{\mathbb{P}^n} \times \text{Seg } n$  into  $\mathbb{N}$  is defined by

(Def. 10) for every element  $x$  of  $2^{\mathbb{P}^n} \times \text{Seg } n$  and for every finite subset  $A$  of  $\mathbb{P}$  and for every natural number  $k$  such that  $x = \langle A, k \rangle$  holds  $it(x) = \prod(A, 1)$ -bag  $\cdot k^2$ .

Now we state the proposition:

$$(69) \quad \left( \sum_{\alpha=0}^{\kappa} (\text{Basel-seq})(\alpha) \right)_{\kappa \in \mathbb{N}}(n) \geq 0.$$

PROOF: For every natural number  $n$ ,  $(\text{Basel-seq})(n) \geq 0$  by [7, (31)].  $\square$

## 7. ON RECIPROALS OF PRODUCTS OF PRIME NUMBERS

Let  $n$  be a natural number. The functor  $\text{ReciProducts}(n)$  yielding a subset of  $\mathbb{R}$  is defined by the term

(Def. 11) the set of all  $\frac{1}{\prod \text{Sgm } X}$  where  $X$  is a subset of  $\mathbb{P}_n$ .

Let us note that  $\text{ReciProducts}(n)$  is finite.

Now we state the propositions:

$$(70) \quad \text{ReciProducts}(0) = \{1\}.$$

PROOF: the set of all  $\frac{1}{\prod \text{Sgm } X}$  where  $X$  is a subset of  $\mathbb{P}_0 = \{1\}$ .  $\square$

(71) Let us consider a prime number  $p$ . If  $p > 2$ , then  $\text{ReciProducts}(p+1) = \text{ReciProducts}(p)$ . The theorem is a consequence of (39) and (36).

(72) Let us consider a natural number  $p$ . Suppose  $p+1$  is not a prime number. Then  $\text{ReciProducts}(p+1) = \text{ReciProducts}(p)$ . The theorem is a consequence of (36).

$$(73) \quad \text{ReciProducts}(1) = \{1\}.$$

PROOF: the set of all  $\frac{1}{\prod \text{Sgm } X}$  where  $X$  is a subset of  $\mathbb{P}_1 = \{1\}$ .  $\square$

$$(74) \text{ ReciProducts}(2) = \{\frac{1}{2}, 1\}.$$

PROOF:  $\{2\} \subseteq \mathbb{P}_2$ .  $\text{ReciProducts}(2) \subseteq \{\frac{1}{2}, 1\}$ .  $\square$

Let us consider a natural number  $n$ .

$$(75) \text{ ReciProducts}(n) \subseteq \text{ReciProducts}(n+1).$$

$$(76) \text{ Suppose } n+1 \text{ is a prime number. Then } \text{ReciProducts}(n+1) = \text{ReciProducts}(n) \cup \left\{ \frac{1}{\prod_{\text{Sgm } X}} \right\}, \text{ where } X \text{ is a subset of } \mathbb{P}_{n+1} : n+1 \in X \}.$$

PROOF:  $\text{ReciProducts}(n+1) \subseteq \text{ReciProducts}(n) \cup \left\{ \frac{1}{\prod_{\text{Sgm } X}} \right\}$ , where  $X$  is a subset of  $\mathbb{P}_{n+1} : n+1 \in X$ .  $\text{ReciProducts}(n) \cup \left\{ \frac{1}{\prod_{\text{Sgm } X}} \right\}$ , where  $X$  is a subset of  $\mathbb{P}_{n+1} : n+1 \in X$   $\subseteq \text{ReciProducts}(n+1)$ .  $\square$

$$(77) \text{ Suppose } n+1 \text{ is a prime number. Then } \text{ReciProducts}(n+1) = \left\{ \frac{1}{\prod_{\text{Sgm } X}} \right\}, \text{ where } X \text{ is a subset of } \mathbb{P}_n : n+1 \notin X \} \cup \left\{ \frac{1}{\prod_{\text{Sgm } X}} \right\}, \text{ where } X \text{ is a subset of } \mathbb{P}_{n+1} : n+1 \in X \}.$$

PROOF:  $\text{ReciProducts}(n+1) \subseteq \left\{ \frac{1}{\prod_{\text{Sgm } X}} \right\}$ , where  $X$  is a subset of  $\mathbb{P}_n : n+1 \notin X$   $\cup \left\{ \frac{1}{\prod_{\text{Sgm } X}} \right\}$ , where  $X$  is a subset of  $\mathbb{P}_{n+1} : n+1 \in X$ .  $\left\{ \frac{1}{\prod_{\text{Sgm } X}} \right\}$ , where  $X$  is a subset of  $\mathbb{P}_n : n+1 \notin X$   $\cup \left\{ \frac{1}{\prod_{\text{Sgm } X}} \right\}$ , where  $X$  is a subset of  $\mathbb{P}_{n+1} : n+1 \in X$   $\subseteq \text{ReciProducts}(n+1)$ .  $\square$

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