

Klein-Beltrami Model. Part II

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Summary. Tim Makarios (with Isabelle/HOL¹) and John Harrison (with HOL-Light²) have shown that “the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski’s axioms except his Euclidean axiom” [2, 3, 15, 4].

With the Mizar system [1], [10] we use some ideas are taken from Tim Makarios’ MSc thesis [12] for formalized some definitions (like the tangent) and lemmas necessary for the verification of the independence of the parallel postulate. This work can be also treated as a further development of Tarski’s geometry in the formal setting [9].

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1. BELTRAMI-CAYLEY-KLEIN DISK MODEL

The BK-model yielding a non empty subset of the projective space over \mathcal{E}_T^3 is defined by the term

(Def. 1) the interior of the conic for 1, 1, -1 , 0, 0 and 0.

Now we state the propositions:

- (1) The BK-model misses the absolute.
- (2) Let us consider an element P of the projective space over \mathcal{E}_T^3 , and a non zero element u of \mathcal{E}_T^3 . Suppose $P =$ the direction of u and $P \in$ the BK-model. Then $u(3) \neq 0$.

¹https://www.isa-afp.org/entries/Tarskis_Geometry.html

²<https://github.com/jrh13/hol-light/blob/master/100/independence.ml>

Let P be an element of the BK-model. The functor BK-to-REAL2(P) yielding an element of the inside of circle(0,0,1) is defined by

(Def. 2) there exists a non zero element u of \mathcal{E}_T^3 such that the direction of $u = P$ and $u(3) = 1$ and $it = [u(1), u(2)]$.

Let Q be an element of the inside of circle(0,0,1). The functor REAL2-to-BK(Q) yielding an element of the BK-model is defined by

(Def. 3) there exists an element P of \mathcal{E}_T^2 such that $P = Q$ and $it =$ the direction of $[(P)_1, (P)_2, 1]$.

Now we state the propositions:

(3) Let us consider an element P of the BK-model.

Then $\text{REAL2-to-BK}(\text{BK-to-REAL2}(P)) = P$.

PROOF: Consider u being a non zero element of \mathcal{E}_T^3 such that the direction of $u = P$ and $u(3) = 1$ and $\text{BK-to-REAL2}(P) = [u(1), u(2)]$. Consider Q being an element of \mathcal{E}_T^2 such that $Q = \text{BK-to-REAL2}(P)$ and $\text{REAL2-to-BK}(\text{BK-to-REAL2}(P)) =$ the direction of $[(Q)_1, (Q)_2, 1]$. $[(Q)_1, (Q)_2, 1]$ and u are proportional. \square

(4) Let us consider elements P, Q of the BK-model. Then $P = Q$ if and only if $\text{BK-to-REAL2}(P) = \text{BK-to-REAL2}(Q)$.

(5) Let us consider an element Q of the inside of circle(0,0,1).

Then $\text{BK-to-REAL2}(\text{REAL2-to-BK}(Q)) = Q$.

(6) Let us consider elements P, Q of the BK-model, and elements P_1, P_2, P_3 of the absolute. Suppose $P \neq Q$ and $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, Q and P_3 are collinear. Then

(i) $P_3 = P_1$, or

(ii) $P_3 = P_2$.

PROOF: $P_3 = P_1$ or $P_3 = P_2$. \square

(7) Let us consider an element P of the BK-model, an element Q of the projective space over \mathcal{E}_T^3 , and a non zero element v of \mathcal{E}_T^3 . Suppose $P \neq Q$ and $Q =$ the direction of v and $v(3) = 1$. Then there exists an element P_1 of the absolute such that P, Q and P_1 are collinear.

PROOF: Consider u being a non zero element of \mathcal{E}_T^3 such that the direction of $u = P$ and $u(3) = 1$ and $\text{BK-to-REAL2}(P) = [u(1), u(2)]$. Reconsider $s = [u(1), u(2)]$, $t = [v(1), v(2)]$ as a point of \mathcal{E}_T^2 . Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s$, $T = t$, $X = [a, b]$ as an element of \mathcal{R}^2 . Reconsider $w_1 = \frac{-2 \cdot |(t-s, s-[a,b])| + \sqrt{\Delta(\sum^2(T-S)), 2 \cdot |(t-s, s-[a,b])|, \sum^2(S-X) - r^2}}{2 \cdot \sum^2(T-S)}$

as a real number. $s \neq t$. Consider e_1 being a point of \mathcal{E}_T^2 such that

$\{e_1\} = \text{HalfLine}(s, t) \cap \text{circle}(a, b, r)$ and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $f = [(e_1)_1, (e_1)_2, 1]$ as an element of \mathcal{E}_T^3 . Reconsider $e_3 = f$ as a non zero element of \mathcal{E}_T^3 . $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}_T^3}$. \square

(8) Let us consider an element P of the BK-model, and a line L of Inc-ProjSp (the real projective plane). Then there exists an element Q of the projective space over \mathcal{E}_T^3 such that

- (i) $P \neq Q$, and
- (ii) $Q \in L$.

(9) Let us consider real numbers a, b, c, d, e , and elements u, v, w of \mathcal{E}_T^3 . Suppose $u = [a, b, e]$ and $v = [c, d, 0]$ and $w = [a + c, b + d, e]$. Then $\langle |u, v, w| \rangle = 0$.

(10) Let us consider real numbers a, b , and a non zero real number c . Then $[a, b, c]$ is a non zero element of \mathcal{E}_T^3 .

(11) Let us consider elements u, v of \mathcal{E}_T^3 , and real numbers a, b, c, d, e . Suppose $u = [a, b, c]$ and $v = [d, e, 0]$ and u and v are proportional. Then $c = 0$.

(12) Let us consider elements P, Q, R of the projective space over \mathcal{E}_T^3 , and non zero elements u, v, w of \mathcal{E}_T^3 . Suppose $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $(u)_3 \neq 0$ and $(v)_3 = 0$ and $w = [(u)_1 + (v)_1, (u)_2 + (v)_2, (u)_3]$. Then

- (i) $R \neq P$, and
- (ii) $R \neq Q$.

(13) Let us consider a line L of Inc-ProjSp(the real projective plane), and elements P, Q of the projective space over \mathcal{E}_T^3 . If $P \neq Q$ and $P, Q \in L$, then $L = \text{Line}(P, Q)$.

(14) Let us consider a line L of Inc-ProjSp(the real projective plane), elements P, Q of the projective space over \mathcal{E}_T^3 , and non zero elements u, v of \mathcal{E}_T^3 . Suppose $P, Q \in L$ and $P =$ the direction of u and $Q =$ the direction of v and $(u)_3 \neq 0$ and $(v)_3 = 0$. Then

- (i) $P \neq Q$, and
- (ii) the direction of $[(u)_1 + (v)_1, (u)_2 + (v)_2, (u)_3] \in L$.

PROOF: $P \neq Q$. Reconsider $w = [(u)_1 + (v)_1, (u)_2 + (v)_2, (u)_3]$ as a non zero element of \mathcal{E}_T^3 . $\langle |u, v, w| \rangle = 0$. \square

(15) Let us consider elements u, v, w of \mathcal{E}_T^3 . Suppose $(v)_3 = 0$ and $w = [(u)_1 + (v)_1, (u)_2 + (v)_2, (u)_3]$. Then $\langle |u, v, w| \rangle = 0$.

(16) Let us consider a line L of Inc-ProjSp(the real projective plane), an element P of the projective space over \mathcal{E}_T^3 , and a non zero element u of \mathcal{E}_T^3 .

Suppose $P =$ the direction of u and $P \in L$ and $u(3) \neq 0$. Then there exists an element Q of the projective space over \mathcal{E}_T^3 and there exists a non zero element v of \mathcal{E}_T^3 such that $Q =$ the direction of v and $Q \in L$ and $P \neq Q$ and $v(3) \neq 0$. The theorem is a consequence of (15).

- (17) Let us consider an element P of the BK-model, and a line L of Inc-ProjSp (the real projective plane). Suppose $P \in L$. Then there exists an element Q of the projective space over \mathcal{E}_T^3 such that
 - (i) $P \neq Q$, and
 - (ii) $Q \in L$, and
 - (iii) for every non zero element u of \mathcal{E}_T^3 such that $Q =$ the direction of u holds $u(3) \neq 0$.

The theorem is a consequence of (16).

- (18) Let us consider non zero elements u, v of \mathcal{E}_T^3 , and a non zero real number k . Suppose $u = k \cdot v$. Then the direction of $u =$ the direction of v .
- (19) Let us consider an element P of the BK-model, and an element Q of the projective space over \mathcal{E}_T^3 . Suppose $P \neq Q$. Then there exists an element P_1 of the absolute such that P, Q and P_1 are collinear.

PROOF: Reconsider $L = \text{Line}(P, Q)$ as a line of Inc-ProjSp(the real projective plane). Consider R being an element of the projective space over \mathcal{E}_T^3 such that $P \neq R$ and $R \in L$ and for every non zero element u of \mathcal{E}_T^3 such that $R =$ the direction of u holds $u(3) \neq 0$. Consider u being a non zero element of \mathcal{E}_T^3 such that the direction of $u = P$ and $u(3) = 1$ and $\text{BK-to-REAL2}(P) = [u(1), u(2)]$. Consider v' being an element of \mathcal{E}_T^3 such that v' is not zero and the direction of $v' = R$. Reconsider $k = \frac{1}{(v')_3}$ as a non zero real number. $k \cdot v'$ is not zero. Reconsider $v = k \cdot v'$ as a non zero element of \mathcal{E}_T^3 . the direction of $v = R$ and $v(3) = 1$. Reconsider $s = [u(1), u(2)], t = [v(1), v(2)]$ as a point of \mathcal{E}_T^2 . Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s, T = t, X = [a, b]$ as an element of \mathcal{R}^2 . Reconsider $w_1 = \frac{-2 \cdot |(t-s, s-[a,b])| + \sqrt{\Delta(\sum(2(T-S)), 2 \cdot |(t-s, s-[a,b])|, \sum(2(S-X)) - r^2)}}{2 \cdot \sum(2(T-S))}$

as a real number. $s \neq t$. Consider e_1 being a point of \mathcal{E}_T^2 such that $\{e_1\} = \text{HalfLine}(s, t) \cap \text{circle}(a, b, r)$ and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $f = [(e_1)_1, (e_1)_2, 1]$ as an element of \mathcal{E}_T^3 . Reconsider $e_3 = f$ as a non zero element of \mathcal{E}_T^3 . $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}_T^3}$. \square

- (20) Let us consider elements P, Q of the BK-model. Suppose $P \neq Q$. Then there exist elements P_1, P_2 of the absolute such that
 - (i) $P_1 \neq P_2$, and
 - (ii) P, Q and P_1 are collinear, and

(iii) P, Q and P_2 are collinear.

PROOF: Consider u being a non zero element of \mathcal{E}_T^3 such that the direction of $u = P$ and $u(3) = 1$ and $\text{BK-to-REAL2}(P) = [u(1), u(2)]$. Consider v being a non zero element of \mathcal{E}_T^3 such that the direction of $v = Q$ and $v(3) = 1$ and $\text{BK-to-REAL2}(Q) = [v(1), v(2)]$. Reconsider $s = [u(1), u(2)], t = [v(1), v(2)]$ as a point of \mathcal{E}_T^2 . Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s, T = t, X = [a, b]$ as an element of \mathcal{R}^2 . Reconsider $w_1 = \frac{-2 \cdot |(t-s, s-[a,b])| + \sqrt{\Delta(\sum^2(T-S)), 2 \cdot |(t-s, s-[a,b])|, \sum^2(S-X)-r^2)}}{2 \cdot (\sum^2(T-S))}$ as a real number. Consider e_1 being a point of \mathcal{E}_T^2 such that $\{e_1\} = \text{HalfLine}(s, t) \cap \text{circle}(a, b, r)$ and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $w_2 = \frac{-2 \cdot |(s-t, t-[a,b])| + \sqrt{\Delta(\sum^2(S-T)), 2 \cdot |(s-t, t-[a,b])|, \sum^2(T-X)-r^2)}}{2 \cdot (\sum^2(S-T))}$ as a real number. Consider e_2 being a point of \mathcal{E}_T^2 such that $\{e_2\} = \text{HalfLine}(t, s) \cap \text{circle}(a, b, r)$ and $e_2 = (1 - w_2) \cdot t + w_2 \cdot s$. Reconsider $f = [(e_1)_1, (e_1)_2, 1]$ as an element of \mathcal{E}_T^3 . Reconsider $e_3 = f$ as a non zero element of \mathcal{E}_T^3 . Reconsider $P_1 =$ the direction of e_3 as a point of the projective space over \mathcal{E}_T^3 . $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}_T^3}$. Reconsider $g = [(e_2)_1, (e_2)_2, 1]$ as an element of \mathcal{E}_T^3 . Reconsider $e_4 = g$ as a non zero element of \mathcal{E}_T^3 . Reconsider $P_2 =$ the direction of e_4 as a point of the projective space over \mathcal{E}_T^3 . $1 \cdot e_4 + (-(1 - w_2)) \cdot v + (-w_2) \cdot u = 0_{\mathcal{E}_T^3}$. $P_1 \neq P_2$. \square

(21) Let us consider elements P, Q, R of the real projective plane, non zero elements u, v, w of \mathcal{E}_T^3 , and real numbers a, b, c, d . Suppose $P \in$ the BK-model and $Q \in$ the absolute and $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $u = [a, b, 1]$ and $v = [c, d, 1]$ and $w = [\frac{a+c}{2}, \frac{b+d}{2}, 1]$. Then

- (i) $R \in$ the BK-model, and
- (ii) $R \neq P$, and
- (iii) P, R and Q are collinear.

PROOF: Reconsider $P_6 = P$ as an element of the BK-model. Consider u_2 being a non zero element of \mathcal{E}_T^3 such that the direction of $u_2 = P_6$ and $u_2(3) = 1$ and $\text{BK-to-REAL2}(P_6) = [u_2(1), u_2(2)]$. Consider p being a point of \mathcal{E}_T^2 such that $[v(1), v(2)] = p$ and $|p - [0, 0]| = 1$. Reconsider $R_1 = [w(1), w(2)]$ as an element of \mathcal{E}_T^2 . $|R_1 - [0, 0]|^2 < 1$. Consider P_1 being an element of \mathcal{E}_T^2 such that $P_1 = R_1$ and $\text{REAL2-to-BK}(R_1) =$ the direction of $[(P_1)_1, (P_1)_2, 1]$. $P \neq R$ by [13, (29)]. \square

(22) Let us consider elements P, Q of the real projective plane. Suppose $P \in$ the absolute and $Q \in$ the BK-model. Then there exists an element R of the real projective plane such that

- (i) $R \in$ the BK-model, and
- (ii) $Q \neq R$, and
- (iii) R, Q and P are collinear.

The theorem is a consequence of (21).

(23) Let us consider a line L of Inc-ProjSp(the real projective plane), points p, q of Inc-ProjSp(the real projective plane), and elements P, Q of the real projective plane. Suppose $p = P$ and $q = Q$ and $P \in$ the BK-model and $Q \in$ the absolute and q lies on L and p lies on L . Then there exist points p_1, p_2 of Inc-ProjSp(the real projective plane) and there exist elements P_1, P_2 of the real projective plane such that $p_1 = P_1$ and $p_2 = P_2$ and $P_1 \neq P_2$ and $P_1, P_2 \in$ the absolute and p_1 lies on L and p_2 lies on L . The theorem is a consequence of (1), (22), and (20).

(24) Let us consider an element P of the BK-model, and an element Q of the absolute. Then there exists an element R of the absolute such that

- (i) $Q \neq R$, and
- (ii) Q, P and R are collinear.

The theorem is a consequence of (1) and (23).

(25) Let us consider an element P of the BK-model, and a non zero element u of \mathcal{E}_T^3 . Suppose $P =$ the direction of u and $u(3) = 1$. Then $(u(1))^2 + (u(2))^2 < 1$.

(26) Let us consider elements P_1, P_2 of the absolute, an element Q of the BK-model, and non zero elements u, v, w of \mathcal{E}_T^3 . Suppose the direction of $u = P_1$ and the direction of $v = P_2$ and the direction of $w = Q$ and $u(3) = 1$ and $v(3) = 1$ and $w(3) = 1$ and $v(1) = -u(1)$ and $v(2) = -u(2)$ and P_1, Q and P_2 are collinear. Then there exists a real number a such that

- (i) $-1 < a < 1$, and
- (ii) $w(1) = a \cdot u(1)$, and
- (iii) $w(2) = a \cdot u(2)$.

The theorem is a consequence of (25).

2. TANGENT

Let P be an element of the absolute. The functor $\text{PoleInfty}(P)$ yielding an element of the real projective plane is defined by

(Def. 4) there exists a non zero element u of \mathcal{E}_T^3 such that $P =$ the direction of u and $u(3) = 1$ and $(u(1))^2 + (u(2))^2 = 1$ and $it =$ the direction of $[-u(2), u(1), 0]$.

Now we state the propositions:

(27) Let us consider an element P of the absolute. Then $P \neq \text{PoleInfty}(P)$.

(28) Let us consider elements P_1, P_2 of the absolute. Suppose $\text{PoleInfty}(P_1) = \text{PoleInfty}(P_2)$. Then

(i) $P_1 = P_2$, or

(ii) there exists a non zero element u of \mathcal{E}_T^3 such that $P_1 =$ the direction of u and $P_2 =$ the direction of $[-(u)_1, -(u)_2, 1]$ and $(u)_3 = 1$.

PROOF: Consider u_1 being a non zero element of \mathcal{E}_T^3 such that $P_1 =$ the direction of u_1 and $u_1(3) = 1$ and $u_1(1)^2 + u_1(2)^2 = 1$ and $\text{PoleInfty}(P_1) =$ the direction of $[-u_1(2), u_1(1), 0]$. Consider u_2 being a non zero element of \mathcal{E}_T^3 such that $P_2 =$ the direction of u_2 and $u_2(3) = 1$ and $(u_2(1))^2 + (u_2(2))^2 = 1$ and $\text{PoleInfty}(P_2) =$ the direction of $[-u_2(2), u_2(1), 0]$. Reconsider $w_1 = [-u_1(2), u_1(1), 0]$ as a non zero element of \mathcal{E}_T^3 . Reconsider $w_2 = [-u_2(2), u_2(1), 0]$ as a non zero element of \mathcal{E}_T^3 . Consider a being a real number such that $a \neq 0$ and $w_1 = a \cdot w_2$. If $a = 1$, then $P_1 = P_2$. If $a = -1$, then there exists a non zero element u of \mathcal{E}_T^3 such that $P_1 =$ the direction of u and $P_2 =$ the direction of $[-(u)_1, -(u)_2, 1]$ and $(u)_3 = 1$. \square

Let P be an element of the absolute. The functor $\text{tangent}(P)$ yielding a line of the real projective plane is defined by

(Def. 5) there exists an element p of the real projective plane such that $p = P$ and $it = \text{Line}(p, \text{PoleInfty}(P))$.

Let us consider an element P of the absolute. Now we state the propositions:

(29) $P \in \text{tangent}(P)$.

(30) $\text{tangent}(P) \cap (\text{the absolute}) = \{P\}$.

PROOF: $\{P\} \subseteq \text{tangent}(P) \cap (\text{the absolute})$. $\text{tangent}(P) \cap (\text{the absolute}) \subseteq \{P\}$. \square

(31) Let us consider elements P_1, P_2 of the absolute. If $\text{tangent}(P_1) = \text{tangent}(P_2)$, then $P_1 = P_2$. The theorem is a consequence of (30).

(32) Let us consider elements P, Q of the absolute. Then there exists an element R of the real projective plane such that

- (i) $R \in \text{tangent}(P)$, and
 - (ii) $R \in \text{tangent}(Q)$.
- (33) Let us consider elements P_1, P_2 of the absolute. Suppose $P_1 \neq P_2$. Then there exists an element P of the real projective plane such that $\text{tangent}(P_1) \cap \text{tangent}(P_2) = \{P\}$. The theorem is a consequence of (31).
- (34) Let us consider a square matrix M over \mathbb{R} of dimension 3, an element P of the absolute, an element Q of the real projective plane, non zero elements u, v of \mathcal{E}_T^3 , and finite sequences f_3, f_7 of elements of \mathbb{R} . Suppose $M = \text{symmetric3}(1, 1, -1, 0, 0, 0)$ and $P =$ the direction of u and $Q =$ the direction of v and $u = f_3$ and $v = f_7$ and $Q \in \text{tangent}(P)$. Then $\text{SumAllQuadraticForm}(f_7, M, f_3) = 0$.

PROOF: Consider p being an element of the real projective plane such that $p = P$ and $\text{tangent}(P) = \text{Line}(p, \text{PoleInfty}(P))$. Consider w being a non zero element of \mathcal{E}_T^3 such that $P =$ the direction of w and $w(3) = 1$ and $(w(1))^2 + (w(2))^2 = 1$ and $\text{PoleInfty}(P) =$ the direction of $[-w(2), w(1), 0]$. Consider a_1 being a real number such that $a_1 \neq 0$ and $w = a_1 \cdot u$. $w(1) = a_1 \cdot ((u)_1)$ and $w(2) = a_1 \cdot ((u)_2)$ and $w(3) = a_1 \cdot ((u)_3)$. $\text{len } f_3 =$ width M and $\text{len } f_7 = \text{len } M$ and $\text{len } f_3 = \text{len } M$ and $\text{len } f_7 =$ width M and $\text{len } f_3 > 0$ and $\text{len } f_7 > 0$. \square

- (35) Let us consider elements P, Q, R of the absolute, and points P_1, P_2, P_3, P_4 of the real projective plane. Suppose P, Q, R are mutually different and $P_1 = P$ and $P_2 = Q$ and $P_3 = R$ and $P_4 \in \text{tangent}(P)$ and $P_4 \in \text{tangent}(Q)$. Then
- (i) P_1, P_2 and P_3 are not collinear, and
 - (ii) P_1, P_2 and P_4 are not collinear, and
 - (iii) P_1, P_3 and P_4 are not collinear, and
 - (iv) P_2, P_3 and P_4 are not collinear.

PROOF: $P_4 \notin$ the absolute. Consider p being an element of the real projective plane such that $p = P$ and $\text{tangent}(P) = \text{Line}(p, \text{PoleInfty}(P))$. Consider q being an element of the real projective plane such that $q = Q$ and $\text{tangent}(Q) = \text{Line}(q, \text{PoleInfty}(Q))$. P_1, P_2 and P_4 are not collinear. P_1, P_3 and P_4 are not collinear. P_2, P_3 and P_4 are not collinear. \square

- (36) Let us consider elements P, Q of the absolute, an element R of the real projective plane, and non zero elements u, v, w of \mathcal{E}_T^3 . Suppose $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $R \in \text{tangent}(P)$ and $R \in \text{tangent}(Q)$ and $u(3) = 1$ and $v(3) = 1$ and $w(3) = 0$. Then
- (i) $P = Q$, or

(ii) $u(1) = -v(1)$ and $u(2) = -v(2)$.

The theorem is a consequence of (34).

(37) Let us consider an element P of the absolute, an element R of the real projective plane, and a non zero element u of \mathcal{E}_T^3 . Suppose $R \in \text{tangent}(P)$ and $R =$ the direction of u and $u(3) = 0$. Then $R = \text{PoleInfty}(P)$. The theorem is a consequence of (34).

(38) Let us consider a non zero real number a , and an invertible square matrix N over \mathbb{R}_F of dimension 3. Suppose $N = \text{symmetric3}(a, a, -a, 0, 0, 0)$. Then (the homography of N) $^\circ$ (the absolute) = the absolute.

PROOF: (The homography of N) $^\circ$ (the absolute) \subseteq the absolute by [8, (8)]. The absolute \subseteq (the homography of N) $^\circ$ (the absolute) by [11, (4), (3)], [7, (89)]. \square

(39) Let us consider a non zero element r_1 of \mathbb{R}_F , and invertible square matrices M, O over \mathbb{R}_F of dimension 3. Suppose $O = \text{symmetric3}(1, 1, -1, 0, 0, 0)$ and $M = r_1 \cdot O$. Then (the homography of M) $^\circ$ (the absolute) = the absolute. PROOF: $r_1 \neq 0$ by [14, (34)]. \square

(40) Let us consider an element P of the absolute. Then $\text{tangent}(P)$ misses the BK-model. The theorem is a consequence of (29), (23), and (30).

(41) Let us consider elements P, P_3, P_4 of the real projective plane, elements P_1, P_2 of the absolute, and an element Q of the real projective plane. Suppose $P_1 \neq P_2$ and $P_3 = P_1$ and $P_4 = P_2$ and $P \in$ the BK-model and P, P_3 and P_4 are collinear and $Q \in \text{tangent}(P_1)$ and $Q \in \text{tangent}(P_2)$. Then there exists an element R of the real projective plane such that

- (i) $R \in$ the absolute, and
- (ii) P, Q and R are collinear.

The theorem is a consequence of (40), (7), (37), (28), and (26).

(42) Let us consider elements P, R, S of the real projective plane, and an element Q of the absolute. Suppose $P \in$ the BK-model and $R \in \text{tangent}(Q)$ and P, S and R are collinear and $R \neq S$. Then $Q \neq S$. The theorem is a consequence of (29), (23), and (30).

3. SUBGROUP OF K -ISOMETRY

Let h be an element of EnsHomography3 . We say that h is K -isometry if and only if

(Def. 6) there exists an invertible square matrix N over \mathbb{R}_F of dimension 3 such that $h =$ the homography of N and (the homography of N) $^\circ$ (the absolute) = the absolute.

Now we state the proposition:

(43) Let us consider an element h of EnsHomography3 .

Suppose $h =$ the homography of $I_{\mathbb{R}_F}^{3 \times 3}$. Then h is K -isometry.

The set of K -isometries yielding a non empty subset of EnsHomography3 is defined by the term

(Def. 7) $\{h, \text{ where } h \text{ is an element of } \text{EnsHomography3} : h \text{ is } K\text{-isometry}\}$.

The subgroup of K -isometries yielding a strict subgroup of GroupHomography3 is defined by

(Def. 8) the carrier of $it =$ the set of K -isometries.

Now we state the propositions:

(44) Let us consider an element h of the set of K -isometries, and an invertible square matrix N over \mathbb{R}_F of dimension 3. Suppose $h =$ the homography of N . Then $(\text{the homography of } N)^\circ(\text{the absolute}) =$ the absolute.

(45) (i) the homography of $I_{\mathbb{R}_F}^{3 \times 3} = \mathbf{1}_{\text{GroupHomography3}}$, and

(ii) the homography of $I_{\mathbb{R}_F}^{3 \times 3} = \mathbf{1}_\alpha$,
where α is the subgroup of K -isometries.

(46) Let us consider invertible square matrices N_1, N_2 over \mathbb{R}_F of dimension 3, and elements h_1, h_2 of the subgroup of K -isometries. Suppose $h_1 =$ the homography of N_1 and $h_2 =$ the homography of N_2 . Then

(i) $h_1 \cdot h_2$ is an element of the subgroup of K -isometries, and

(ii) $h_1 \cdot h_2 =$ the homography of $N_1 \cdot N_2$.

(47) Let us consider an invertible square matrix N over \mathbb{R}_F of dimension 3, and an element h of the subgroup of K -isometries.

Suppose $h =$ the homography of N . Then

(i) $h^{-1} =$ the homography of N^\smile , and

(ii) the homography of N^\smile is an element of the subgroup of K -isometries.

The theorem is a consequence of (45).

(48) Let us consider an element s of the projective space over \mathcal{E}_T^3 , and elements p, q, r of the absolute. Suppose p, q, r are mutually different and $s \in \text{tangent}(p) \cap \text{tangent}(q)$. Then there exists an invertible square matrix N over \mathbb{R}_F of dimension 3 such that

(i) $(\text{the homography of } N)^\circ(\text{the absolute}) =$ the absolute, and

(ii) $(\text{the homography of } N)(\text{Dir101}) = p$, and

(iii) $(\text{the homography of } N)(\text{Dirm101}) = q$, and

(iv) $(\text{the homography of } N)(\text{Dir011}) = r$, and

(v) (the homography of N)(Dir010) = s .

PROOF: Reconsider $P_1 = p, P_2 = q, P_3 = r, P_4 = s$ as a point of the real projective plane. P_1, P_2 and P_3 are not collinear and P_1, P_2 and P_4 are not collinear and P_1, P_3 and P_4 are not collinear and P_2, P_3 and P_4 are not collinear. Consider N being an invertible square matrix over \mathbb{R}_F of dimension 3 such that (the homography of N)(Dir101) = P_1 and (the homography of N)(Dir101) = P_2 and (the homography of N)(Dir011) = P_3 and (the homography of N)(Dir010) = P_4 . Consider $n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9$ being elements of \mathbb{R}_F such that $N = \langle \langle n_1, n_2, n_3 \rangle, \langle n_4, n_5, n_6 \rangle, \langle n_7, n_8, n_9 \rangle \rangle$. Reconsider $b = -1$ as an element of \mathbb{R}_F . Reconsider $a = 1$ as an element of \mathbb{R}_F . Reconsider $a = 1, b = -1$ as a non zero element of \mathbb{R}_F . Reconsider $N_1 = \langle \langle a, 0, 0 \rangle, \langle 0, a, 0 \rangle, \langle 0, 0, b \rangle \rangle$ as an invertible square matrix over \mathbb{R}_F of dimension 3. Reconsider $M = N^T \cdot N_1 \cdot N$ as an invertible square matrix over \mathbb{R}_F of dimension 3. Consider $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9$ being elements of \mathbb{R}_F such that $M = \langle \langle v_1, v_2, v_3 \rangle, \langle v_4, v_5, v_6 \rangle, \langle v_7, v_8, v_9 \rangle \rangle$. Reconsider $r_1 = v_1, r_2 = v_2, r_3 = v_3, r_4 = v_5, r_5 = v_6, r_6 = v_9$ as a real number. Consider Q being a point of the projective space over \mathcal{E}_T^3 such that Dir101 = Q and for every element u of \mathcal{E}_T^3 such that u is not zero and $Q =$ the direction of u holds $\text{qfconic}(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. Consider Q being a point of the projective space over \mathcal{E}_T^3 such that Dir101 = Q and for every element u of \mathcal{E}_T^3 such that u is not zero and $Q =$ the direction of u holds $\text{qfconic}(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. Consider Q being a point of the projective space over \mathcal{E}_T^3 such that Dir011 = Q and for every element u of \mathcal{E}_T^3 such that u is not zero and $Q =$ the direction of u holds $\text{qfconic}(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. $r_3 = 0$ and $r_1 = -r_6$ and $r_2 = 0$ and $r_5 = 0$ and $r_1 = r_4$. $r_1 \neq 0$. (The homography of M) $^\circ$ (the absolute) = the absolute. \square

(49) Let us consider elements $p_1, q_1, r_1, p_2, q_2, r_2$ of the absolute, and elements s_1, s_2 of the real projective plane. Suppose p_1, q_1, r_1 are mutually different and p_2, q_2, r_2 are mutually different and $s_1 \in \text{tangent}(p_1) \cap \text{tangent}(q_1)$ and $s_2 \in \text{tangent}(p_2) \cap \text{tangent}(q_2)$. Then there exists an invertible square matrix N over \mathbb{R}_F of dimension 3 such that

- (i) (the homography of N) $^\circ$ (the absolute) = the absolute, and
- (ii) (the homography of N)(p_1) = p_2 , and
- (iii) (the homography of N)(q_1) = q_2 , and
- (iv) (the homography of N)(r_1) = r_2 , and
- (v) (the homography of N)(s_1) = s_2 .

The theorem is a consequence of (48) and (47).

(50) Let us consider elements $p_1, q_1, r_1, p_2, q_2, r_2$ of the absolute. Suppose p_1, q_1, r_1 are mutually different and p_2, q_2, r_2 are mutually different. Then there exists an invertible square matrix N over \mathbb{R}_F of dimension 3 such that

- (i) (the homography of N) $^\circ$ (the absolute) = the absolute, and
- (ii) (the homography of N)(p_1) = p_2 , and
- (iii) (the homography of N)(q_1) = q_2 , and
- (iv) (the homography of N)(r_1) = r_2 .

The theorem is a consequence of (33), (48), and (47).

(51) Let us consider a collinearity space C , and elements p, q, r, s of C . If $\text{Line}(p, q) = \text{Line}(r, s)$, then r, s and p are collinear.

(52) Let us consider a collinearity space C , and elements p, q of C . Then $\text{Line}(p, q) = \text{Line}(q, p)$.

PROOF: $\text{Line}(p, q) \subseteq \text{Line}(q, p)$. $\text{Line}(q, p) \subseteq \text{Line}(p, q)$. \square

(53) Let us consider an invertible square matrix N over \mathbb{R}_F of dimension 3, and elements p, q, r, s of the projective space over \mathcal{E}_T^3 .

Suppose $\text{Line}(\text{(the homography of } N)(p), \text{(the homography of } N)(q)) = \text{Line}(\text{(the homography of } N)(r), \text{(the homography of } N)(s))$. Then

- (i) p, q and r are collinear, and
- (ii) p, q and s are collinear, and
- (iii) r, s and p are collinear, and
- (iv) r, s and q are collinear.

The theorem is a consequence of (51) and (52).

Let us consider an invertible square matrix N over \mathbb{R}_F of dimension 3 and elements $p, q, r, s, t, u, n_1, n_2, n_3, n_4$ of the real projective plane. Now we state the propositions:

(54) Suppose $p \neq q$ and $r \neq s$ and $n_1 \neq n_2$ and $n_3 \neq n_4$ and p, q and t are collinear and r, s and t are collinear and $n_1 = \text{(the homography of } N)(p)$ and $n_2 = \text{(the homography of } N)(q)$ and $n_3 = \text{(the homography of } N)(r)$ and $n_4 = \text{(the homography of } N)(s)$ and n_1, n_2 and u are collinear and n_3, n_4 and u are collinear. Then

- (i) $u = \text{(the homography of } N)(t)$, or
- (ii) $\text{Line}(n_1, n_2) = \text{Line}(n_3, n_4)$.

(55) Suppose $p \neq q$ and $r \neq s$ and $n_1 \neq n_2$ and $n_3 \neq n_4$ and p, q and t are collinear and r, s and t are collinear and $n_1 =$ (the homography of N)(p) and $n_2 =$ (the homography of N)(q) and $n_3 =$ (the homography of N)(r) and $n_4 =$ (the homography of N)(s) and n_1, n_2 and u are collinear and n_3, n_4 and u are collinear and p, q and r are not collinear. Then $u =$ (the homography of N)(t). The theorem is a consequence of (54) and (53).

(56) Let us consider elements p, q of the absolute, and elements a, b of the BK-model. Then there exists an invertible square matrix N over \mathbb{R}_F of dimension 3 such that

- (i) (the homography of N) $^\circ$ (the absolute) = the absolute, and
- (ii) (the homography of N)(a) = b , and
- (iii) (the homography of N)(p) = q .

PROOF: Consider p' being an element of the absolute such that $p \neq p'$ and p, a and p' are collinear. Consider q' being an element of the absolute such that $q \neq q'$ and q, b and q' are collinear. Consider t being an element of the real projective plane such that $\text{tangent}(p) \cap \text{tangent}(p') = \{t\}$. Consider u being an element of the real projective plane such that $\text{tangent}(q) \cap \text{tangent}(q') = \{u\}$. Reconsider $a' = a$ as an element of the real projective plane. Consider R_1 being an element of the real projective plane such that $R_1 \in$ the absolute and a', t and R_1 are collinear. Reconsider $b' = b$ as an element of the real projective plane. Consider R_2 being an element of the real projective plane such that $R_2 \in$ the absolute and b', u and R_2 are collinear. p, p', R_1 are mutually different. Consider N being an invertible square matrix over \mathbb{R}_F of dimension 3 such that (the homography of N) $^\circ$ (the absolute) = the absolute and (the homography of N)(p) = q and (the homography of N)(p') = q' and (the homography of N)(R_1) = R_2 and (the homography of N)(t) = u . Reconsider $p_5 = p, p_6 = p', p_7 = R_1, p_8 = t, p_9 = a, n_1 = q, n_2 = q', n_3 = R_2, n_4 = u, n_5 = b$ as an element of the real projective plane. $n_5 =$ (the homography of N)(p_9). \square

(57) Let us consider elements p, q, r, s of the absolute. Suppose p, q, r are mutually different and q, p, s are mutually different. Then there exists an invertible square matrix N over \mathbb{R}_F of dimension 3 such that

- (i) (the homography of N) $^\circ$ (the absolute) = the absolute, and
- (ii) (the homography of N)(p) = q , and
- (iii) (the homography of N)(q) = p , and
- (iv) (the homography of N)(r) = s , and

- (v) for every element t of the real projective plane such that $t \in \text{tangent}(p) \cap \text{tangent}(q)$ holds (the homography of N)(t) = t .

The theorem is a consequence of (33), (48), and (47).

Let us consider elements P, Q of the BK-model. Now we state the propositions:

- (58) Suppose $P \neq Q$. Then there exist elements P_1, P_2, P_3, P_4 of the absolute and there exists an element P_5 of the projective space over \mathcal{E}_T^3 such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, P_5 and P_3 are collinear and Q, P_5 and P_4 are collinear and P_1, P_2, P_3 are mutually different and P_1, P_2, P_4 are mutually different and $P_5 \in \text{tangent}(P_1) \cap \text{tangent}(P_2)$. The theorem is a consequence of (20), (32), (41), (30), (42), (29), (40), and (7).
- (59) Suppose $P \neq Q$. Then there exists an invertible square matrix N over \mathbb{R}_F of dimension 3 such that
- (i) (the homography of N) $^\circ$ (the absolute) = the absolute, and
 - (ii) (the homography of N)(P) = Q , and
 - (iii) (the homography of N)(Q) = P , and
 - (iv) there exist elements P_1, P_2 of the absolute such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and (the homography of N)(P_1) = P_2 and (the homography of N)(P_2) = P_1 .

PROOF: Consider P_1, P_2, P_3, P_4 being elements of the absolute, P_5 being an element of the projective space over \mathcal{E}_T^3 such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, P_5 and P_3 are collinear and Q, P_5 and P_4 are collinear and P_1, P_2, P_3 are mutually different and P_1, P_2, P_4 are mutually different and $P_5 \in \text{tangent}(P_1) \cap \text{tangent}(P_2)$. Consider N_1 being an invertible square matrix over \mathbb{R}_F of dimension 3 such that (the homography of N_1) $^\circ$ (the absolute) = the absolute and (the homography of N_1)(Dir101) = P_1 and (the homography of N_1)(Dir101) = P_2 and (the homography of N_1)(Dir011) = P_3 and (the homography of N_1)(Dir010) = P_5 . Consider N_2 being an invertible square matrix over \mathbb{R}_F of dimension 3 such that (the homography of N_2) $^\circ$ (the absolute) = the absolute and (the homography of N_2)(Dir101) = P_2 and (the homography of N_2)(Dir101) = P_1 and (the homography of N_2)(Dir011) = P_4 and (the homography of N_2)(Dir010) = P_5 . Reconsider $N = N_2 \cdot (N_1)^\smile$ as an invertible square matrix over \mathbb{R}_F of dimension 3. Reconsider h_1 = the homography of N_1 as an element of EnsHomography3. Reconsider $h_5 = h_1$ as an element of the subgroup of K -isometries. Reconsider h_2 = the homography of N_2 as an element of EnsHomography3. Reconsider

$h_6 = h_2$ as an element of the subgroup of K -isometries. Reconsider $h_3 =$ the homography of $N_1 \smile$ as an element of EnsHomography3 . $h_5^{-1} = h_3$. Reconsider $h_7 = h_3$ as an element of the subgroup of K -isometries. Reconsider $h_4 = h_6 \cdot h_7$ as an element of the subgroup of K -isometries. Consider h being an element of EnsHomography3 such that $h_4 = h$ and h is K -isometry. (the homography of N)(P) = Q and (the homography of N)(Q) = P by [5, (102), (57)], [6, (15)]. \square

4. MAIN LEMMAS

Now we state the propositions:

- (60) Let us consider elements P, Q of the BK-model. Then there exists an element h of the subgroup of K -isometries and there exists an invertible square matrix N over \mathbb{R}_F of dimension 3 such that $h =$ the homography of N and (the homography of N)(P) = Q and (the homography of N)(Q) = P . The theorem is a consequence of (43) and (59).
- (61) Let us consider elements P, Q, R, S, T, U of the BK-model. Suppose there exist elements h_1, h_2 of the subgroup of K -isometries and there exist invertible square matrices N_1, N_2 over \mathbb{R}_F of dimension 3 such that $h_1 =$ the homography of N_1 and $h_2 =$ the homography of N_2 and (the homography of N_1)(P) = R and (the homography of N_1)(Q) = S and (the homography of N_2)(R) = T and (the homography of N_2)(S) = U . Then there exists an element h_3 of the subgroup of K -isometries and there exists an invertible square matrix N_3 over \mathbb{R}_F of dimension 3 such that $h_3 =$ the homography of N_3 and (the homography of N_3)(P) = T and (the homography of N_3)(Q) = U . The theorem is a consequence of (46).
- (62) Let us consider elements P, Q, R of the BK-model, an element h of the subgroup of K -isometries, and an invertible square matrix N over \mathbb{R}_F of dimension 3. Suppose $h =$ the homography of N and (the homography of N)(P) = R and (the homography of N)(Q) = R . Then $P = Q$.

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