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Kleene Algebra of Partial Predicates

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Summary. We show that the set of all partial predicates over a set D together with the disjunction, conjunction, and negation operations, defined in accordance with the truth tables of S.C. Kleene's strong logic of indeterminacy [17], forms a Kleene algebra. A Kleene algebra is a De Morgan algebra [3] (also called quasi-Boolean algebra) which satisfies the condition $x \wedge \neg x \leqslant y \vee \neg y$ (sometimes called the normality axiom). We use the formalization of De Morgan algebras from [8].

The term "Kleene algebra" was introduced by A. Monteiro and D. Brignole in [3]. A similar notion of a "normal i-lattice" had been previously studied by J.A. Kalman [16]. More details about the origin of this notion and its relation to other notions can be found in [24, 4, 1, 2]. It should be noted that there is a different widely known class of algebras, also called Kleene algebras [22, 6], which generalize the algebra of regular expressions, however, the term "Kleene algebra" used in this paper does not refer to them.

Algebras of partial predicates naturally arise in computability theory in the study on partial recursive predicates. They were studied in connection with non-classical logics [17, 5, 18, 32, 29, 30]. A partial predicate also corresponds to the notion of a partial set [26] on a given domain, which represents a (partial) property which for any given element of this domain may hold, not hold, or neither hold nor not hold. The field of all partial sets on a given domain is an algebra with generalized operations of union, intersection, complement, and three constants (0, 1, n which is the fixed point of complement) which can be generalized to an equational class of algebras called DMF-algebras (De Morgan algebras with a single fixed point of involution) [25]. In [27] partial sets and DMF-algebras were considered as a basis for unification of set-theoretic and linguistic approaches to probability.

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Partial predicates over classes of mathematical models of data were used for formalizing semantics of computer programs in the composition-nominative approach to program formalization [31, 28, 33, 15], for formalizing extensions of the Floyd-Hoare logic [7, 9] which allow reasoning about properties of programs in the case of partial pre- and postconditions [23, 20, 19, 21], for formalizing dynamical models with partial behaviors in the context of the mathematical systems theory [11, 13, 14, 12, 10].

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1. Partial Predicates

From now on x denotes an object and D denotes a set.

Let us consider D. The functor $\mathrm{Pr}(D)$ yielding a set is defined by the term

(Def. 1) $D \dot{\rightarrow} \mathit{Boolean}$.

Observe that $\mathrm{Pr}(D)$ is non empty and functional.

A partial predicate of D is a partial function from D to *Boolean*. From now on p denotes a partial predicate of D.

Now we state the propositions:

(1) If $x \in \mathrm{Pr}(D)$, then x is a partial predicate of D.

(2) $p \in \mathrm{Pr}(D)$.

(3) If $x \in \mathrm{dom}\, p$, then $p(x) = \mathit{true}$ or $p(x) = \mathit{false}$.

Let us consider D. The functor $\mathrm{PPneg}(D)$ yielding a function from $\mathrm{Pr}(D)$ into $\mathrm{Pr}(D)$ is defined by

(Def. 2) for every partial predicate p of D, $\mathrm{dom}(\mathit{it}(p)) = \mathrm{dom}\, p$ and for every object d, if $d \in \mathrm{dom}\, p$ and $p(d) = \mathit{true}$, then $\mathit{it}(p)(d) = \mathit{false}$ and if $d \in \mathrm{dom}\, p$ and $p(d) = \mathit{false}$, then $\mathit{it}(p)(d) = \mathit{true}$.

Let us consider p. The functor $\neg p$ yielding a partial predicate of D is defined by the term

(Def. 3) $(\mathrm{PPneg}(D))(p)$.

Let us note that the functor is involutive.

Now we state the propositions:

(4) If $x \in \mathrm{dom}\, p$ and $(\neg p)(x) = \mathit{false}$, then $p(x) = \mathit{true}$. The theorem is a consequence of (3).

(5) If $x \in \mathrm{dom}\, p$ and $(\neg p)(x) = \mathit{true}$, then $p(x) = \mathit{false}$. The theorem is a consequence of (3).

(6) If $x \in \operatorname{dom} \neg p$ and $(\neg p)(x) = \mathit{false}$, then $x \in \operatorname{dom} p$ and $p(x) = \mathit{true}$. The theorem is a consequence of (3).

(7) If $x \in \operatorname{dom} \neg p$ and $(\neg p)(x) = \mathit{true}$, then $x \in \operatorname{dom} p$ and $p(x) = \mathit{false}$. The theorem is a consequence of (3).

In the sequel D denotes a non empty set and p, q, r denote partial predicates of D.

Let us consider D. The functor $\mathrm{PPdisj}(D)$ yielding a function from $\mathrm{Pr}(D) \times \mathrm{Pr}(D)$ into $\mathrm{Pr}(D)$ is defined by

(Def. 4) for every partial predicates p, q of D, $\operatorname{dom} \mathit{it}(p, q) = \{d$, where d is an element of $D : d \in \operatorname{dom} p$ and $p(d) = \mathit{true}$ or $d \in \operatorname{dom} q$ and $q(d) = \mathit{true}$ or $d \in \operatorname{dom} p$ and $p(d) = \mathit{false}$ and $d \in \operatorname{dom} q$ and $q(d) = \mathit{false}\}$ and for every object d, if $d \in \operatorname{dom} p$ and $p(d) = \mathit{true}$ or $d \in \operatorname{dom} q$ and $q(d) = \mathit{true}$, then $\mathit{it}(p, q)(d) = \mathit{true}$ and if $d \in \operatorname{dom} p$ and $p(d) = \mathit{false}$ and $d \in \operatorname{dom} q$ and $q(d) = \mathit{false}$, then $\mathit{it}(p, q)(d) = \mathit{false}$.

Let us consider p and q. The functor $p \vee q$ yielding a partial predicate of D is defined by the term

(Def. 5) $(\mathrm{PPdisj}(D))(p, q)$.

Observe that the functor is commutative and idempotent.

Now we state the propositions:

(8) Suppose $x \in \operatorname{dom}(p \vee q)$. Then

 (i) $x \in \operatorname{dom} p$ and $p(x) = \mathit{true}$, or

 (ii) $x \in \operatorname{dom} q$ and $q(x) = \mathit{true}$, or

 (iii) $x \in \operatorname{dom} p$ and $p(x) = \mathit{false}$ and $x \in \operatorname{dom} q$ and $q(x) = \mathit{false}$.

(9) If $x \in \operatorname{dom} p$ and $x \in \operatorname{dom} q$ and $(p \vee q)(x) = \mathit{true}$, then $p(x) = \mathit{true}$ or $q(x) = \mathit{true}$. The theorem is a consequence of (3).

(10) If $x \in \operatorname{dom}(p \vee q)$ and $(p \vee q)(x) = \mathit{true}$, then $x \in \operatorname{dom} p$ and $p(x) = \mathit{true}$ or $x \in \operatorname{dom} q$ and $q(x) = \mathit{true}$. The theorem is a consequence of (8) and (9).

(11) If $x \in \operatorname{dom} p$ and $(p \vee q)(x) = \mathit{false}$, then $p(x) = \mathit{false}$. The theorem is a consequence of (3).

(12) If $x \in \operatorname{dom} q$ and $(p \vee q)(x) = \mathit{false}$, then $q(x) = \mathit{false}$. The theorem is a consequence of (3).

(13) If $x \in \operatorname{dom}(p \vee q)$ and $(p \vee q)(x) = \mathit{false}$, then $x \in \operatorname{dom} p$ and $p(x) = \mathit{false}$ and $x \in \operatorname{dom} q$ and $q(x) = \mathit{false}$. The theorem is a consequence of (8) and (12).

(14) Associativity law:
$p \vee (q \vee r) = (p \vee q) \vee r$. The theorem is a consequence of (8) and (11).

(15) $(p \vee q) \vee (p \vee r) = (p \vee q) \vee r$. The theorem is a consequence of (14).

Let us consider D, p, and q. The functor $p \wedge q$ yielding a partial predicate of D is defined by the term

(Def. 6) $\neg(\neg p \vee \neg q)$.

Observe that the functor is commutative and idempotent. The functor $p \Rightarrow q$ yielding a partial predicate of D is defined by the term

(Def. 7) $\neg p \vee q$.

Now we state the propositions:

(16) $\operatorname{dom}(p \wedge q) = \{d$, where d is an element of $D : d \in \operatorname{dom} p$ and $p(d) = \mathit{false}$ or $d \in \operatorname{dom} q$ and $q(d) = \mathit{false}$ or $d \in \operatorname{dom} p$ and $p(d) = \mathit{true}$ and $d \in \operatorname{dom} q$ and $q(d) = \mathit{true}\}$. The theorem is a consequence of (5) and (4).

(17) Suppose $x \in \operatorname{dom}(p \wedge q)$. Then

(i) $x \in \operatorname{dom} p$ and $p(x) = \mathit{false}$, or

(ii) $x \in \operatorname{dom} q$ and $q(x) = \mathit{false}$, or

(iii) $x \in \operatorname{dom} p$ and $p(x) = \mathit{true}$ and $x \in \operatorname{dom} q$ and $q(x) = \mathit{true}$.

The theorem is a consequence of (16).

(18) If $x \in \operatorname{dom} p$ and $p(x) = \mathit{true}$ and $x \in \operatorname{dom} q$ and $q(x) = \mathit{true}$, then $(p \wedge q)(x) = \mathit{true}$.

(19) If $x \in \operatorname{dom} p$ and $p(x) = \mathit{false}$, then $(p \wedge q)(x) = \mathit{false}$.

(20) If $x \in \operatorname{dom} q$ and $q(x) = \mathit{false}$, then $(p \wedge q)(x) = \mathit{false}$.

(21) If $x \in \operatorname{dom} p$ and $(p \wedge q)(x) = \mathit{true}$, then $p(x) = \mathit{true}$.

(22) If $x \in \operatorname{dom} q$ and $(p \wedge q)(x) = \mathit{true}$, then $q(x) = \mathit{true}$.

(23) If $x \in \operatorname{dom}(p \wedge q)$ and $(p \wedge q)(x) = \mathit{true}$, then $x \in \operatorname{dom} p$ and $p(x) = \mathit{true}$ and $x \in \operatorname{dom} q$ and $q(x) = \mathit{true}$. The theorem is a consequence of (17) and (19).

(24) If $x \in \operatorname{dom} p$ and $x \in \operatorname{dom} q$ and $(p \wedge q)(x) = \mathit{false}$, then $p(x) = \mathit{false}$ or $q(x) = \mathit{false}$. The theorem is a consequence of (18) and (3).

(25) If $x \in \operatorname{dom}(p \wedge q)$ and $(p \wedge q)(x) = \mathit{false}$, then $x \in \operatorname{dom} p$ and $p(x) = \mathit{false}$ or $x \in \operatorname{dom} q$ and $q(x) = \mathit{false}$. The theorem is a consequence of (17) and (24).

(26) Associativity law:
$p \wedge (q \wedge r) = (p \wedge q) \wedge r$.

(27) $(p \wedge q) \wedge (p \wedge r) = (p \wedge q) \wedge r$.

(28) Meet-absorbing law:
$(p \wedge q) \vee q = q$. The theorem is a consequence of (16), (8), (17), (19), and (3).

(29) Join-absorbing law:
$p \wedge (p \vee q) = p$. The theorem is a consequence of (16), (17), (8), (3), (19), and (18).

(30) Distributivity law:
$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. The theorem is a consequence of (16), (17), (19), (13), (10), (18), (8), (23), and (25).

(31) $\mathrm{dom}(p \Rightarrow q) = \{d$, where d is an element of $D : d \in \mathrm{dom}\, p$ and $p(d) = \mathit{false}$ or $d \in \mathrm{dom}\, q$ and $q(d) = \mathit{true}$ or $d \in \mathrm{dom}\, p$ and $p(d) = \mathit{true}$ and $d \in \mathrm{dom}\, q$ and $q(d) = \mathit{false}\}$. The theorem is a consequence of (5) and (4).

(32) Suppose $x \in \mathrm{dom}(p \Rightarrow q)$. Then

(i) $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{false}$, or

(ii) $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{true}$, or

(iii) $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{true}$ and $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{false}$.

The theorem is a consequence of (31).

(33) If $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{false}$, then $(p \Rightarrow q)(x) = \mathit{true}$.

(34) If $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{true}$, then $(p \Rightarrow q)(x) = \mathit{true}$.

(35) If $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{true}$ and $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{false}$, then $(p \Rightarrow q)(x) = \mathit{false}$.

(36) If $x \in \mathrm{dom}\, p$ and $x \in \mathrm{dom}\, q$ and $(p \Rightarrow q)(x) = \mathit{true}$, then $p(x) = \mathit{false}$ or $q(x) = \mathit{true}$. The theorem is a consequence of (35) and (3).

(37) If $x \in \mathrm{dom}\, p$ and $(p \Rightarrow q)(x) = \mathit{false}$, then $p(x) = \mathit{true}$.

(38) If $x \in \mathrm{dom}\, q$ and $(p \Rightarrow q)(x) = \mathit{false}$, then $q(x) = \mathit{false}$.

(39) If $x \in \mathrm{dom}(p \Rightarrow q)$ and $(p \Rightarrow q)(x) = \mathit{false}$, then $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{true}$ and $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{false}$. The theorem is a consequence of (32) and (33).

(40) If $x \in \mathrm{dom}(p \Rightarrow q)$ and $(p \Rightarrow q)(x) = \mathit{true}$, then $x \in \mathrm{dom}\, p$ and $p(x) = \mathit{false}$ or $x \in \mathrm{dom}\, q$ and $q(x) = \mathit{true}$. The theorem is a consequence of (32) and (35).

(41) $(p \Rightarrow r) \wedge (q \Rightarrow r) = (p \vee q) \Rightarrow r$. The theorem is a consequence of (30).

(42) $(p \Rightarrow r) \vee (q \Rightarrow r) = (p \wedge q) \Rightarrow r$. The theorem is a consequence of (15) and (14).

Let D be a set. The functor $\mathrm{true}_{\mathrm{PP}}(D)$ yielding a partial predicate of D is defined by the term

(Def. 8) $D \longmapsto \mathit{true}$.

Let D be a set. The functor $\mathrm{false}_{\mathrm{PP}}(D)$ yielding a partial predicate of D is defined by the term

(Def. 9) $D \longmapsto \mathit{false}$.

Let us consider a set D. Now we state the propositions:

(43) $\neg\,\mathrm{false}_{\mathrm{PP}}(D) = \mathrm{true}_{\mathrm{PP}}(D)$.

(44) $\neg\,\mathrm{true}_{\mathrm{PP}}(D) = \mathrm{false}_{\mathrm{PP}}(D)$. The theorem is a consequence of (43).

Now we state the propositions:

(45) $p \vee \mathrm{true}_{\mathrm{PP}}(D) = \mathrm{true}_{\mathrm{PP}}(D)$.

(46) $\mathrm{true}_{\mathrm{PP}}(D) \vee p = \mathrm{true}_{\mathrm{PP}}(D)$.

(47) $p \wedge \mathrm{false}_{\mathrm{PP}}(D) = \mathrm{false}_{\mathrm{PP}}(D)$.

(48) $\mathrm{false}_{\mathrm{PP}}(D) \wedge p = \mathrm{false}_{\mathrm{PP}}(D)$.

(49) $p \vee \neg p = \mathrm{true}_{\mathrm{PP}}(D) \restriction \mathrm{dom}\, p$. The theorem is a consequence of (8) and (3).

(50) $\neg p \vee p = \mathrm{true}_{\mathrm{PP}}(D) \restriction \mathrm{dom}\, p$.

(51) $p \wedge \neg p = \mathrm{false}_{\mathrm{PP}}(D) \restriction \mathrm{dom}\, p$. The theorem is a consequence of (16), (17), (3), and (19).

(52) $\neg p \wedge p = \mathrm{false}_{\mathrm{PP}}(D) \restriction \mathrm{dom}\, p$.

(53) $\mathrm{false}_{\mathrm{PP}}(D) \Rightarrow p = \mathrm{true}_{\mathrm{PP}}(D)$. The theorem is a consequence of (43) and (45).

(54) $p \Rightarrow \mathrm{true}_{\mathrm{PP}}(D) = \mathrm{true}_{\mathrm{PP}}(D)$.

(55) $\mathrm{false}_{\mathrm{PP}}(D) \restriction \mathrm{dom}\, p \vee \mathrm{true}_{\mathrm{PP}}(D) \restriction \mathrm{dom}\, q = \mathrm{true}_{\mathrm{PP}}(D) \restriction \mathrm{dom}\, q$.

Let D be a set. The functor $\perp_{\mathrm{PP}}(D)$ yielding a partial predicate of D is defined by the term

(Def. 10) $\emptyset$.

Now we state the propositions:

(56) Let us consider a set D. Then $\neg\perp_{\mathrm{PP}}(D) = \perp_{\mathrm{PP}}(D)$.

(57) $\perp_{\mathrm{PP}}(D) \vee \mathrm{true}_{\mathrm{PP}}(D) = \mathrm{true}_{\mathrm{PP}}(D)$.

(58) $\perp_{\mathrm{PP}}(D) \wedge \mathrm{false}_{\mathrm{PP}}(D) = \mathrm{false}_{\mathrm{PP}}(D)$.

(59) $\perp_{\mathrm{PP}}(D) \Rightarrow \mathrm{true}_{\mathrm{PP}}(D) = \mathrm{true}_{\mathrm{PP}}(D)$. The theorem is a consequence of (56) and (57).

2. Algebra of Partial Connectives with (strong) Kleene Logical Connectives

Let us consider D. The functors: $\bigwedge_D$ and $\bigvee_D$ yielding binary operations on $\mathrm{Pr}(D)$ are defined by conditions

(Def. 11) for every partial predicates p, q of D, $\bigwedge_D(p, q) = p \wedge q$,

(Def. 12) for every partial predicates p, q of D, $\bigvee_D(p, q) = p \vee q$,

respectively.The functor $\bar{\cdot}_D$ yielding a unary operation on $\mathrm{Pr}(D)$ is defined by

(Def. 13) for every partial predicate p of D, $it(p) = \neg p$.

The functor PartPredLatt(D) yielding a strict ortholattice structure is defined by the term

(Def. 14) $\langle \mathrm{Pr}(D), \bigvee_D, \bigwedge_D, \bar{\cdot}_D \rangle$.

Let D be a non empty set, f, g be binary operations on D, and h be a unary operation on D. One can verify that $\langle D, f, g, h \rangle$ is non empty.

Let us consider D. Let us note that PartPredLatt(D) is non empty and constituted functions and there exists a lattice structure which is non empty and constituted functions and there exists an ortholattice structure which is strict, non empty, and constituted functions.

Let us consider D. One can verify that PartPredLatt(D) is lattice-like and PartPredLatt(D) is bounded and PartPredLatt(D) is de Morgan and join-idempotent and has idempotent element.

Now we state the propositions:

(60) $\top_{\mathrm{PartPredLatt}(D)} = \mathrm{true}_{\mathrm{PP}}(D)$.

(61) $\bot_{\mathrm{PartPredLatt}(D)} = \mathrm{false}_{\mathrm{PP}}(D)$.

Let L be a non empty, constituted functions lattice structure and a, b be elements of L. We say that a is a partial complement of b if and only if

(Def. 15) $a \sqcup b = \top_L \restriction \mathrm{dom}\, b$ and $b \sqcup a = \top_L \restriction \mathrm{dom}\, b$ and $a \sqcap b = \bot_L \restriction \mathrm{dom}\, b$ and $b \sqcap a = \bot_L \restriction \mathrm{dom}\, b$.

We say that L is partially complemented if and only if

(Def. 16) for every element b of L, there exists an element a of L such that a is a partial complement of b.

Let L be a constituted functions, non empty lattice structure. We say that L is partially Boolean if and only if

(Def. 17) L is bounded, partially complemented, and distributive.

One can verify that every constituted functions, non empty lattice structure which is partially Boolean is also bounded, partially complemented, and distributive and every constituted functions, non empty lattice structure which is bounded, partially complemented, and distributive is also partially Boolean.

Now we state the proposition:

(62) Let us consider elements a, b of PartPredLatt(D). If $a = p$ and $b = \neg p$, then b is a partial complement of a. The theorem is a consequence of (60), (49), (61), and (51).

Let us consider D. Note that PartPredLatt(D) is partially Boolean.

Now we state the proposition:

(63) Distributivity law:
$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.

Let L be a non empty ortholattice structure. We say that L is Kleene if and only if

(Def. 18) for every elements x, y of L, $x \sqcap x^{\mathrm{c}} \sqsubseteq y \sqcup y^{\mathrm{c}}$.

Let us observe that every meet-absorbing, join-absorbing, meet-commutative, non empty ortholattice structure which is Boolean and well-complemented is also Kleene.

Let us consider D. Observe that PartPredLatt(D) is Kleene and there exists a non empty, constituted functions lattice structure which is partially Boolean, join-idempotent, and lattice-like and there exists a non empty ortholattice structure which is Kleene, de Morgan, join-idempotent, lattice-like, and strict and has idempotent element and there exists a non empty, constituted functions ortholattice structure which is partially Boolean, Kleene, de Morgan, join-idempotent, lattice-like, and strict and has idempotent element.

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