FORMALIZED MATHEMATICS
Vol. 26, No. 1, Pages 49–67, 2018
DOI: 10.2478/forma-2018-0005

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Fubini's Theorem for Non-Negative or Non-Positive Functions

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Summary. The goal of this article is to show Fubini's theorem for non-negative or non-positive measurable functions [10], [2], [3], using the Mizar system [1], [9]. We formalized Fubini's theorem in our previous article [5], but in that case we showed the Fubini's theorem for measurable sets and it was not enough as the integral does not appear explicitly.

On the other hand, the theorems obtained in this paper are more general and it can be easily extended to a general integrable function. Furthermore, it also can be easy to extend to functional space L^p [12]. It should be mentioned also that Hölzl and Heller [11] have introduced the Lebesgue integration theory in Isabelle/HOL and have proved Fubini's theorem there.

MSC: 28A35 03B35

Keywords: Fubini's theorem; extended real-valued non-negative (or non-positive) measurable function

MML identifier: `MESFUN12`, version: `8.1.07 5.47.1318`

1. Extended Real-Valued Characteristic Function

Let A, X be sets and e be an extended real. The functor $\chi_{e,A,X}$ yielding a function from X into $\overline{\mathbb{R}}$ is defined by

(Def. 1) for every object x such that $x \in X$ holds if $x \in A$, then $it(x) = e$ and if $x \notin A$, then $it(x) = 0$.

Now we state the propositions:

(1) Let us consider a non empty set X, a set A, and a real number r. Then $r \cdot \chi_{A,X} = \chi_{r,A,X}$.

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ISSN 1426–2630(Print), 1898-9934(Online)

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(2) Let us consider a non empty set X, and a set A. Then

(i) $\chi_{+\infty,A,X} = \overline{\chi}_{A,X}$, and

(ii) $\chi_{-\infty,A,X} = -\overline{\chi}_{A,X}$.

(3) Let us consider sets X, A. Then $\chi_{A,X}$ is without $+\infty$ and without $-\infty$.

(4) Let us consider a non empty set X, a set A, and a real number r. Then

(i) $\operatorname{rng} \chi_{r,A,X} \subseteq \{0, r\}$, and

(ii) $\chi_{r,A,X}$ is without $+\infty$ and without $-\infty$.

The theorem is a consequence of (3) and (1).

(5) Let us consider a non empty set X, a σ-field S of subsets of X, a non empty partial function f from X to $\overline{\mathbb{R}}$, and a σ-measure M on S. Suppose f is simple function in S. Then there exists a non empty finite sequence E of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ and there exists a finite sequence r of elements of $\mathbb{R}$ such that E and a are representation of f and for every natural number n, $a(n) = r(n)$ and $f \restriction E(n) = \chi_{r(n),E(n),X} \restriction E(n)$ and if $E(n) = \emptyset$, then $r(n) = 0$.

Proof: Consider E being a finite sequence of separated subsets of S, b being a finite sequence of elements of $\overline{\mathbb{R}}$ such that E and b are representation of f. For every natural number n such that $E(n) \neq \emptyset$ holds $b(n) \in \mathbb{R}$ by [8, (32)]. Define $\mathcal{Q}[\text{natural number}, \text{object}] \equiv$ if $E(\$_1) \neq \emptyset$, then $\$_2 = b(\$_1)$ and if $E(\$_1) = \emptyset$, then $\$_2 = 0$. For every natural number n such that $n \in \operatorname{Seg} \operatorname{len} E$ there exists an element a of $\overline{\mathbb{R}}$ such that $\mathcal{Q}[n, a]$. Consider a being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} a = \operatorname{Seg} \operatorname{len} E$ and for every natural number n such that $n \in \operatorname{Seg} \operatorname{len} E$ holds $\mathcal{Q}[n, a(n)]$. Define $\mathcal{R}[\text{natural number}, \text{object}] \equiv \$_2 = a(\$_1)$. For every natural number n such that $n \in \operatorname{Seg} \operatorname{len} E$ there exists an element r of $\mathbb{R}$ such that $\mathcal{R}[n, r]$. Consider r being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} r = \operatorname{Seg} \operatorname{len} E$ and for every natural number n such that $n \in \operatorname{Seg} \operatorname{len} E$ holds $\mathcal{R}[n, r(n)]$. For every natural number n such that $n \in \operatorname{dom} E$ for every object x such that $x \in E(n)$ holds $f(x) = a(n)$. For every natural number n, $a(n) = r(n)$ and $f \restriction E(n) = \chi_{r(n),E(n),X} \restriction E(n)$ and if $E(n) = \emptyset$, then $r(n) = 0$. $\square$

Let F be a finite sequence-like function. Let us observe that F is disjoint valued if and only if the condition (Def. 2) is satisfied.

(Def. 2) for every natural numbers m, n such that $m, n \in \operatorname{dom} F$ and $m \neq n$ holds $F(m)$ misses $F(n)$.

Now we state the propositions:

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(6) Let us consider a non empty set X, a σ-field S of subsets of X, and elements E_1, E_2 of S. Suppose E_1 misses E_2. Then $\langle E_1, E_2 \rangle$ is a finite sequence of separated subsets of S.

(7) Let us consider a non empty set X, subsets A_1, A_2 of X, and real numbers r_1, r_2. Then $\langle \chi_{r_1,A_1,X}, \chi_{r_2,A_2,X} \rangle$ is a summable finite sequence of elements of $\overline{\mathbb{R}}^X$. The theorem is a consequence of (4).

(8) Let us consider a non empty set X, and a summable finite sequence F of elements of $\overline{\mathbb{R}}^X$. Suppose $\operatorname{len} F \geqslant 2$. Then $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/2} = F_{/1} + F_{/2}$.

(9) Let us consider a non empty set X, and a function f from X into $\overline{\mathbb{R}}$. Then $f + (X \longmapsto 0_{\overline{\mathbb{R}}}) = f$.

(10) Let us consider a non empty set X, and a summable finite sequence F of elements of $\overline{\mathbb{R}}^X$. Then

  (i) $\operatorname{dom} F = \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$, and

  (ii) for every natural number n such that $n \in \operatorname{dom} F$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n} = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$, and

  (iii) for every natural number n and for every element x of X such that $1 \leqslant n < \operatorname{len} F$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n+1}(x) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n}(x) + F_{/n+1}(x)$.

Proof: For every natural number n and for every element x of X such that $1 \leqslant n < \operatorname{len} F$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n+1}(x) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n}(x) + F_{/n+1}(x)$. □

(11) Let us consider a non empty set X, a σ-field S of subsets of X, a function f from X into $\overline{\mathbb{R}}$, a finite sequence E of separated subsets of S, and a summable finite sequence F of elements of $\overline{\mathbb{R}}^X$. Suppose $\operatorname{dom} E = \operatorname{dom} F$ and $\operatorname{dom} f = \bigcup \operatorname{rng} E$ and for every natural number n such that $n \in \operatorname{dom} F$ there exists a real number r such that $F_{/n} = r \cdot \chi_{E(n),X}$ and $f = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/\operatorname{len} F}$. Then

  (i) for every element x of X and for every natural numbers m, n such that $m, n \in \operatorname{dom} F$ and $x \in E(m)$ and $m \neq n$ holds $F_{/n}(x) = 0$, and

  (ii) for every element x of X and for every natural numbers m, n such that $m, n \in \operatorname{dom} F$ and $x \in E(m)$ and $n < m$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n}(x) = 0$, and

  (iii) for every element x of X and for every natural numbers m, n such that $m, n \in \operatorname{dom} F$ and $x \in E(m)$ and $n \geqslant m$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n}(x) = f(x)$, and

  (iv) for every element x of X and for every natural number m such that $m \in \operatorname{dom} F$ and $x \in E(m)$ holds $F_{/m}(x) = f(x)$, and

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(v) f is simple function in S.

Proof: For every element x of X and for every natural numbers m, n such that m, $n \in \operatorname{dom} F$ and $x \in E(m)$ and $m \neq n$ holds $F_{/n}(x) = 0$. For every element x of X and for every natural numbers m, n such that m, $n \in \operatorname{dom} F$ and $x \in E(m)$ and $n < m$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n}(x) = 0$. For every element x of X and for every natural numbers m, n such that m, $n \in \operatorname{dom} F$ and $x \in E(m)$ and $n \geqslant m$ holds $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/n}(x) = f(x)$. For every element x of X and for every natural number m such that $m \in \operatorname{dom} F$ and $x \in E(m)$ holds $F_{/m}(x) = f(x)$. For every element x of X such that $x \in \operatorname{dom} f$ holds $|f(x)| < +\infty$ by [7, (41)]. For every natural number n and for every elements x, y of X such that $n \in \operatorname{dom} E$ and x, $y \in E(n)$ holds $f(x) = f(y)$. □

(12) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and an element E of S. Then $\chi_{E,X}$ is simple function in S.

Proof: Reconsider $E_2 = X \setminus E$ as an element of S. Reconsider $E_3 = \langle E, E_2 \rangle$ as a finite sequence of separated subsets of S. $1 \cdot \chi_{E,X} = \chi_{1,E,X}$ and $0 \cdot \chi_{E_2,X} = \chi_{0,E_2,X}$. Reconsider $F = \langle 1 \cdot \chi_{E,X}, 0 \cdot \chi_{E_2,X} \rangle$ as a summable finite sequence of elements of $\overline{\mathbb{R}}^X$. For every natural number n such that $n \in \operatorname{dom} F$ there exists a real number r such that $F_{/n} = r \cdot \chi_{E_3(n),X}$. $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/\operatorname{len} F} = F_{/1} + F_{/2}$. $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})_{/\operatorname{len} F} = \chi_{E,X}$. □

(13) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, elements A, B of S, and an extended real e. Then $\chi_{e,A,X}$ is measurable on B. The theorem is a consequence of (2) and (1).

(14) Let us consider a set X, subsets A_1, A_2 of X, and an extended real e. Then $\chi_{e,A_1,X} \restriction A_2 = \chi_{e,A_1 \cap A_2,X} \restriction A_2$.

(15) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, elements A, B, C of S, and an extended real e. If $C \subseteq B$, then $\chi_{e,A,X} \restriction B$ is measurable on C. The theorem is a consequence of (13).

(16) Let us consider a set X, subsets A_1, A_2 of X, an extended real e, and an object x. If A_1 misses A_2, then $(\chi_{e,A_1,X} \restriction A_2)(x) = 0$.

(17) Let us consider a set X, a subset A of X, and an extended real e. Then

(i) if $e \geqslant 0$, then $\chi_{e,A,X}$ is non-negative, and

(ii) if $e \leqslant 0$, then $\chi_{e,A,X}$ is non-positive.

(18) Let us consider sets A, X, and a subset B of X. Then $\operatorname{dom}(\chi_{A,X} \restriction B) = B$.

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2. Some Properties of Integration

Now we state the propositions:

(19) Let us consider a non empty set X, a σ-field S of subsets of X, and a partial function f from X to $\overline{\mathbb{R}}$. If f is empty, then f is simple function in S.

Proof: Reconsider $E_4 = \emptyset$ as an element of S. Reconsider $F = \langle E_4 \rangle$ as a finite sequence of separated subsets of S. For every natural number n and for every elements x, y of X such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $f(x) = f(y)$. $\square$

(20) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and elements E_1, E_2 of S. Then $\int \chi_{E_1,X} \restriction E_2 \, \mathrm{d}M = M(E_1 \cap E_2)$.

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, elements E_1, E_2 of S, and partial functions f, g from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(21) Suppose $E_1 = \operatorname{dom} f$ and f is non-negative and f is measurable on E_1 and $E_2 = \operatorname{dom} g$ and g is non-negative and g is measurable on E_2. Then $\int f + g \, \mathrm{d}M = \int f \restriction \operatorname{dom}(f+g) \, \mathrm{d}M + \int g \restriction \operatorname{dom}(f+g) \, \mathrm{d}M$.

(22) Suppose $E_1 = \operatorname{dom} f$ and f is non-positive and f is measurable on E_1 and $E_2 = \operatorname{dom} g$ and g is non-positive and g is measurable on E_2. Then $\int f + g \, \mathrm{d}M = \int f \restriction \operatorname{dom}(f+g) \, \mathrm{d}M + \int g \restriction \operatorname{dom}(f+g) \, \mathrm{d}M$. The theorem is a consequence of (21).

(23) Suppose $E_1 = \operatorname{dom} f$ and f is non-negative and f is measurable on E_1 and $E_2 = \operatorname{dom} g$ and g is non-positive and g is measurable on E_2. Then

(i) $\int f - g \, \mathrm{d}M = \int f \restriction \operatorname{dom}(f-g) \, \mathrm{d}M - \int g \restriction \operatorname{dom}(f-g) \, \mathrm{d}M$, and

(ii) $\int g - f \, \mathrm{d}M = \int g \restriction \operatorname{dom}(g-f) \, \mathrm{d}M - \int f \restriction \operatorname{dom}(g-f) \, \mathrm{d}M$.

The theorem is a consequence of (21).

(24) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, a partial function f from X to $\overline{\mathbb{R}}$, and a real number r. Suppose $E = \operatorname{dom} f$ and f is non-positive or non-negative and f is measurable on E. Then $\int r \cdot f \, \mathrm{d}M = r \cdot \int f \, \mathrm{d}M$.

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3. Sections of Partial Function

Now we state the proposition:

(25) Let us consider non empty sets X, Y, an element A of $2^{X \times Y}$, and sets x, y. Suppose $x \in X$ and $y \in Y$. Then

(i) $\langle x, y \rangle \in A$ iff $x \in \mathrm{Ysection}(A, y)$, and

(ii) $\langle x, y \rangle \in A$ iff $y \in \mathrm{Xsection}(A, x)$.

Let X_1, X_2 be non empty sets, Y be a set, f be a partial function from $X_1 \times X_2$ to Y, and x be an element of X_1. The functor $\mathrm{ProjPMap1}(f, x)$ yielding a partial function from X_2 to Y is defined by

(Def. 3) $\mathrm{dom}\, it = \mathrm{Xsection}(\mathrm{dom}\, f, x)$ and for every element y of X_2 such that $\langle x, y \rangle \in \mathrm{dom}\, f$ holds $it(y) = f(x, y)$.

Let y be an element of X_2. The functor $\mathrm{ProjPMap2}(f, y)$ yielding a partial function from X_1 to Y is defined by

(Def. 4) $\mathrm{dom}\, it = \mathrm{Ysection}(\mathrm{dom}\, f, y)$ and for every element x of X_1 such that $\langle x, y \rangle \in \mathrm{dom}\, f$ holds $it(x) = f(x, y)$.

Now we state the propositions:

(26) Let us consider non empty sets X_1, X_2, a set Y, a partial function f from $X_1 \times X_2$ to Y, an element x of X_1, and an element y of X_2. Then

(i) if $x \in \mathrm{dom}\, \mathrm{ProjPMap2}(f, y)$, then $(\mathrm{ProjPMap2}(f, y))(x) = f(x, y)$, and

(ii) if $y \in \mathrm{dom}\, \mathrm{ProjPMap1}(f, x)$, then $(\mathrm{ProjPMap1}(f, x))(y) = f(x, y)$.

(27) Let us consider non empty sets X_1, X_2, Y, a function f from $X_1 \times X_2$ into Y, an element x of X_1, and an element y of X_2. Then

(i) $\mathrm{ProjPMap1}(f, x) = \mathrm{curry}(f, x)$, and

(ii) $\mathrm{ProjPMap2}(f, y) = \mathrm{curry}'(f, y)$.

The theorem is a consequence of (26).

(28) Let us consider non empty sets X, Y, Z, a partial function f from $X \times Y$ to Z, an element x of X, an element y of Y, and a set A. Then

(i) $\mathrm{Xsection}(f^{-1}(A), x) = (\mathrm{ProjPMap1}(f, x))^{-1}(A)$, and

(ii) $\mathrm{Ysection}(f^{-1}(A), y) = (\mathrm{ProjPMap2}(f, y))^{-1}(A)$.

(29) Let us consider non empty sets X_1, X_2, an element x of X_1, an element y of X_2, a real number r, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Then

(i) $\mathrm{ProjPMap1}(r \cdot f, x) = r \cdot \mathrm{ProjPMap1}(f, x)$, and

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(ii) $\mathrm{ProjPMap2}(r \cdot f, y) = r \cdot \mathrm{ProjPMap2}(f, y)$.

(30) Let us consider non empty sets X_1, X_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element x of X_1, and an element y of X_2. Suppose for every element z of $X_1 \times X_2$ such that $z \in \operatorname{dom} f$ holds $f(z) = 0$. Then

(i) $(\mathrm{ProjPMap2}(f, y))(x) = 0$, and

(ii) $(\mathrm{ProjPMap1}(f, x))(y) = 0$.

(31) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, an element x of X_1, an element y of X_2, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose f is simple function in $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Then

(i) $\mathrm{ProjPMap1}(f, x)$ is simple function in S_2, and

(ii) $\mathrm{ProjPMap2}(f, y)$ is simple function in S_1.

PROOF: Consider F being a finite sequence of separated subsets of σ(Meas$-$Rect$(S_1, S_2))$ such that $\operatorname{dom} f = \bigcup \operatorname{rng} F$ and for every natural number n and for every elements z_1, z_2 of $X_1 \times X_2$ such that $n \in \operatorname{dom} F$ and z_1, $z_2 \in F(n)$ holds $f(z_1) = f(z_2)$. Define $\mathcal{H}(\text{natural number}) = \mathrm{MeasurableXsection}(F(\$_1), x)$. Consider H being a finite sequence of elements of S_2 such that $\operatorname{len} H = \operatorname{len} F$ and for every natural number n such that $n \in \operatorname{dom} H$ holds $H(n) = \mathcal{H}(n)$. Reconsider $F_1 = F$ as a finite sequence of elements of $2^{X_1 \times X_2}$. Reconsider $F_2 = H$ as a finite sequence of elements of 2^{X_2}. For every natural number n such that $n \in \operatorname{dom} F_2$ holds $F_2(n) = \mathrm{Xsection}(F_1(n), x)$. For every natural number n and for every elements y_1, y_2 of X_2 such that $n \in \operatorname{dom} H$ and y_1, $y_2 \in H(n)$ holds $(\mathrm{ProjPMap1}(f, x))(y_1) = (\mathrm{ProjPMap1}(f, x))(y_2)$. Define $\mathcal{G}(\text{natural number}) = \mathrm{MeasurableYsection}(F(\$_1), y)$. Consider G being a finite sequence of elements of S_1 such that $\operatorname{len} G = \operatorname{len} F$ and for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) = \mathcal{G}(n)$. Reconsider $G_1 = G$ as a finite sequence of elements of 2^{X_1}. For every natural number n such that $n \in \operatorname{dom} G_1$ holds $G_1(n) = \mathrm{Ysection}(F_1(n), y)$. For every natural number n and for every elements x_1, x_2 of X_1 such that $n \in \operatorname{dom} G$ and x_1, $x_2 \in G(n)$ holds $(\mathrm{ProjPMap2}(f, y))(x_1) = (\mathrm{ProjPMap2}(f, y))(x_2)$. $\square$

Let us consider non empty sets X_1, X_2, an element x of X_1, an element y of X_2, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Now we state the propositions:

(32) If f is non-negative, then $\mathrm{ProjPMap1}(f, x)$ is non-negative and $\mathrm{ProjPMap2}(f, y)$ is non-negative.

PROOF: For every object q such that $q \in \operatorname{dom} \mathrm{ProjPMap1}(f, x)$ holds $0 \leqslant (\mathrm{ProjPMap1}(f, x))(q)$. For every object p such that $p \in \operatorname{dom} \mathrm{ProjPMap2}(f, y)$ holds $0 \leqslant (\mathrm{ProjPMap2}(f, y))(p)$. $\square$

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(33) If f is non-positive, then $\mathrm{ProjPMap1}(f,x)$ is non-positive and $\mathrm{ProjPMap2}(f,y)$ is non-positive.
PROOF: For every set q such that $q \in \mathrm{dom}\,\mathrm{ProjPMap1}(f,x)$ holds $0 \geqslant (\mathrm{ProjPMap1}(f,x))(q)$. For every set p such that $p \in \mathrm{dom}\,\mathrm{ProjPMap2}(f,y)$ holds $0 \geqslant (\mathrm{ProjPMap2}(f,y))(p)$ by [6, (8)]. □

(34) Let us consider non empty sets X_1, X_2, an element x of X_1, an element y of X_2, a subset A of $X_1 \times X_2$, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Then

(i) $\mathrm{ProjPMap1}(f \restriction A, x) = \mathrm{ProjPMap1}(f,x) \restriction \mathrm{Xsection}(A,x)$, and

(ii) $\mathrm{ProjPMap2}(f \restriction A, y) = \mathrm{ProjPMap2}(f,y) \restriction \mathrm{Ysection}(A,y)$.

The theorem is a consequence of (25).

(35) Let us consider non empty sets X_1, X_2, a subset A of $X_1 \times X_2$, an element x of X_1, and an element y of X_2. Then

(i) $\mathrm{ProjPMap1}(\overline{\chi}_{A,X_1 \times X_2}, x) = \overline{\chi}_{\mathrm{Xsection}(A,x),X_2}$, and

(ii) $\mathrm{ProjPMap2}(\overline{\chi}_{A,X_1 \times X_2}, y) = \overline{\chi}_{\mathrm{Ysection}(A,y),X_1}$.

The theorem is a consequence of (27) and (25).

(36) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, partial functions f, g from X to $\overline{\mathbb{R}}$, and an element E of S. Suppose $f \restriction E = g \restriction E$ and $E \subseteq \mathrm{dom}\,f$ and $E \subseteq \mathrm{dom}\,g$ and f is measurable on E. Then g is measurable on E.

(37) Let us consider non empty sets X_1, X_2, a subset A of $X_1 \times X_2$, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element x of X_1, an element y of X_2, and a sequence F of partial functions from $X_1 \times X_2$ into $\overline{\mathbb{R}}$. Suppose $A \subseteq \mathrm{dom}\,f$ and for every natural number n, $\mathrm{dom}(F(n)) = A$ and for every element z of $X_1 \times X_2$ such that $z \in A$ holds $F\#z$ is convergent and $\lim(F\#z) = f(z)$. Then

(i) there exists a sequence F_1 of partial functions from X_1 into $\overline{\mathbb{R}}$ with the same dom such that for every natural number n, $F_1(n) = \mathrm{ProjPMap2}(F(n),y)$ and for every element x of X_1 such that $x \in \mathrm{Ysection}(A,y)$ holds $F_1\#x$ is convergent and $(\mathrm{ProjPMap2}(f,y))(x) = \lim(F_1\#x)$, and

(ii) there exists a sequence F_2 of partial functions from X_2 into $\overline{\mathbb{R}}$ with the same dom such that for every natural number n, $F_2(n) = \mathrm{ProjPMap1}(F(n),x)$ and for every element y of X_2 such that $y \in \mathrm{Xsection}(A,x)$ holds $F_2\#y$ is convergent and $(\mathrm{ProjPMap1}(f,x))(y) = \lim(F_2\#y)$.

PROOF: Define $\mathcal{R}[\text{element of } \mathbb{N}, \text{object}] \equiv \$_2 = \mathrm{ProjPMap2}(F(\$_1), y)$. For every element n of $\mathbb{N}$, there exists an element f of $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$ such that

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$\mathcal{R}[n, f]$. There exists a sequence F_1 of partial functions from X_1 into $\overline{\mathbb{R}}$ with the same dom such that for every natural number n, $F_1(n) = \mathrm{ProjPMap2}(F(n), y)$ and for every element x of X_1 such that $x \in \mathrm{Ysection}(A, y)$ holds $F_1\#x$ is convergent and $(\mathrm{ProjPMap2}(f, y))(x) = \lim(F_1\#x)$. Define $\mathcal{Q}[\text{element of } \mathbb{N}, \text{object}] \equiv \$_2 = \mathrm{ProjPMap1}(F(\$_1), x)$. For every element n of $\mathbb{N}$, there exists an element f of $X_2 \dot{\rightarrow} \overline{\mathbb{R}}$ such that $\mathcal{Q}[n, f]$. Consider F_2 being a sequence of $X_2 \dot{\rightarrow} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{Q}[n, F_2(n)]$. For every natural number n, $\mathrm{dom}(F_2(n)) = \mathrm{Xsection}(A, x)$. For every natural numbers m, n, $\mathrm{dom}(F_2(m)) = \mathrm{dom}(F_2(n))$. For every natural number n, $F_2(n) = \mathrm{ProjPMap1}(F(n), x)$. For every element y of X_2 such that $y \in \mathrm{Xsection}(A, x)$ holds $F_2\#y$ is convergent and $(\mathrm{ProjPMap1}(f, x))(y) = \lim(F_2\#y)$. □

(38) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, a σ-measure M_2 on S_2, an element A of S_1, an element B of S_2, and an element x of X_1. Then $M_2(B \cap \mathrm{MeasurableXsection}(E, x)) \cdot (\chi_{A,X_1}(x)) = \int \mathrm{ProjPMap1}(\chi_{A \times B, X_1 \times X_2} \restriction E, x)\, \mathrm{d}M_2$.
PROOF: Set $C_1 = \chi_{A \times B, X_1 \times X_2} \restriction E$. $\mathrm{ProjPMap1}(\chi_{A \times B, X_1 \times X_2}, x) = \mathrm{curry}(\chi_{A \times B, X_1 \times X_2}, x)$. $\mathrm{ProjPMap1}(C_1, x) = \mathrm{ProjPMap1}(\chi_{A \times B, X_1 \times X_2}, x) \restriction \mathrm{Xsection}(E, x)$. For every element y of X_2, $(\mathrm{ProjPMap1}(C_1, x))(y) = (\chi_{A,X_1} \restriction \mathrm{MeasurableYsection}(E, y))(x) \cdot (\chi_{B,X_2}(y))$. □

(39) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, a σ-measure M_1 on S_1, an element A of S_1, an element B of S_2, and an element y of X_2. Then $M_1(A \cap \mathrm{MeasurableYsection}(E, y)) \cdot (\chi_{B,X_2}(y)) = \int \mathrm{ProjPMap2}(\chi_{A \times B, X_1 \times X_2} \restriction E, y)\, \mathrm{d}M_1$.
PROOF: Set $C_1 = \chi_{A \times B, X_1 \times X_2} \restriction E$. $\mathrm{ProjPMap2}(\chi_{A \times B, X_1 \times X_2}, y) = \mathrm{curry}'(\chi_{A \times B, X_1 \times X_2}, y)$. $\mathrm{ProjPMap2}(C_1, y) = \mathrm{ProjPMap2}(\chi_{A \times B, X_1 \times X_2}, y) \restriction \mathrm{Ysection}(E, y)$. For every element x of X_1, $(\mathrm{ProjPMap2}(C_1, y))(x) = (\chi_{B,X_2} \restriction \mathrm{MeasurableXsection}(E, x))(y) \cdot (\chi_{A,X_1}(x))$ by [4, (2)]. □

(40) Let us consider non empty sets X_1, X_2, an element x of X_1, an element y of X_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an extended real e. Then

(i) $\langle x, y \rangle \in \mathrm{dom}\, f$ and $f(x, y) = e$ iff $y \in \mathrm{dom}\, \mathrm{ProjPMap1}(f, x)$ and $(\mathrm{ProjPMap1}(f, x))(y) = e$, and

(ii) $\langle x, y \rangle \in \mathrm{dom}\, f$ and $f(x, y) = e$ iff $x \in \mathrm{dom}\, \mathrm{ProjPMap2}(f, y)$ and $(\mathrm{ProjPMap2}(f, y))(x) = e$.

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The theorem is a consequence of (25) and (26).

(41) Let us consider non empty sets X_1, X_2, sets A, Z, a partial function f from $X_1 \times X_2$ to Z, and an element x of X_1. Then $\mathrm{Xsection}(f^{-1}(A), x) = (\mathrm{ProjPMap1}(f, x))^{-1}(A)$.

(42) Let us consider non empty sets X_1, X_2, sets A, Z, a partial function f from $X_1 \times X_2$ to Z, and an element y of X_2. Then $\mathrm{Ysection}(f^{-1}(A), y) = (\mathrm{ProjPMap2}(f, y))^{-1}(A)$.

(43) Let us consider non empty sets X_1, X_2, subsets A, B of $X_1 \times X_2$, and a set p. Then

(i) $\mathrm{Xsection}(A \setminus B, p) = \mathrm{Xsection}(A, p) \setminus \mathrm{Xsection}(B, p)$, and

(ii) $\mathrm{Ysection}(A \setminus B, p) = \mathrm{Ysection}(A, p) \setminus \mathrm{Ysection}(B, p)$.

(44) Let us consider non empty sets X_1, X_2, an element x of X_1, an element y of X_2, and partial functions f_1, f_2 from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Then

(i) $\mathrm{ProjPMap1}(f_1 + f_2, x) = \mathrm{ProjPMap1}(f_1, x) + \mathrm{ProjPMap1}(f_2, x)$, and

(ii) $\mathrm{ProjPMap1}(f_1 - f_2, x) = \mathrm{ProjPMap1}(f_1, x) - \mathrm{ProjPMap1}(f_2, x)$, and

(iii) $\mathrm{ProjPMap2}(f_1 + f_2, y) = \mathrm{ProjPMap2}(f_1, y) + \mathrm{ProjPMap2}(f_2, y)$, and

(iv) $\mathrm{ProjPMap2}(f_1 - f_2, y) = \mathrm{ProjPMap2}(f_1, y) - \mathrm{ProjPMap2}(f_2, y)$.

The theorem is a consequence of (42), (41), (43), (26), and (40).

(45) Let us consider non empty sets X_1, X_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an element x of X_1. Then

(i) $\mathrm{ProjPMap1}(\max_+(f), x) = \max_+(\mathrm{ProjPMap1}(f, x))$, and

(ii) $\mathrm{ProjPMap1}(\max_-(f), x) = \max_-(\mathrm{ProjPMap1}(f, x))$.

(46) Let us consider non empty sets X_1, X_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an element y of X_2. Then

(i) $\mathrm{ProjPMap2}(\max_+(f), y) = \max_+(\mathrm{ProjPMap2}(f, y))$, and

(ii) $\mathrm{ProjPMap2}(\max_-(f), y) = \max_-(\mathrm{ProjPMap2}(f, y))$.

(47) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element x of X_1, an element y of X_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose $E \subseteq \mathrm{dom}\, f$ and f is measurable on E. Then

(i) $\mathrm{ProjPMap1}(f, x)$ is measurable on $\mathrm{MeasurableXsection}(E, x)$, and

(ii) $\mathrm{ProjPMap2}(f, y)$ is measurable on $\mathrm{MeasurableYsection}(E, y)$.

The theorem is a consequence of (45) and (46).

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Let X_1, X_2, Y be non empty sets, F be a sequence of partial functions from $X_1 \times X_2$ into Y, and x be an element of X_1. The functor $\mathrm{ProjPMap1}(F, x)$ yielding a sequence of partial functions from X_2 into Y is defined by

(Def. 5) for every natural number n, $it(n) = \mathrm{ProjPMap1}(F(n), x)$.

Let y be an element of X_2. The functor $\mathrm{ProjPMap2}(F, y)$ yielding a sequence of partial functions from X_1 into Y is defined by

(Def. 6) for every natural number n, $it(n) = \mathrm{ProjPMap2}(F(n), y)$.

(48) Let us consider non empty sets X_1, X_2, a subset E of $X_1 \times X_2$, an element x of X_1, and an element y of X_2. Then

(i) $\mathrm{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) = \chi_{\mathrm{Xsection}(E,x), X_2}$, and

(ii) $\mathrm{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) = \chi_{\mathrm{Ysection}(E,y), X_1}$.

The theorem is a consequence of (25) and (27).

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and an extended real e. Now we state the propositions:

(49) $\int \chi_{e,E,X} \, \mathrm{d}M = e \cdot M(E)$. The theorem is a consequence of (2), (12), and (1).

(50) $\int \chi_{e,E,X} \restriction E \, \mathrm{d}M = e \cdot M(E)$. The theorem is a consequence of (15), (2), (13), (49), (16), (1), and (12).

(51) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, elements E_1, E_2 of S, and an extended real e. Then $\int \chi_{e,E_1,X} \restriction E_2 \, \mathrm{d}M = e \cdot M(E_1 \cap E_2)$. The theorem is a consequence of (14), (17), (13), (16), (15), and (50).

(52) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, an element x of X_1, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_2 is σ-finite. Then

(i) $(\mathrm{Yvol}(E, M_2))(x) = \int \mathrm{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) \, \mathrm{d}M_2$, and

(ii) $(\mathrm{Yvol}(E, M_2))(x) = \int^{+} \mathrm{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) \, \mathrm{d}M_2$, and

(iii) $(\mathrm{Yvol}(E, M_2))(x) = \int' \mathrm{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) \, \mathrm{d}M_2$.

The theorem is a consequence of (48), (12), and (27).

(53) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, an element y of X_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite. Then

(i) $(\mathrm{Xvol}(E, M_1))(y) = \int \mathrm{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) \, \mathrm{d}M_1$, and

(ii) $(\mathrm{Xvol}(E, M_1))(y) = \int^{+} \mathrm{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) \, \mathrm{d}M_1$, and

(iii) $(\mathrm{Xvol}(E, M_1))(y) = \int' \mathrm{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) \, \mathrm{d}M_1$.

The theorem is a consequence of (48), (12), and (27).

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(54) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and a real number r. Then $\int r \cdot \chi_{E,X} \,\mathrm{d}M = r \cdot \int \chi_{E,X} \,\mathrm{d}M$. The theorem is a consequence of (12).

(55) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, an element y of X_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and a real number r. Suppose M_1 is σ-finite. Then

(i) $(r \cdot \mathrm{Xvol}(E, M_1))(y) = \int \mathrm{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) \,\mathrm{d}M_1$, and

(ii) if $r \geqslant 0$, then $(r \cdot \mathrm{Xvol}(E, M_1))(y) = \int^{+} \mathrm{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) \,\mathrm{d}M_1$.

Proof: Set $p_2 = \mathrm{ProjPMap2}(\chi_{E,X_1 \times X_2}, y)$. $\chi_{r,E,X_1 \times X_2} = r \cdot \chi_{E,X_1 \times X_2}$. $\mathrm{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) = r \cdot p_2$. p_2 is non-negative. $\chi_{E,X_1 \times X_2}$ is simple function in $\sigma(\mathrm{MeasRect}(S_1, S_2))$. $\int \mathrm{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) \,\mathrm{d}M_1 = r \cdot (\int' p_2 \,\mathrm{d}M_1)$. If $r \geqslant 0$, then $(r \cdot \mathrm{Xvol}(E, M_1))(y) = \int^{+} \mathrm{ProjPMap2}(\chi_{r,E,X_1 \times X_2}, y) \,\mathrm{d}M_1$. □

(56) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, an element x of X_1, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and a real number r. Suppose M_2 is σ-finite. Then

(i) $(r \cdot \mathrm{Yvol}(E, M_2))(x) = \int \mathrm{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) \,\mathrm{d}M_2$, and

(ii) if $r \geqslant 0$, then $(r \cdot \mathrm{Yvol}(E, M_2))(x) = \int^{+} \mathrm{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) \,\mathrm{d}M_2$.

Proof: Set $p_2 = \mathrm{ProjPMap1}(\chi_{E,X_1 \times X_2}, x)$. $\chi_{r,E,X_1 \times X_2} = r \cdot \chi_{E,X_1 \times X_2}$. $\mathrm{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) = r \cdot p_2$. p_2 is non-negative. $\chi_{E,X_1 \times X_2}$ is simple function in $\sigma(\mathrm{MeasRect}(S_1, S_2))$. $\int \mathrm{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) \,\mathrm{d}M_2 = r \cdot (\int' p_2 \,\mathrm{d}M_2)$. If $r \geqslant 0$, then $(r \cdot \mathrm{Yvol}(E, M_2))(x) = \int^{+} \mathrm{ProjPMap1}(\chi_{r,E,X_1 \times X_2}, x) \,\mathrm{d}M_2$. □

(57) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose $\mathrm{dom}\, f \in S$ and for every element x of X such that $x \in \mathrm{dom}\, f$ holds $0 = f(x)$. Then

(i) for every element E of S such that $E \subseteq \mathrm{dom}\, f$ holds f is measurable on E, and

(ii) $\int f \,\mathrm{d}M = 0$.

The theorem is a consequence of (15) and (50).

(58) Let us consider non empty sets X_1, X_2, Y, a sequence F of partial functions from $X_1 \times X_2$ into Y, an element x of X_1, and an element y of X_2. Suppose F has the same dom. Then

(i) $\mathrm{ProjPMap1}(F, x)$ has the same dom, and

(ii) $\mathrm{ProjPMap2}(F, y)$ has the same dom.

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4. Fubini's Theorem for Non-negative or Non-positive Functions

Let X_1, X_2 be non empty sets, S_1 be a σ-field of subsets of X_1, M_1 be a σ-measure on S_1, and f be a partial function from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. The functor $\mathrm{Integral1}(M_1, f)$ yielding a function from X_2 into $\overline{\mathbb{R}}$ is defined by

(Def. 7) for every element y of X_2, $it(y) = \int \mathrm{ProjPMap2}(f, y)\, \mathrm{d}M_1$.

Let S_2 be a σ-field of subsets of X_2 and M_2 be a σ-measure on S_2. The functor $\mathrm{Integral2}(M_2, f)$ yielding a function from X_1 into $\overline{\mathbb{R}}$ is defined by

(Def. 8) for every element x of X_1, $it(x) = \int \mathrm{ProjPMap1}(f, x)\, \mathrm{d}M_2$.

Now we state the propositions:

(59) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element V of S_2. Suppose M_1 is σ-finite and f is non-negative or non-positive and $E = \mathrm{dom}\, f$ and f is measurable on E. Then $\mathrm{Integral1}(M_1, f)$ is measurable on V.

(60) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element U of S_1. Suppose M_2 is σ-finite and f is non-negative or non-positive and $E = \mathrm{dom}\, f$ and f is measurable on E. Then $\mathrm{Integral2}(M_2, f)$ is measurable on U.

(61) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, an element y of X_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite. Then $(\mathrm{Xvol}(E, M_1))(y) = \int \chi_{\mathrm{MeasurableYsection}(E,y), X_1}\, \mathrm{d}M_1$.

(62) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, an element x of X_1, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_2 is σ-finite. Then $(\mathrm{Yvol}(E, M_2))(x) = \int \chi_{\mathrm{MeasurableXsection}(E,x), X_2}\, \mathrm{d}M_2$.

(63) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element x of X_1, and an element y of X_2. Then

(i) $\mathrm{ProjPMap1}(\chi_{E, X_1 \times X_2}, x) = \chi_{\mathrm{MeasurableXsection}(E,x), X_2}$, and

(ii) $\mathrm{ProjPMap2}(\chi_{E, X_1 \times X_2}, y) = \chi_{\mathrm{MeasurableYsection}(E,y), X_1}$.

(64) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and an element

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E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite. Then $\mathrm{Xvol}(E, M_1) = \mathrm{Integral1}(M_1, \chi_{E,X_1 \times X_2})$. The theorem is a consequence of (61) and (63).

(65) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_2 is σ-finite. Then $\mathrm{Yvol}(E, M_2) = \mathrm{Integral2}(M_2, \chi_{E,X_1 \times X_2})$. The theorem is a consequence of (62) and (63).

Let X_1, X_2 be non empty sets, S_1 be a σ-field of subsets of X_1, S_2 be a σ-field of subsets of X_2, M_1 be a σ-measure on S_1, and M_2 be a σ-measure on S_2. The functor $\mathrm{ProdMeas}(M_1, M_2)$ yielding a σ-measure on $\sigma(\mathrm{MeasRect}(S_1, S_2))$ is defined by the term

(Def. 9) $\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2)$.

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and elements E_1, E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Now we state the propositions:

(66) Suppose $E_1 = \mathrm{dom}\, f$ and f is non-negative and f is measurable on E_1. Then

(i) $\mathrm{Integral1}(M_1, f)$ is non-negative, and

(ii) $\mathrm{Integral1}(M_1, f \restriction E_2)$ is non-negative, and

(iii) $\mathrm{Integral2}(M_2, f)$ is non-negative, and

(iv) $\mathrm{Integral2}(M_2, f \restriction E_2)$ is non-negative.

The theorem is a consequence of (47) and (32).

(67) Suppose $E_1 = \mathrm{dom}\, f$ and f is non-positive and f is measurable on E_1. Then

(i) $\mathrm{Integral1}(M_1, f)$ is non-positive, and

(ii) $\mathrm{Integral1}(M_1, f \restriction E_2)$ is non-positive, and

(iii) $\mathrm{Integral2}(M_2, f)$ is non-positive, and

(iv) $\mathrm{Integral2}(M_2, f \restriction E_2)$ is non-positive.

The theorem is a consequence of (47) and (33).

(68) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, elements E_1, E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element V of S_2. Suppose M_1 is σ-finite and f is non-negative or non-positive and $E_1 = \mathrm{dom}\, f$ and f is measurable on E_1. Then $\mathrm{Integral1}(M_1, f \restriction E_2)$ is measurable on V. The theorem is a consequence of (59).

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(69) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, elements E_1, E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element U of S_1. Suppose M_2 is σ-finite and f is non-negative or non-positive and $E_1 = \mathrm{dom}\, f$ and f is measurable on E_1. Then $\mathrm{Integral2}(M_2, f \restriction E_2)$ is measurable on U. The theorem is a consequence of (60).

(70) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element y of X_2. Suppose $E = \mathrm{dom}\, f$ and f is non-negative or non-positive and f is measurable on E and for every element x of X_1 such that $x \in \mathrm{dom}\,\mathrm{ProjPMap2}(f, y)$ holds $(\mathrm{ProjPMap2}(f, y))(x) = 0$. Then $(\mathrm{Integral1}(M_1, f))(y) = 0$. The theorem is a consequence of (47), (32), and (33).

(71) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element x of X_1. Suppose $E = \mathrm{dom}\, f$ and f is non-negative or non-positive and f is measurable on E and for every element y of X_2 such that $y \in \mathrm{dom}\,\mathrm{ProjPMap1}(f, x)$ holds $(\mathrm{ProjPMap1}(f, x))(y) = 0$. Then $(\mathrm{Integral2}(M_2, f))(x) = 0$. The theorem is a consequence of (47), (32), and (33).

(72) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, elements E, E_1, E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose $E = \mathrm{dom}\, f$ and f is non-negative or non-positive and f is measurable on E and E_1 misses E_2. Then

(i) $\mathrm{Integral1}(M_1, f \restriction (E_1 \cup E_2)) =$
$\mathrm{Integral1}(M_1, f \restriction E_1) + \mathrm{Integral1}(M_1, f \restriction E_2)$, and

(ii) $\mathrm{Integral2}(M_2, f \restriction (E_1 \cup E_2)) =$
$\mathrm{Integral2}(M_2, f \restriction E_1) + \mathrm{Integral2}(M_2, f \restriction E_2)$.

(73) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose $E = \mathrm{dom}\, f$ and f is measurable on E. Then

(i) $\mathrm{Integral1}(M_1, -f) = -\mathrm{Integral1}(M_1, f)$, and

(ii) $\mathrm{Integral2}(M_2, -f) = -\mathrm{Integral2}(M_2, f)$.

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The theorem is a consequence of (29) and (47).

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, partial functions f, g from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and elements E_1, E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Now we state the propositions:

(74) Suppose $E_1 = \mathrm{dom}\, f$ and f is non-negative and f is measurable on E_1 and $E_2 = \mathrm{dom}\, g$ and g is non-negative and g is measurable on E_2. Then

(i) $\mathrm{Integral1}(M_1, f+g) =$
$\mathrm{Integral1}(M_1, f \restriction \mathrm{dom}(f+g)) + \mathrm{Integral1}(M_1, g \restriction \mathrm{dom}(f+g))$, and

(ii) $\mathrm{Integral2}(M_2, f+g) =$
$\mathrm{Integral2}(M_2, f \restriction \mathrm{dom}(f+g)) + \mathrm{Integral2}(M_2, g \restriction \mathrm{dom}(f+g))$.

Proof: Set $f_1 = f \restriction (A \cap B)$. Set $g_1 = g \restriction (A \cap B)$. $\mathrm{Integral1}(M_1, f_1)$ is non-negative and $\mathrm{Integral1}(M_1, g_1)$ is non-negative and $\mathrm{Integral2}(M_2, f_1)$ is non-negative and $\mathrm{Integral2}(M_2, g_1)$ is non-negative. For every element y of X_2, $(\mathrm{Integral1}(M_1, f_1) + \mathrm{Integral1}(M_1, g_1))(y) = (\mathrm{Integral1}(M_1, f+g))(y)$. For every element x of X_1, $(\mathrm{Integral2}(M_2, f_1) + \mathrm{Integral2}(M_2, g_1))(x) = (\mathrm{Integral2}(M_2, f+g))(x)$. $\square$

(75) Suppose $E_1 = \mathrm{dom}\, f$ and f is non-positive and f is measurable on E_1 and $E_2 = \mathrm{dom}\, g$ and g is non-positive and g is measurable on E_2. Then

(i) $\mathrm{Integral1}(M_1, f+g) =$
$\mathrm{Integral1}(M_1, f \restriction \mathrm{dom}(f+g)) + \mathrm{Integral1}(M_1, g \restriction \mathrm{dom}(f+g))$, and

(ii) $\mathrm{Integral2}(M_2, f+g) =$
$\mathrm{Integral2}(M_2, f \restriction \mathrm{dom}(f+g)) + \mathrm{Integral2}(M_2, g \restriction \mathrm{dom}(f+g))$.

The theorem is a consequence of (73) and (74).

(76) Suppose $E_1 = \mathrm{dom}\, f$ and f is non-negative and f is measurable on E_1 and $E_2 = \mathrm{dom}\, g$ and g is non-positive and g is measurable on E_2. Then

(i) $\mathrm{Integral1}(M_1, f-g) =$
$\mathrm{Integral1}(M_1, f \restriction \mathrm{dom}(f-g)) - \mathrm{Integral1}(M_1, g \restriction \mathrm{dom}(f-g))$, and

(ii) $\mathrm{Integral1}(M_1, g-f) =$
$\mathrm{Integral1}(M_1, g \restriction \mathrm{dom}(g-f)) - \mathrm{Integral1}(M_1, f \restriction \mathrm{dom}(g-f))$, and

(iii) $\mathrm{Integral2}(M_2, f-g) =$
$\mathrm{Integral2}(M_2, f \restriction \mathrm{dom}(f-g)) - \mathrm{Integral2}(M_2, g \restriction \mathrm{dom}(f-g))$, and

(iv) $\mathrm{Integral2}(M_2, g-f) =$
$\mathrm{Integral2}(M_2, g \restriction \mathrm{dom}(g-f)) - \mathrm{Integral2}(M_2, f \restriction \mathrm{dom}(g-f))$.

The theorem is a consequence of (74) and (73).

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(77) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite and M_2 is σ-finite. Then

(i) $\int \mathrm{Yvol}(E, M_2)\,\mathrm{d}M_1 = \int \chi_{E,X_1\times X_2}\,\mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2)$, and

(ii) $\int \mathrm{Xvol}(E, M_1)\,\mathrm{d}M_2 = \int \chi_{E,X_1\times X_2}\,\mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2)$.

(78) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and a real number r. Suppose $E = \mathrm{dom}\, f$ and f is non-negative or non-positive and f is measurable on E. Then

(i) $\mathrm{Integral1}(M_1, r \cdot f) = r \cdot \mathrm{Integral1}(M_1, f)$, and

(ii) $\mathrm{Integral2}(M_2, r \cdot f) = r \cdot \mathrm{Integral2}(M_2, f)$.

The theorem is a consequence of (32), (33), (29), and (47).

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Now we state the propositions:

(79) (i) $\mathrm{Integral1}(M_1, \chi_{E,X_1\times X_2}\restriction E) = \mathrm{Integral1}(M_1, \chi_{E,X_1\times X_2})$, and

(ii) $\mathrm{Integral2}(M_2, \chi_{E,X_1\times X_2}\restriction E) = \mathrm{Integral2}(M_2, \chi_{E,X_1\times X_2})$.

The theorem is a consequence of (34) and (48).

(80) (i) $\mathrm{Integral1}(M_1, \overline{\chi}_{E,X_1\times X_2}\restriction E) = \mathrm{Integral1}(M_1, \overline{\chi}_{E,X_1\times X_2})$, and

(ii) $\mathrm{Integral2}(M_2, \overline{\chi}_{E,X_1\times X_2}\restriction E) = \mathrm{Integral2}(M_2, \overline{\chi}_{E,X_1\times X_2})$.

The theorem is a consequence of (34), (35), (2), (50), and (49).

(81) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an extended real e. Then

(i) $\mathrm{Integral1}(M_1, \chi_{e,E,X_1\times X_2}\restriction E) = \mathrm{Integral1}(M_1, \chi_{e,E,X_1\times X_2})$, and

(ii) $\mathrm{Integral2}(M_2, \chi_{e,E,X_1\times X_2}\restriction E) = \mathrm{Integral2}(M_2, \chi_{e,E,X_1\times X_2})$.

The theorem is a consequence of (1), (78), (79), (2), and (80).

(82) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite and M_2 is σ-finite. Then

(i) $\int \chi_{E,X_1\times X_2}\,\mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral1}(M_1, \chi_{E,X_1\times X_2})\,\mathrm{d}M_2$, and

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(ii) $\int \chi_{E,X_1\times X_2}\restriction E \,\mathrm{d}\,\mathrm{ProdMeas}(M_1,M_2) = \int \mathrm{Integral1}(M_1,\chi_{E,X_1\times X_2}\restriction E)\,\mathrm{d}M_2$, and

(iii) $\int \chi_{E,X_1\times X_2} \,\mathrm{d}\,\mathrm{ProdMeas}(M_1,M_2) = \int \mathrm{Integral2}(M_2,\chi_{E,X_1\times X_2})\,\mathrm{d}M_1$, and

(iv) $\int \chi_{E,X_1\times X_2}\restriction E \,\mathrm{d}\,\mathrm{ProdMeas}(M_1,M_2) = \int \mathrm{Integral2}(M_2,\chi_{E,X_1\times X_2}\restriction E)\,\mathrm{d}M_1$.

The theorem is a consequence of (64), (77), (79), and (65).

(83) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1,S_2))$, and a real number r. Suppose M_1 is σ-finite and M_2 is σ-finite. Then

(i) $\int \chi_{r,E,X_1\times X_2} \,\mathrm{d}\,\mathrm{ProdMeas}(M_1,M_2) = \int \mathrm{Integral1}(M_1,\chi_{r,E,X_1\times X_2})\,\mathrm{d}M_2$, and

(ii) $\int \chi_{r,E,X_1\times X_2}\restriction E \,\mathrm{d}\,\mathrm{ProdMeas}(M_1,M_2) = \int \mathrm{Integral1}(M_1,\chi_{r,E,X_1\times X_2}\restriction E)\,\mathrm{d}M_2$, and

(iii) $\int \chi_{r,E,X_1\times X_2} \,\mathrm{d}\,\mathrm{ProdMeas}(M_1,M_2) = \int \mathrm{Integral2}(M_2,\chi_{r,E,X_1\times X_2})\,\mathrm{d}M_1$, and

(iv) $\int \chi_{r,E,X_1\times X_2}\restriction E \,\mathrm{d}\,\mathrm{ProdMeas}(M_1,M_2) = \int \mathrm{Integral2}(M_2,\chi_{r,E,X_1\times X_2}\restriction E)\,\mathrm{d}M_1$.

The theorem is a consequence of (1), (12), (64), (82), (78), (81), and (65).

(84) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element A of $\sigma(\mathrm{MeasRect}(S_1,S_2))$, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose M_1 is σ-finite and M_2 is σ-finite and f is non-negative or non-positive and $A = \mathrm{dom}\, f$ and f is measurable on A. Then

(i) $\int f \,\mathrm{d}\,\mathrm{ProdMeas}(M_1,M_2) = \int \mathrm{Integral1}(M_1,f)\,\mathrm{d}M_2$, and

(ii) $\int f \,\mathrm{d}\,\mathrm{ProdMeas}(M_1,M_2) = \int \mathrm{Integral2}(M_2,f)\,\mathrm{d}M_1$.

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Received March 27, 2018

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