Introduction to Stochastic Finance: Random Variables and Arbitrage Theory

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Summary. Using the Mizar system [1], [5], we start to show, that the Call-Option, the Put-Option and the Straddle (more generally defined as in the literature) are random variables ([4], p. 15), see (Def. 1) and (Def. 2). Next we construct and prove the simple random variables ([2], p. 14) in (Def. 8).

In the third section, we introduce the definition of arbitrage opportunity, see (Def. 12). Next we show, that this definition can be characterized in a different way (Lemma 1.3. in [4], p. 5), see (17). In our formalization for Lemma 1.3 we make the assumption that $\varphi$ is a sequence of real numbers (there are only finitely many valued of interest, the values of $\varphi$ in $R^d$). For the definition of almost sure with probability 1 see p. 6 in [2]. Last we introduce the risk-neutral probability (Definition 1.4, p. 6 in [4]), here see (Def. 16).

We give an example in real world: Suppose you have some assets like bonds (riskless assets). Then we can fix our price for these bonds with $x$ for today and $x \cdot (1 + r)$ for tomorrow, $r$ is the interest rate. So we simply assume, that in every possible market evolution of tomorrow we have a determinated value. Then every probability measure of $\Omega_{fut1}$ is a risk-neutral measure, see (21). This example shows the existence of some risk-neutral measure. If you find more than one of them, you can determine – with an additional condition to the probability measures – whether a market model is arbitrage free or not (see Theorem 1.6. in [4], p. 6.)

A short graph for (21):
Suppose we have a portfolio with many (in this example infinitely many) assets. For asset $d$ we have the price $\pi(d)$ for today, and the price $\pi(d) \cdot (1 + r)$ for tomorrow with some interest rate $r > 0$.

Let $G$ be a sequence of random variables on $\Omega_{fut1}$, Borel sets. So you have many functions $f_k : \{1, 2, 3, 4\} \to R$ with $G(k) = f_k$ and $f_k$ is a random variable of $\Omega_{fut1}$, Borel sets. For every $f_k$ we have $f_k(w) = \pi(k) \cdot (1 + r)$ for $w \in \{1, 2, 3, 4\}$.
Only one scenario: \[
\{ w_{21} = \{1, 2\}, w_{22} = \{3, 4\}, \]
for all \( d \in \mathbb{N} \) holds \( \pi(d) \)
\[
\begin{cases} 
  f_d(w) = G(d)(w) = \pi(d) \cdot (1 + r), \\
  w \in w_{21} \text{ or } w \in w_{22}, \\
  r > 0 \text{ is the interest rate.}
\end{cases}
\]

Here, every probability measure of \( \Omega_{fut1} \) is a risk-neutral measure.

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1. Put-Option, Call-Option and Straddle are Random Variables

From now on \( \Omega \) denotes a non empty set and \( F \) denotes a \( \sigma \)-field of subsets of \( \Omega \).

Now we state the propositions:

1. \( ]0, +\infty[ \) is an element of the Borel sets.
2. Let us consider a random variable \( R \) of \( F \) and the Borel sets, an element \( K \) of \( \mathbb{R} \), and a function \( g \) from \( \Omega \) into \( \mathbb{R} \). Suppose \( g = \chi_{(R-(\Omega \mapsto K))^{-1}(\{0, +\infty\}, \Omega)} \). Then \( \text{Call-Option}(R, K) = g \cdot (R - (\Omega \mapsto K)) \).
3. Let us consider a random variable \( R \) of \( F \) and the Borel sets, and a real number \( K \). Then \( (\Omega \mapsto K) - R \) is a random variable of \( F \) and the Borel sets.
4. Let us consider an element \( A \) of \( F \). Then \( \chi_{A, \Omega} \) is a random variable of \( F \) and the Borel sets.
5. \( \chi_{\Omega, \Omega} \) is random variable on \( F \) and the Borel sets. The theorem is a consequence of (4).
6. Let us consider random variables \( f, R \) of \( F \) and the Borel sets, and a real number \( K \). Then \( (f - R)^{-1}(\{0, +\infty\}) \) is an element of \( F \). The theorem is a consequence of (1).

Let us consider \( \Omega \) and \( F \). Let \( R \) be a random variable of \( F \) and the Borel sets and \( K \) be a real number. Let us note that the functor \( \text{Call-Option}(R, K) \) yields a random variable of \( F \) and the Borel sets. The functor \( \text{Put-Option}(R, K) \) yielding a function from \( \Omega \) into \( \mathbb{R} \) is defined by

(Def. 1) for every element \( w \) of \( \Omega \), if \( ((\Omega \mapsto K) - R)(w) \geq 0 \), then \( it(w) = ((\Omega \mapsto K) - R)(w) \) and if \( ((\Omega \mapsto K) - R)(w) < 0 \), then \( it(w) = 0 \).
Now we state the proposition:

(7) Let us consider a random variable $R$ of $F$ and the Borel sets, a real number $K$, and a function $g$ from $\Omega$ into $\mathbb{R}$. Suppose $g = \chi_{((\Omega \mapsto -\rightarrow K) - R)^{-1}([0, +\infty]), \Omega}$. Then \( \text{Put-Option}(R, K) = g \cdot (\Omega \mapsto K - R) \).

Let us consider $\Omega$ and $F$. Let $R$ be a random variable of $F$ and the Borel sets and $K$ be a real number. Note that the functor $\text{Put-Option}(R, K)$ yields a random variable of $F$ and the Borel sets.

2. Simple Random Variables

Let us consider $\Omega$ and $F$. Let $R$ be a random variable of $F$ and the Borel sets and $K$ be a real number. The functor $\text{Straddle}(R, K)$ yielding a random variable of $F$ and the Borel sets is defined by the term

(Def. 2) $\text{Put-Option}(R, K) + \text{Call-Option}(R, K)$.

Now we state the proposition:

(8) Let us consider a random variable $R$ of $F$ and the Borel sets, a real number $K$, and an element $w$ of $\Omega$. Then \((\text{Straddle}(R, K))(w) = |(R - (\Omega \mapsto K))(w)|\).

Let us consider $\Omega$ and $F$. The functors: the set of constants $F$ and the set of $\chi_F$ yielding sets are defined by terms

(Def. 3) \{ $f$, where $f$ is a function from $\Omega$ into $\mathbb{R}$ : $f$ is random variable on $F$ and the Borel sets and constant \},

(Def. 4) \{ $\chi_{A, \Omega}$, where $A$ is an element of $F$ : $\chi_{A, \Omega}$ is random variable on $F$ and the Borel sets \},

respectively. Let $X$ be a set. We say that $X$ is $F$-random membered if and only if

(Def. 5) for every object $x$ such that $x \in X$ there exists a function $f$ from $\Omega$ into $\mathbb{R}$ such that $f = x$ and $f$ is random variable on $F$ and the Borel sets.

Observe that the set of constants $F$ is non empty and the set of $\chi_F$ is non empty and the set of constants $F$ is $F$-random membered and the set of $\chi_F$ is $F$-random membered and there exists a set which is $F$-random membered and non empty.

Let $D$ be an $F$-random membered, non empty set, $C_1$ be a sequence of $D$, and $n$ be a natural number. The change type of $C_1$ and $n$ yielding a random variable of $F$ and the Borel sets is defined by the term

(Def. 6) $C_1(n)$.

Let $C_2$ be a sequence of $D$ and $w$ be an element of $\Omega$. The change all types of $C_2$ and $w$ yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by
(Def. 7) for every natural number $n$, $it(n) = (\text{the change type of } C_2 \text{ and } n)(w)$.

Let $D_1, D_2$ be $F$-random membered, non empty sets, $C_1$ be a sequence of $D_1$, $C_2$ be a sequence of $D_2$, and $n$ be a natural number. The simple $RV$ of $C_1$, $C_2$ and $n$ yielding a function from $\Omega$ into $\mathbb{R}$ is defined by

(Def. 8) for every element $w$ of $\Omega$, $it(w) = (\sum_{\alpha=0}^{\kappa} ((\text{the change all types of } C_2 \text{ and } w) \cdot (\text{the change all types of } C_1 \text{ and } w))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

Observe that the simple $RV$ of $C_1$, $C_2$ and $n$ yields a random variable of $F$ and the Borel sets.

3. Arbitrage Theory: Definition and Alternative Representation

From now on $\phi$ denotes a sequence of real numbers and $\pi$ denotes a price function.

Let us consider $\Omega$ and $F$. Let $q$ be a natural number and $G$ be a sequence of the set of random variables on $F$ and the Borel sets. The change element to functions $G$ and $q$ yielding a real-valued random variable on $F$ is defined by the term

(Def. 9) $G(q)$.

Let us consider $\phi$. Let $n$ be a natural number. The functors: the first $AO$-set of $\phi$, $\Omega$, $F$, $G$ and $n$ and the second $AO$-set of $\phi$, $\Omega$, $F$, $G$ and $n$ yielding elements of $F$ are defined by terms

(Def. 10) (the $RV$-portfolio value for future extension of $\phi$, $F$, $G$ and $n$)$^{-1}([0, +\infty[)$,
(Def. 11) (the $RV$-portfolio value for future extension of $\phi$, $F$, $G$ and $n$)$^{-1}([0, +\infty[)$,
respectively. Let $P$ be a probability on $F$ and $\pi$ be a price function. We say that there exists an $AO$ w.r.t. $P$, $G$, $\pi$ and $n$ if and only if

(Def. 12) there exists a sequence $\phi$ of real numbers such that the buy portfolio extension of $\phi$, $\pi$, and $n \leq 0$ and $P(\text{the first } AO\text{-set of } \phi, \Omega, F, G \text{ and } n) = 1$ and $P(\text{the second } AO\text{-set of } \phi, \Omega, F, G \text{ and } n) > 0$.

Let $r$ be a real number. The first $RV$ of $r$ yielding an element of the set of random variables on $\Omega_{now}$ and the Borel sets is defined by the term

(Def. 13) $\{1, 2, 3, 4\} \mapsto -\rightarrow r$.

Let $\pi$ be a price function and $d$ be a natural number. The first $RV$ of $\pi$, $r$ and $d$ yielding an element of the set of random variables on $\Omega_{fut1}$ and the Borel sets is defined by the term

(Def. 14) the first $RV$ of $\pi(d) \cdot (1 + r)$.

Now we state the propositions:

(9) There exists a sequence $G$ of the set of random variables on $\Omega_{now}$ and the Borel sets such that
Let us consider a probability $P$ on $\Omega_{\text{now}}$, and a sequence $G$ of the set of random variables on $\Omega_{\text{now}}$ and the Borel sets. Suppose $G(0) = \{1, 2, 3, 4\} \mapsto 1$ and $G(1) = \{1, 2, 3, 4\} \mapsto 5$ and for every natural number $k$ such that $k > 1$ holds $G(k) = \{1, 2, 3, 4\} \mapsto 0$. Then there exists a price function $\pi$ such that there exists an $\mathcal{AO}$ w.r.t. $P$, $G$, $\pi$ and 1.

**Proof:** Define $\mathcal{U}$ (natural number) $= (\$1 = 0 \mapsto$ the first $\mathcal{RV}$ of 1, $(\$1 = 1 \mapsto$ the first $\mathcal{RV}$ of 5, the first $\mathcal{RV}$ of 0)). Consider $f$ being a sequence of the set of random variables on $\Omega_{\text{now}}$ and the Borel sets such that for every element $d$ of $\mathbb{N}$, $f(d) = \mathcal{U}(d)$. $f(0) = (0 = 0 \mapsto$ the first $\mathcal{RV}$ of 1, $(0 = 1 \mapsto$ the first $\mathcal{RV}$ of 5, the first $\mathcal{RV}$ of 0)). $f(1) = (1 = 0 \mapsto$ the first $\mathcal{RV}$ of 1, $(1 = 1 \mapsto$ the first $\mathcal{RV}$ of 5, the first $\mathcal{RV}$ of 0)). For every natural number $k$ such that $k > 1$ holds $f(k) = \{1, 2, 3, 4\} \mapsto 0$. □

Let us consider a natural number $n$, a real number $r$, and a sequence $G$ of the set of random variables on $F$ and the Borel sets. Then $\{w$, where $w$ is an element of $\Omega$ : the portfolio value for future extension of $n, \varphi, F, G$ and $w \geq 0\} = \{\text{the } \mathcal{RV}\text{-portfolio value for future extension of } \varphi, F, G\text{ and } n\}^{-1}([0, +\infty[)$. The theorem is a consequence of (1).

Let us consider natural numbers $d$, $d_1$, a real number $r$, and a sequence $G$ of the set of random variables on $F$ and the Borel sets.

Suppose $d_1 = d - 1$. Then $\{w$, where $w$ is an element of $\Omega$ : the portfolio value for future of $d, \varphi, F, G$ and $w \geq (1 + r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)\} = ((\text{the } \mathcal{RV}\text{-portfolio value for future of } \varphi, F, G \text{ and } d_1) - (\Omega \mapsto (1 + r) \cdot (\text{the buy portfolio of } \varphi, \pi, \text{ and } d)))^{-1}([0, +\infty[)$.
\[ d_1 - (\Omega \mapsto (1 + r) \cdot \text{(the buy portfolio of } \varphi, \pi, \text{ and } d_1))^{-1}([0, +\infty[). \] For every object \( x, x \in S_1 \) iff \( x \in S_2. \)

(13) \((\text{The } \mathcal{R} \mathcal{V}-\text{portfolio value for future of } \varphi, F, G \text{ and } d_1) - (\Omega \mapsto (1 + r) \cdot \text{(the buy portfolio of } \varphi, \pi, \text{ and } d_1))^{-1}([0, +\infty[) \) is an event of \( F. \)

(14) Let us consider a natural number \( d, \) a real number \( r, \) and a sequence \( G \) of the set of random variables on \( F \) and the Borel sets. Then \( \{w, \text{ where } w \text{ is an element of } \Omega : \text{ the portfolio value for future extension of } d, \varphi, F, G \text{ and } w > 0\} \) = \((\text{the } \mathcal{R} \mathcal{V}-\text{portfolio value for future extension of } \varphi, F, G \text{ and } d)^{-1}([0, +\infty[). \) The theorem is a consequence of (1).

Let us consider natural numbers \( d, d_1, \) a real number \( r, \) and a sequence \( G \) of the set of random variables on \( F \) and the Borel sets.

(15) Suppose \( d_1 = d - 1. \) Then \( \{w, \text{ where } w \text{ is an element of } \Omega : \text{ the portfolio value for future of } d, \varphi, F, G \text{ and } w > (1 + r) \cdot \text{(the buy portfolio of } \varphi, \pi, \text{ and } d_1) - (\Omega \mapsto (1 + r) \cdot \text{(the buy portfolio of } \varphi, \pi, \text{ and } d_1)\})^{-1}([0, +\infty[). \) \)

(16) \((\text{The } \mathcal{R} \mathcal{V}-\text{portfolio value for future of } \varphi, F, G \text{ and } d_1) - (\Omega \mapsto (1 + r) \cdot \text{(the buy portfolio of } \varphi, \pi, \text{ and } d_1))^{-1}([0, +\infty[) \) is an event of \( F. \)

(17) Let us consider a price function \( \pi, \) and natural numbers \( d, d_1. \) Suppose \( d > 0 \) and \( d_1 = d - 1. \) Let us consider a probability \( P \) on \( F, \) and a real number \( r. \) Suppose \( r > -1. \) Let us consider a sequence \( G \) of the set of random variables on \( F \) and the Borel sets. Suppose the element of \( F, \) the Borel sets, \( G, \) and \( 0 = \Omega \mapsto 1 + r. \) Then there exists an \( \mathcal{A} \mathcal{O} \) w.r.t. \( P, G, \pi \) and \( d \) if and only if there exists a sequence \( \varphi_1 \) of real numbers such that \( P(\text{(the } \mathcal{R} \mathcal{V}-\text{portfolio value for future of } \varphi_1, F, G \text{ and } d_1) - (\Omega \mapsto (1 + r) \cdot \text{(the buy portfolio of } \varphi_1, \pi, \text{ and } d_1)\})^{-1}([0, +\infty[) = 1 \) and \( P(\text{(the } \mathcal{R} \mathcal{V}-\text{portfolio value for future of } \varphi_1, F, G \text{ and } d_1) - (\Omega \mapsto (1 + r) \cdot \text{(the buy portfolio of } \varphi_1, \pi, \text{ and } d_1)\})^{-1}([0, +\infty[) > 0. \)

\text{Proof: If there exists an } \mathcal{A} \mathcal{O} \text{ w.r.t. } P, G, \pi \text{ and } d, \text{ then there exists a sequence } \varphi_1 \text{ of real numbers such that } P(\text{(the } \mathcal{R} \mathcal{V}-\text{portfolio value for future of } \varphi_1, F, G \text{ and } d_1) - (\Omega \mapsto (1 + r) \cdot \text{(the buy portfolio of } \varphi_1, \pi, \text{ and } d_1)\})^{-1}([0, +\infty[) = 1 \) and \( P(\text{(the } \mathcal{R} \mathcal{V}-\text{portfolio value for future of } \varphi_1, F, G \text{ and } d_1) - (\Omega \mapsto (1 + r) \cdot \text{(the buy portfolio of } \varphi_1, \pi, \text{ and } d_1)\})^{-1}([0, +\infty[) > 0. \) Define } \mathcal{U}(\text{natural number}) = (S_1 = 0 \rightarrow -(\text{the buy portfolio of } \varphi_1, \pi, \text{ and } d), \varphi_1(S_1))((\mathbb{R}). \text{ Consider } \varphi \text{ being a se-}

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4. Risk-Neutral Probability Measure

Let us consider Ω and F. Let R be a real-valued random variable on F and r be a real number. The r-discounted value of R yielding a real-valued random variable on F is defined by the term \( R \cdot \frac{1}{1+r} \).

Let \( \pi \) be a price function and G be a sequence of the set of random variables on F and the Borel sets. We say that there exists a risk neutral measure w.r.t. G, \( \pi \) and \( r \) if and only if

\[
\text{(Def. 16)} \quad \text{there exists a probability } P \text{ on } F \text{ such that for every natural number } d, \quad \pi(d) = E_P\{\text{the } r\text{-discounted value of (the change element to functions } G \text{ and } d)\}.
\]

From now on \( P \) denotes a probability on \( \Omega_{fut1} \).

Now we state the propositions:

(18) Let us consider a real number \( r \). Suppose \( r > 0 \). Let us consider a price function \( \pi \), and a natural number \( d \). Then there exists a real-valued random variable \( f \) on \( \Omega_{fut1} \) such that

(i) \( f = \{1, 2, 3, 4\} \rightarrow \pi(d) \cdot (1 + r) \), and

(ii) \( f \) is integrable on \( P2M(P) \), and

(iii) \( f \) is simple function in \( \Omega_{fut1} \).

\textbf{Proof:} Set \( \Omega_2 = \{1, 2, 3, 4\} \). Define \( U(\text{element of } \Omega_2) = \pi(d) \cdot (1 + r)(\in \mathbb{R}) \). Consider \( f \) being a function from \( \Omega_2 \) into \( \mathbb{R} \) such that for every element \( d \) of \( \Omega_2 \), \( f(d) = U(d) \). Set \( g = \Omega_2 \rightarrow \pi(d) \cdot (1 + r)(\in \mathbb{R}) \). For every object \( x \) such that \( x \in \text{dom } f \) holds \( f(x) = g(x) \). \( f \) is integrable on \( P2M(P) \) by [6 (9), (3)], [3 (12)]. □

(19) Let us consider a natural number \( n \), and a real number \( r \). Suppose \( r > 0 \). Let us consider a price function \( \pi \), a natural number \( d \), and a real-valued random variable \( R \) on \( \Omega_{fut1} \). Suppose \( R = \{1, 2, 3, 4\} \rightarrow \pi(d) \cdot (1 + r) \) and \( R \) is integrable on \( P2M(P) \) and \( R \) is simple function in \( \Omega_{fut1} \). Then \( \pi(d) = E_P\{\text{the } r\text{-discounted value of } R\} \).
Let us consider a real number $r$. Suppose $r > 0$. Let us consider a price function $\pi$. Then there exists a sequence $G$ of the set of random variables on $\Omega_{fut1}$ and the Borel sets such that for every natural number $d$, $G(d) = \{1, 2, 3, 4\} \mapsto \pi(d) \cdot (1 + r)$ and the change element to functions $G$ and $d$ is integrable on P2M($P$) and the change element to functions $G$ and $d$ is simple function in $\Omega_{fut1}$.

**Proof:** Define $U$(natural number) = the first RV of $\pi$, $r$ and $S_1$. Consider $g$ being a sequence of the set of random variables on $\Omega_{fut1}$ and the Borel sets such that for every element $d$ of $\mathbb{N}$, $g(d) = U(d)$. There exists a real-valued random variable $R$ on $\Omega_{fut1}$ such that $R = \{1, 2, 3, 4\} \mapsto \pi(d) \cdot (1 + r)(\in \mathbb{R})$ and $R$ is integrable on P2M($P$) and $R$ is simple function in $\Omega_{fut1}$. □

Let us consider a real number $r$. Suppose $r > 0$. Let us consider a price function $\pi$, and a sequence $G$ of the set of random variables on $\Omega_{fut1}$ and the Borel sets. Suppose for every natural number $d$, $G(d) = \{1, 2, 3, 4\} \mapsto \pi(d) \cdot (1 + r)$ and the change element to functions $G$ and $d$ is integrable on P2M($P$) and the change element to functions $G$ and $d$ is simple function in $\Omega_{fut1}$. Then

(i) there exists a risk neutral measure w.r.t. $G$, $\pi$ and $r$, and

(ii) for every natural number $s$, $\pi(s) = E_P\{\text{the } r\text{-discounted value of } G(s)\}$.

The theorem is a consequence of (19).

**References**


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