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Implicit Function Theorem. Part I[1]

Kazuhisa Nakasho
Osaka University
Osaka, Japan

Yuichi Futa
Tokyo University of Technology
Tokyo, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we formalize in Mizar [1], [3] the existence and uniqueness part of the implicit function theorem. In the first section, some composition properties of Lipschitz continuous linear function are discussed. In the second section, a definition of closed ball and theorems of several properties of open and closed sets in Banach space are described. In the last section, we formalized the existence and uniqueness of continuous implicit function in Banach space using Banach fixed point theorem. We referred to [7], [8], and [2] in this formalization.

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1. Properties of Lipschitz Continuous Linear Function

From now on S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, Z denotes a subset of S, and i, n denote natural numbers.

Now we state the propositions:

(1) Let us consider real normed spaces X, Y, a point x of X, a point y of Y, and a point z of $X \times Y$. Suppose $z = \langle x, y \rangle$. Then $\|z\| = \sqrt{\|x\|^{\mathbf{2}} + \|y\|^{\mathbf{2}}}$.

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(2) Let us consider real normed spaces X, Y, a point x of X, and a point z of $X \times Y$. Suppose $z = \langle x, 0_Y \rangle$. Then $\|z\| = \|x\|$. The theorem is a consequence of (1).

(3) Let us consider real normed spaces X, Y, a point y of Y, and a point z of $X \times Y$. Suppose $z = \langle 0_X, y \rangle$. Then $\|z\| = \|y\|$. The theorem is a consequence of (1).

(4) Let us consider real normed spaces X, Y, Z, W, a Lipschitzian linear operator f from Z into W, a Lipschitzian linear operator g from Y into Z, and a Lipschitzian linear operator h from X into Y. Then $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

(5) Let us consider real normed spaces X, Y, Z, a Lipschitzian linear operator g from X into Y, a Lipschitzian linear operator f from Y into Z, and a Lipschitzian linear operator h from X into Z. Then $h = f \cdot g$ if and only if for every vector x of X, $h(x) = f(g(x))$.

(6) Let us consider real normed spaces X, Y, and a Lipschitzian linear operator f from X into Y. Then

(i) $f \cdot \mathrm{id}_\alpha = f$, and

(ii) $\mathrm{id}_\beta \cdot f = f$,

where α is the carrier of X and β is the carrier of Y.

(7) Let us consider real normed spaces X, Y, Z, W, an element f of $\mathrm{BdLinOps}(Z, W)$, an element g of $\mathrm{BdLinOps}(Y, Z)$, and an element h of $\mathrm{BdLinOps}(X, Y)$. Then $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

(8) Let us consider real normed spaces X, Y, and an element f of $\mathrm{BdLinOps}(X, Y)$. Then

(i) $f \cdot \mathrm{FuncUnit}(X) = f$, and

(ii) $\mathrm{FuncUnit}(Y) \cdot f = f$.

The theorem is a consequence of (6).

(9) Let us consider real normed spaces X, Y, Z, an element f of the real norm space of bounded linear operators from Y into Z, and elements g, h of the real norm space of bounded linear operators from X into Y. Then $f \cdot (g + h) = f \cdot g + f \cdot h$.

PROOF: Set $m_1 = \mathrm{PartFuncs}(f, Y, Z)$. Set $m_2 = \mathrm{PartFuncs}(g, X, Y)$. Set $m_4 = \mathrm{PartFuncs}(h, X, Y)$. Set $m_3 = \mathrm{PartFuncs}(g + h, X, Y)$. For every vector x of X, $(m_1 \cdot m_3)(x) = (m_1 \cdot m_2)(x) + (m_1 \cdot m_4)(x)$ by [9, (35)], (5). □

(10) Let us consider real normed spaces X, Y, Z, an element f of the real norm space of bounded linear operators from X into Y, and elements g, h

of the real norm space of bounded linear operators from Y into Z. Then $(g+h)\cdot f = g\cdot f + h\cdot f$.
PROOF: Set $m_1 = \mathrm{PartFuncs}(f, X, Y)$. Set $m_2 = \mathrm{PartFuncs}(g, Y, Z)$. Set $m_4 = \mathrm{PartFuncs}(h, Y, Z)$. Set $m_3 = \mathrm{PartFuncs}(g+h, Y, Z)$. For every vector x of X, $(m_3 \cdot m_1)(x) = (m_2 \cdot m_1)(x) + (m_4 \cdot m_1)(x)$. $\square$

(11) Let us consider real normed spaces X, Y, Z, an element f of the real norm space of bounded linear operators from Y into Z, an element g of the real norm space of bounded linear operators from X into Y, and real numbers a, b. Then $(a\cdot b)\cdot(f\cdot g) = a\cdot f\cdot(b\cdot g)$.
PROOF: Set $m_1 = \mathrm{PartFuncs}(f, Y, Z)$. Set $m_2 = \mathrm{PartFuncs}(g, X, Y)$. Set $m_5 = \mathrm{PartFuncs}(a\cdot f, Y, Z)$. Set $m_6 = \mathrm{PartFuncs}(b\cdot g, X, Y)$. For every vector x of X, $(m_5 \cdot m_6)(x) = a\cdot b\cdot(m_1\cdot m_2)(x)$. $\square$

(12) Let us consider real normed spaces X, Y, Z, an element f of the real norm space of bounded linear operators from Y into Z, an element g of the real norm space of bounded linear operators from X into Y, and a real number a. Then $a\cdot(f\cdot g) = (a\cdot f)\cdot g$. The theorem is a consequence of (11).

2. Properties of Open and Closed Sets in Banach Space

Let M be a real normed space, p be an element of M, and r be a real number. The functor $\overline{\mathrm{Ball}}(p, r)$ yielding a subset of M is defined by the term

(Def. 1) $\{q$, where q is an element of $M : \|p-q\| \leqslant r\}$.

Let us consider an element p of S and a real number r. Now we state the propositions:

(13) If $0 < r$, then $p \in \mathrm{Ball}(p, r)$ and $p \in \overline{\mathrm{Ball}}(p, r)$.

(14) If $0 < r$, then $\mathrm{Ball}(p, r) \neq \emptyset$ and $\overline{\mathrm{Ball}}(p, r) \neq \emptyset$.

Let us consider a real normed space M, an element p of M, and real numbers r_1, r_2. Now we state the propositions:

(15) Suppose $r_1 \leqslant r_2$. Then

(i) $\overline{\mathrm{Ball}}(p, r_1) \subseteq \overline{\mathrm{Ball}}(p, r_2)$, and

(ii) $\mathrm{Ball}(p, r_1) \subseteq \overline{\mathrm{Ball}}(p, r_2)$, and

(iii) $\mathrm{Ball}(p, r_1) \subseteq \mathrm{Ball}(p, r_2)$.

(16) If $r_1 < r_2$, then $\overline{\mathrm{Ball}}(p, r_1) \subseteq \mathrm{Ball}(p, r_2)$.

Let us consider an element p of S and a real number r. Now we state the propositions:

(17) $\mathrm{Ball}(p, r) = \{y$, where y is a point of $S : \|y - p\| < r\}$.
PROOF: Define $\mathcal{F}(\text{object}) = \$_1$. Define $\mathcal{P}[\text{element of } S] \equiv \|p - \$_1\| < r$. Define $\mathcal{Q}[\text{element of } S] \equiv \|\$_1 - p\| < r$. $\{\mathcal{F}(y)$, where y is an element of the carrier of $S : \mathcal{P}[y]\} = \{\mathcal{F}(y)$, where y is an element of the carrier of $S : \mathcal{Q}[y]\}$. □

(18) $\overline{\mathrm{Ball}}(p, r) = \{y$, where y is a point of $S : \|y - p\| \leqslant r\}$.
PROOF: Define $\mathcal{F}(\text{object}) = \$_1$. Define $\mathcal{P}[\text{element of } S] \equiv \|p - \$_1\| \leqslant r$. Define $\mathcal{Q}[\text{element of } S] \equiv \|\$_1 - p\| \leqslant r$. $\{\mathcal{F}(y)$, where y is an element of the carrier of $S : \mathcal{P}[y]\} = \{\mathcal{F}(y)$, where y is an element of the carrier of $S : \mathcal{Q}[y]\}$. □

(19) If $0 < r$, then $\mathrm{Ball}(p, r)$ is a neighbourhood of p. The theorem is a consequence of (17).

Let X be a real normed space, x be a point of X, and r be a real number. One can check that $\mathrm{Ball}(x, r)$ is open and $\overline{\mathrm{Ball}}(x, r)$ is closed.

Now we state the propositions:

(20) Let us consider a real normed space X, and a subset V of X. Then V is open if and only if for every point x of X such that $x \in V$ there exists a real number r such that $r > 0$ and $\mathrm{Ball}(x, r) \subseteq V$.

(21) Let us consider real normed spaces X, Y, a point x of X, a point y of Y, and a point z of $X \times Y$. Suppose $z = \langle x, y \rangle$. Then

(i) $\|x\| \leqslant \|z\|$, and

(ii) $\|y\| \leqslant \|z\|$.

The theorem is a consequence of (1).

(22) Let us consider real normed spaces X, Y, a point x of X, a point y of Y, a point z of $X \times Y$, and a real number r_1. Suppose $0 < r_1$ and $z = \langle x, y \rangle$. Then there exists a real number r_2 such that

(i) $0 < r_2 < r_1$, and

(ii) $\mathrm{Ball}(x, r_2) \times \mathrm{Ball}(y, r_2) \subseteq \mathrm{Ball}(z, r_1)$.

PROOF: $\mathrm{Ball}(x, r_2) \times \mathrm{Ball}(y, r_2) \subseteq \mathrm{Ball}(z, r_1)$. □

(23) Let us consider real normed spaces X, Y, a point x of X, a point y of Y, and a subset V of $X \times Y$. Suppose V is open and $\langle x, y \rangle \in V$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) $\mathrm{Ball}(x, r) \times \mathrm{Ball}(y, r) \subseteq V$.

The theorem is a consequence of (20) and (22).

(24) Let us consider real normed spaces X, Y, a point x of X, a point y of Y, a subset V of $X \times Y$, and a real number r. Suppose $V = \mathrm{Ball}(x, r) \times \mathrm{Ball}(y, r)$. Then V is open.

PROOF: For every point z of $X \times Y$ such that $z \in V$ there exists a real number s such that $s > 0$ and $\mathrm{Ball}(z, s) \subseteq V$ by [5, (18)]. $\square$

(25) Let us consider real normed spaces E, F, a linear operator Q from E into F, and a point v of E. If Q is one-to-one, then $Q(v) = 0_F$ iff $v = 0_E$.

Let us consider a real normed space E, subsets X, Y of E, and a point v of E. Now we state the propositions:

(26) If X is open and $Y = \{x + v$, where x is a point of $E : x \in X\}$, then Y is open.

PROOF: Define $\mathcal{C}(\text{point of } E) = 1 \cdot \$_1 + -v$. Consider H being a function from E into E such that for every point p of E, $H(p) = \mathcal{C}(p)$. For every object s, $s \in H^{-1}(X)$ iff $s \in Y$. $\square$

(27) If X is open and $Y = \{x - v$, where x is a point of $E : x \in X\}$, then Y is open.

PROOF: Set $w = -v$. $\{x + w$, where x is a point of $E : x \in X\} \subseteq$ the carrier of E. Define $\mathcal{F}(\text{point of } E) = \$_1 + w$. Define $\mathcal{G}(\text{point of } E) = \$_1 - v$. Define $\mathcal{P}[\text{point of } E] \equiv \$_1 \in X$. $\{\mathcal{F}(v_1)$, where v_1 is an element of the carrier of $E : \mathcal{P}[v_1]\} = \{\mathcal{G}(v_2)$, where v_2 is an element of the carrier of $E : \mathcal{P}[v_2]\}$. $\square$

3. EXISTENCE AND UNIQUENESS OF CONTINUOUS IMPLICIT FUNCTION

Now we state the propositions:

(28) Let us consider a real Banach space X, a non empty subset S of X, and a partial function f from X to X. Suppose S is closed and $\mathrm{dom}\, f = S$ and $\mathrm{rng}\, f \subseteq S$ and there exists a real number k such that $0 < k < 1$ and for every points x, y of X such that $x, y \in S$ holds $\|f_x - f_y\| \leqslant k \cdot \|x - y\|$. Then

(i) there exists a point x_0 of X such that $x_0 \in S$ and $f(x_0) = x_0$, and

(ii) for every points x_0, y_0 of X such that $x_0, y_0 \in S$ and $f(x_0) = x_0$ and $f(y_0) = y_0$ holds $x_0 = y_0$.

PROOF: Consider x_0 being an object such that $x_0 \in S$. Consider K being a real number such that $0 < K$ and $K < 1$ and for every points x, y of X such that $x, y \in S$ holds $\|f_x - f_y\| \leqslant K \cdot \|x - y\|$. Define $\mathcal{G}(\text{set}, \text{set}) = f(\$_2)$. Consider g being a function such that $\mathrm{dom}\, g = \mathbb{N}$ and $g(0) = x_0$ and for every natural number n, $g(n+1) = \mathcal{G}(n, g(n))$. Define $\mathcal{P}[\text{natural number}] \equiv$

$g(\$_1) \in S$ and $g(\$_1)$ is an element of X. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number n, $\mathcal{P}[n]$. For every object n such that $n \in \mathbb{N}$ holds $g(n) \in$ the carrier of X. For every natural number n, $\|g(n+1) - g(n)\| \leqslant \|g(1) - g(0)\| \cdot (K^n)$. For every natural numbers k, n, $\|g(n+k) - g(n)\| \leqslant \|g(1) - g(0)\| \cdot (\frac{K^n - K^{n+k}}{1-K})$. For every natural numbers k, n, $\|g(n+k) - g(n)\| \leqslant \|g(1) - g(0)\| \cdot (\frac{K^n}{1-K})$. For every real number e such that $e > 0$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\|(g\uparrow 1)(m) - f_{\lim g}\| < e$. For every points x_0, y_0 of X such that x_0, $y_0 \in S$ and $f(x_0) = x_0$ and $f(y_0) = y_0$ holds $x_0 = y_0$. $\square$

(29) Let us consider a real normed space E, a real Banach space F, a non empty subset E_0 of E, a non empty subset F_0 of F, and a partial function F_1 from $E \times F$ to F. Suppose F_0 is closed and $E_0 \times F_0 \subseteq \operatorname{dom} F_1$ and for every point x of E and for every point y of F such that $x \in E_0$ and $y \in F_0$ holds $F_1(x, y) \in F_0$ and for every point y of F such that $y \in F_0$ for every point x_0 of E such that $x_0 \in E_0$ for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every point x_1 of E such that $x_1 \in E_0$ and $\|x_1 - x_0\| < d$ holds $\|{F_1}_{\langle x_1, y \rangle} - {F_1}_{\langle x_0, y \rangle}\| < e$ and there exists a real number k such that $0 < k < 1$ and for every point x of E such that $x \in E_0$ for every points y_1, y_2 of F such that y_1, $y_2 \in F_0$ holds $\|{F_1}_{\langle x, y_1 \rangle} - {F_1}_{\langle x, y_2 \rangle}\| \leqslant k \cdot \|y_1 - y_2\|$. Then

(i) for every point x of E such that $x \in E_0$ holds there exists a point y of F such that $y \in F_0$ and $F_1(x, y) = y$ and for every points y_1, y_2 of F such that y_1, $y_2 \in F_0$ and $F_1(x, y_1) = y_1$ and $F_1(x, y_2) = y_2$ holds $y_1 = y_2$, and

(ii) for every point x_0 of E and for every point y_0 of F such that $x_0 \in E_0$ and $y_0 \in F_0$ and $F_1(x_0, y_0) = y_0$ for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every point x_1 of E and for every point y_1 of F such that $x_1 \in E_0$ and $y_1 \in F_0$ and $F_1(x_1, y_1) = y_1$ and $\|x_1 - x_0\| < d$ holds $\|y_1 - y_0\| < e$.

Proof: Consider k being a real number such that $0 < k < 1$ and for every point x of E such that $x \in E_0$ for every points y_1, y_2 of F such that y_1, $y_2 \in F_0$ holds $\|{F_1}_{\langle x, y_1 \rangle} - {F_1}_{\langle x, y_2 \rangle}\| \leqslant k \cdot \|y_1 - y_2\|$. For every point x of E such that $x \in E_0$ holds there exists a point y of F such that $y \in F_0$ and $F_1(x, y) = y$ and for every points y_1, y_2 of F such that y_1, $y_2 \in F_0$ and $F_1(x, y_1) = y_1$ and $F_1(x, y_2) = y_2$ holds $y_1 = y_2$. For every point x_0 of E and for every point y_0 of F such that $x_0 \in E_0$ and $y_0 \in F_0$ and $F_1(x_0, y_0) = y_0$ for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every point x_1 of E and for every point y_1 of F such that $x_1 \in E_0$ and $y_1 \in F_0$ and $F_1(x_1, y_1) = y_1$ and $\|x_1 - x_0\| < d$ holds $\|y_1 - y_0\| < e$. □

(30) Let us consider a real normed space E, a real Banach space F, a non empty subset A of E, a non empty subset B of F, and a partial function F_1 from $E \times F$ to F. Suppose B is closed and $A \times B \subseteq \operatorname{dom} F_1$ and for every point x of E and for every point y of F such that $x \in A$ and $y \in B$ holds $F_1(x, y) \in B$ and for every point y of F such that $y \in B$ for every point x_0 of E such that $x_0 \in A$ for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every point x_1 of E such that $x_1 \in A$ and $\|x_1 - x_0\| < d$ holds $\|F_{1\langle x_1, y\rangle} - F_{1\langle x_0, y\rangle}\| < e$ and there exists a real number k such that $0 < k < 1$ and for every point x of E such that $x \in A$ for every points y_1, y_2 of F such that $y_1, y_2 \in B$ holds $\|F_{1\langle x, y_1\rangle} - F_{1\langle x, y_2\rangle}\| \leqslant k \cdot \|y_1 - y_2\|$. Then

(i) there exists a partial function g from E to F such that g is continuous on A and $\operatorname{dom} g = A$ and $\operatorname{rng} g \subseteq B$ and for every point x of E such that $x \in A$ holds $F_1(x, g(x)) = g(x)$, and

(ii) for every partial functions g_1, g_2 from E to F such that $\operatorname{dom} g_1 = A$ and $\operatorname{rng} g_1 \subseteq B$ and $\operatorname{dom} g_2 = A$ and $\operatorname{rng} g_2 \subseteq B$ and for every point x of E such that $x \in A$ holds $F_1(x, g_1(x)) = g_1(x)$ and for every point x of E such that $x \in A$ holds $F_1(x, g_2(x)) = g_2(x)$ holds $g_1 = g_2$.

PROOF: There exists a partial function g from E to F such that g is continuous on A and $\operatorname{dom} g = A$ and $\operatorname{rng} g \subseteq B$ and for every point x of E such that $x \in A$ holds $F_1(x, g(x)) = g(x)$ by (29), [4, (19)]. For every object x such that $x \in \operatorname{dom} g_1$ holds $g_1(x) = g_2(x)$. □

Let us consider real normed spaces E, F and points s_1, s_2 of $E \times F$. Now we state the propositions:

(31) If $(s_1)_{\mathbf{2}} = (s_2)_{\mathbf{2}}$, then $\operatorname{reproj1}(s_1) = \operatorname{reproj1}(s_2)$.

(32) If $(s_1)_{\mathbf{1}} = (s_2)_{\mathbf{1}}$, then $\operatorname{reproj2}(s_1) = \operatorname{reproj2}(s_2)$.

(33) Let us consider a real normed space E, a real number r, and points z, y_1, y_2 of E. Suppose $y_1, y_2 \in \overline{\operatorname{Ball}}(z, r)$. Then $[y_1, y_2] \subseteq \overline{\operatorname{Ball}}(z, r)$.

(34) Let us consider a real normed space E, points x, b of E, and a neighbourhood N of x. Then $\{z - b$, where z is a point of $E : z \in N\}$ is neighbourhood of $x - b$ and neighbourhood of $x + b$.

PROOF: Consider g being a real number such that $0 < g$ and $\{y$, where y is a point of $E : \|y - x\| < g\} \subseteq N$. $\{z - b$, where z is a point of $E : z \in N\} \subseteq$ the carrier of E. $\{z + b$, where z is a point of $E : z \in N\} \subseteq$ the carrier of E. $\{y$, where y is a point of $E : \|y - (x - b)\| < g\} \subseteq \{z - b$, where z is a point of $E : z \in N\}$. $\{y$, where y is a point of

$E : \|y - (x+b)\| < g\} \subseteq \{z+b$, where z is a point of $E : z \in N\}$. □

Let us consider real normed spaces E, G, a real Banach space F, a subset Z of $E \times F$, a partial function f from $E \times F$ to G, a point a of E, a point b of F, a point c of G, and a point z of $E \times F$. Now we state the propositions:

(35) Suppose Z is open and $\operatorname{dom} f = Z$ and f is continuous on Z and f is partially differentiable on Z w.r.t. 2 and $f \restriction^2 Z$ is continuous on Z and $z = \langle a,\, b\rangle$ and $z \in Z$ and $f(a,b) = c$ and partdiff(f,z) w.r.t. 2 is one-to-one and (partdiff(f,z) w.r.t. $2)^{-1}$ is a Lipschitzian linear operator from G into F. Then there exist real numbers r_1, r_2 such that

(i) $0 < r_1$, and

(ii) $0 < r_2$, and

(iii) $\mathrm{Ball}(a, r_1) \times \overline{\mathrm{Ball}}(b, r_2) \subseteq Z$, and

(iv) for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ there exists a point y of F such that $y \in \overline{\mathrm{Ball}}(b, r_2)$ and $f(x,y) = c$, and

(v) for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ for every points y_1, y_2 of F such that y_1, $y_2 \in \overline{\mathrm{Ball}}(b, r_2)$ and $f(x, y_1) = c$ and $f(x, y_2) = c$ holds $y_1 = y_2$, and

(vi) there exists a partial function g from E to F such that g is continuous on $\mathrm{Ball}(a, r_1)$ and $\operatorname{dom} g = \mathrm{Ball}(a, r_1)$ and $\operatorname{rng} g \subseteq \overline{\mathrm{Ball}}(b, r_2)$ and $g(a) = b$ and for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ holds $f(x, g(x)) = c$, and

(vii) for every partial functions g_1, g_2 from E to F such that $\operatorname{dom} g_1 = \mathrm{Ball}(a, r_1)$ and $\operatorname{rng} g_1 \subseteq \overline{\mathrm{Ball}}(b, r_2)$ and for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ holds $f(x, g_1(x)) = c$ and $\operatorname{dom} g_2 = \mathrm{Ball}(a, r_1)$ and $\operatorname{rng} g_2 \subseteq \overline{\mathrm{Ball}}(b, r_2)$ and for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ holds $f(x, g_2(x)) = c$ holds $g_1 = g_2$.

PROOF: Consider Q_1 being a Lipschitzian linear operator from G into F such that $Q_1 =$ (partdiff(f,z) w.r.t. $2)^{-1}$. Reconsider $Q = Q_1$ as a point of the real norm space of bounded linear operators from G into F. Reconsider $z_1 = \langle a,\, 0_F\rangle$ as a point of $E \times F$. Reconsider $e_0 = \langle 0_E,\, b\rangle$ as a point of $E \times F$. Define $\mathcal{C}$(point of $E \times F$) $= 1 \cdot \$_1 + -e_0$. Consider H being a function from the carrier of $E \times F$ into the carrier of $E \times F$ such that for every point p of $E \times F$, $H(p) = \mathcal{C}(p)$. For every point x of E and for every point y of F, $H(x,y) = \langle x,\, y-b\rangle$. Define $\mathcal{D}$(point of $E \times F$) $= 1 \cdot \$_1 + e_0$. Consider K being a function from the carrier of $E \times F$ into the carrier of $E \times F$ such that for every point p of $E \times F$, $K(p) = \mathcal{D}(p)$. For every point p of $E \times F$, $K \cdot H(p) = p$. For every point p of $E \times F$, $H \cdot K(p) = p$. Reconsider $Z_1 = H^{\circ} Z$ as a subset of $E \times F$. For every point x of E and for every point y of F, $\langle x, y+b\rangle \in Z$ iff $\langle x, y\rangle \in Z_1$. Reconsider $e_0 = \langle 0_E, b\rangle$ as a point of $E \times F$. For every object p, $p \in Z_1$ iff $p \in \{y - e_0$, where y is a point of $E \times F : y \in Z\}$. Z_1 is open. Define $\mathcal{J}[\text{object}, \text{object}] \equiv$ there exists a point x of E and there exists a point y of F such that $\$_1 = \langle x, y\rangle$ and $\$_2 = f_{\langle x, y+b\rangle} - c$. For every object p such that $p \in Z_1$ there exists an object w such that $w \in$ the carrier of G and $\mathcal{J}[p, w]$. Consider f_1 being a function from Z_1 into G such that for every object p such that $p \in Z_1$ holds $\mathcal{J}[p, f_1(p)]$. For every point x of E and for every point y of F such that $\langle x, y\rangle \in Z_1$ holds $f_1(x, y) = f_{\langle x, y+b\rangle} - c$. Define $\mathcal{O}[\text{object}, \text{object}] \equiv$ there exists a point x of E and there exists a point y of F such that $\$_1 = \langle x, y\rangle$ and $\$_2 = Q(f_1(x, y))$. For every object p such that $p \in Z_1$ there exists an object w such that $w \in$ the carrier of F and $\mathcal{O}[p, w]$. Consider f_2 being a function from Z_1 into F such that for every object p such that $p \in Z_1$ holds $\mathcal{O}[p, f_2(p)]$. For every point x of E and for every point y of F such that $\langle x, y\rangle \in Z_1$ holds $f_2(x, y) = Q(f_1(x, y))$. Define $\mathcal{U}[\text{object}, \text{object}] \equiv$ there exists a point x of E and there exists a point y of F such that $\$_1 = \langle x, y\rangle$ and $\$_2 = y - f_{2\langle x, y\rangle}$. For every object p such that $p \in Z_1$ there exists an object w such that $w \in$ the carrier of F and $\mathcal{U}[p, w]$.

Consider F_1 being a function from Z_1 into F such that for every object p such that $p \in Z_1$ holds $\mathcal{U}[p, F_1(p)]$. For every point x of E and for every point y of F such that $\langle x, y\rangle \in Z_1$ holds $F_1(x, y) = y - f_{2\langle x, y\rangle}$. For every point z_0 of $E \times F$ and for every real number r such that $z_0 \in Z_1$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point z_1 of $E \times F$ such that $z_1 \in Z_1$ and $\|z_1 - z_0\| < s$ holds $\|F_{1z_1} - F_{1z_0}\| < r$. For every point w_0 of $E \times F$ such that $w_0 \in Z$ holds $f \cdot (\text{reproj2}(w_0))$ is differentiable in $(w_0)_{\mathbf{2}}$. For every point w_0 of $E \times F$ such that $w_0 \in Z$ there exists a neighbourhood N of $(w_0)_{\mathbf{2}}$ such that $N \subseteq \operatorname{dom} f \cdot (\text{reproj2}(w_0))$ and there exists a rest R of F, G such that for every point w_1 of F such that $w_1 \in N$ holds $f \cdot (\text{reproj2}(w_0))_{w_1} - f \cdot (\text{reproj2}(w_0))_{(w_0)_{\mathbf{2}}} = f \cdot (\text{reproj2}(w_0))'((w_0)_{\mathbf{2}})(w_1 - (w_0)_{\mathbf{2}}) + R_{w_1 - (w_0)_{\mathbf{2}}}$. For every point z_0 of $E \times F$ such that $z_0 \in Z_1$ holds $F_1 \cdot (\text{reproj2}(z_0))$ is differentiable in $(z_0)_{\mathbf{2}}$ and there exist points L_0, I of the real norm space of bounded linear operators from F into F such that $L_0 = Q \cdot ((f \restriction^2 Z)_{z_0 + e_0})$ and $I = \mathrm{id}_\alpha$ and $F_1 \cdot (\text{reproj2}(z_0))'((z_0)_{\mathbf{2}}) = I - L_0$, where α is the carrier of F. $\operatorname{dom}(F_1 \restriction^2 Z_1) = Z_1$ and for every point z of $E \times F$ such that $z \in Z_1$ there exist points L, I of the real norm space of bounded linear operators from F into F such that $L = Q \cdot ((f \restriction^2 Z)_{z + e_0})$ and $I = \mathrm{id}_\alpha$ and $(F_1 \restriction^2 Z_1)_z = I - L$, where α is the carrier of F. Set $F_2 = F_1 \restriction^2 Z_1$. For every point z_0 of $E \times F$ and for every real number r such that $z_0 \in Z_1$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point z_1 of $E \times F$ such that $z_1 \in Z_1$ and $\|z_1 - z_0\| < s$ holds $\|F_{2z_1} - F_{2z_0}\| < r$. $F_1(a, 0_F) = 0_F$ by [6, (3)]. Reconsider $a_0 = \langle a, 0_F \rangle$ as a point of $E \times F$. Consider r_4 being a real number such that $0 < r_4$ and for every point s of $E \times F$ such that $s \in Z_1$ and $\|s - a_0\| < r_4$ holds $\|(F_1 \restriction^2 Z_1)_s - (F_1 \restriction^2 Z_1)_{a_0}\| < \frac{1}{4}$. Consider r_5 being a real number such that $0 < r_5$ and $\mathrm{Ball}(a_0, r_5) \subseteq Z_1$. Reconsider $r_6 = \min(r_4, r_5)$ as a real number. $\mathrm{Ball}(a_0, r_6) \subseteq \mathrm{Ball}(a_0, r_5)$.

Consider r_1 being a real number such that $0 < r_1 < r_6$ and $\mathrm{Ball}(a, r_1) \times \mathrm{Ball}(0_F, r_1) \subseteq \mathrm{Ball}(a_0, r_6)$. For every point x of $E \times F$ such that $x \in Z_1$ holds $(F_1 \restriction^2 Z_1)_x = F_1 \cdot (\mathrm{reproj2}(x))'((x)_{\mathbf{2}})$. $a \in \mathrm{Ball}(a, r_1)$. $0_F \in \mathrm{Ball}(0_F, r_1)$. Reconsider $a_0 = \langle a, 0_F \rangle$ as a point of $E \times F$. Consider L_1, I_1 being points of the real norm space of bounded linear operators from F into F such that $L_1 = Q \cdot ((f \restriction^2 Z)_{a_0 + e_0})$ and $I_1 = \mathrm{id}_\alpha$ and $(F_1 \restriction^2 Z_1)_{a_0} = I_1 - L_1$, where α is the carrier of F. For every point x of E and for every point y of F such that $x \in \mathrm{Ball}(a, r_1)$ and $y \in \mathrm{Ball}(0_F, r_1)$ holds $\|(F_1 \restriction^2 Z_1)_{\langle x, y \rangle}\| < \frac{1}{4}$. Reconsider $r_2 = \frac{r_1}{2}$ as a real number. Consider a_2 being a real number such that $0 < a_2$ and for every point s of $E \times F$ such that $s \in Z_1$ and $\|s - a_0\| < a_2$ holds $\|F_{1s} - F_{1a_0}\| < (\frac{1}{2}) \cdot r_2$. Consider a_4 being a real number such that $0 < a_4 < a_2$ and $\mathrm{Ball}(a, a_4) \times \mathrm{Ball}(0_F, a_4) \subseteq \mathrm{Ball}(a_0, a_2)$. Reconsider $a_3 = \min(a_2, a_4)$ as a real number. $\mathrm{Ball}(a, a_3) \subseteq \mathrm{Ball}(a, a_4)$. Reconsider $a_1 = \min(a_3, r_1)$ as a real number. $\mathrm{Ball}(a, a_1) \subseteq \mathrm{Ball}(a, r_1)$. $\mathrm{Ball}(a, a_1) \subseteq \mathrm{Ball}(a, a_3)$. For every point x of E such that $x \in \mathrm{Ball}(a, a_1)$ holds $\|F_{1\langle x, 0_F \rangle}\| \leqslant (\frac{1}{2}) \cdot r_2$. Reconsider $r_0 = \min(\frac{r_1}{2}, a_1)$ as a real number. $\mathrm{Ball}(a, r_0) \subseteq \mathrm{Ball}(a, r_1)$. For every point x of E such that $x \in \mathrm{Ball}(a, r_0)$ holds $\|F_{1\langle x, 0_F \rangle}\| \leqslant (\frac{1}{2}) \cdot r_2$. $\overline{\mathrm{Ball}}(0_F, r_2) \subseteq \mathrm{Ball}(0_F, r_1)$. For every point x of E such that $x \in \mathrm{Ball}(a, r_0)$ for every points y_1, y_2 of F such that $y_1, y_2 \in \overline{\mathrm{Ball}}(0_F, r_2)$ holds $\|F_{1\langle x, y_1 \rangle} - F_{1\langle x, y_2 \rangle}\| \leqslant (\frac{1}{2}) \cdot \|y_1 - y_2\|$. For every point x of E and for every point y of F such that $x \in \mathrm{Ball}(a, r_0)$ and $y \in \overline{\mathrm{Ball}}(0_F, r_2)$ holds $F_1(x, y) \in \overline{\mathrm{Ball}}(0_F, r_2)$. $\mathrm{Ball}(a, r_0) \neq \emptyset$. $\overline{\mathrm{Ball}}(0_F, r_2) \neq \emptyset$. For every point y of F such that $y \in \overline{\mathrm{Ball}}(0_F, r_2)$ for every point x_0 of E such that $x_0 \in \mathrm{Ball}(a, r_0)$ for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every point x_1 of E such that $x_1 \in \mathrm{Ball}(a, r_0)$ and $\|x_1 - x_0\| < d$ holds $\|F_{1\langle x_1, y \rangle} - F_{1\langle x_0, y \rangle}\| < e$.

Consider Ψ being a partial function from E to F such that Ψ is continuous on $\mathrm{Ball}(a, r_0)$ and $\mathrm{dom}\,\Psi = \mathrm{Ball}(a, r_0)$ and $\mathrm{rng}\,\Psi \subseteq \overline{\mathrm{Ball}}(0_F, r_2)$ and for every point x of E such that $x \in \mathrm{Ball}(a, r_0)$ holds $F_1(x, \Psi(x)) = \Psi(x)$. For every object z, $z \in \overline{\mathrm{Ball}}(b, r_2)$ iff $z \in \{y + b$, where y is a point of $F : y \in \overline{\mathrm{Ball}}(0_F, r_2)\}$. For every object y, $y \in \mathrm{Ball}(a, r_0) \times \overline{\mathrm{Ball}}(b, r_2)$

iff there exists an object x such that $x \in \operatorname{dom} K$ and $x \in \operatorname{Ball}(a, r_0) \times \overline{\operatorname{Ball}}(0_F, r_2)$ and $y = K(x)$. Define $\mathcal{W}(\text{object}) = \Psi_{\$_1} + b$. For every object y such that $y \in \operatorname{Ball}(a, r_0)$ holds $\mathcal{W}(y) \in \overline{\operatorname{Ball}}(b, r_2)$. Consider E_3 being a function from $\operatorname{Ball}(a, r_0)$ into $\overline{\operatorname{Ball}}(b, r_2)$ such that for every object y such that $y \in \operatorname{Ball}(a, r_0)$ holds $E_3(y) = \mathcal{W}(y)$. $\overline{\operatorname{Ball}}(b, r_2) \neq \emptyset$. For every point x_0 of E and for every real number r such that $x_0 \in \operatorname{Ball}(a, r_0)$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of E such that $x_1 \in \operatorname{Ball}(a, r_0)$ and $\|x_1 - x_0\| < s$ holds $\|E_{3x_1} - E_{3x_0}\| < r$. For every point x of E such that $x \in \operatorname{Ball}(a, r_0)$ holds $f(x, E_3(x)) = c$. For every point x of E such that $x \in \operatorname{Ball}(a, r_0)$ there exists a point y of F such that $y \in \overline{\operatorname{Ball}}(b, r_2)$ and $f(x, y) = c$. For every point x of E such that $x \in \operatorname{Ball}(a, r_0)$ for every points y_1, y_2 of F such that y_1, $y_2 \in \overline{\operatorname{Ball}}(b, r_2)$ and $f(x, y_1) = c$ and $f(x, y_2) = c$ holds $y_1 = y_2$. $a \in \operatorname{Ball}(a, r_0)$ and $b \in \overline{\operatorname{Ball}}(b, r_2)$. $E_3(a) \in \operatorname{rng} E_3$. $f(a, E_3(a)) = c$. For every partial functions E_1, E_2 from E to F such that $\operatorname{dom} E_1 = \operatorname{Ball}(a, r_0)$ and $\operatorname{rng} E_1 \subseteq \overline{\operatorname{Ball}}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_0)$ holds $f(x, E_1(x)) = c$ and $\operatorname{dom} E_2 = \operatorname{Ball}(a, r_0)$ and $\operatorname{rng} E_2 \subseteq \overline{\operatorname{Ball}}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_0)$ holds $f(x, E_2(x)) = c$ holds $E_1 = E_2$. $\square$

(36) Suppose Z is open and $\operatorname{dom} f = Z$ and f is continuous on Z and f is partially differentiable on Z w.r.t. 2 and $f \restriction^2 Z$ is continuous on Z and $z = \langle a, b \rangle$ and $z \in Z$ and $f(a, b) = c$ and $\operatorname{partdiff}(f, z)$ w.r.t. 2 is one-to-one and $(\operatorname{partdiff}(f, z)\text{ w.r.t. }2)^{-1}$ is a Lipschitzian linear operator from G into F. Then there exist real numbers r_1, r_2 such that

(i) $0 < r_1$, and

(ii) $0 < r_2$, and

(iii) $\operatorname{Ball}(a, r_1) \times \overline{\operatorname{Ball}}(b, r_2) \subseteq Z$, and

(iv) for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ there exists a point y of F such that $y \in \operatorname{Ball}(b, r_2)$ and $f(x, y) = c$, and

(v) for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ for every points y_1, y_2 of F such that y_1, $y_2 \in \operatorname{Ball}(b, r_2)$ and $f(x, y_1) = c$ and $f(x, y_2) = c$ holds $y_1 = y_2$, and

(vi) there exists a partial function g from E to F such that g is continuous on $\operatorname{Ball}(a, r_1)$ and $\operatorname{dom} g = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g \subseteq \operatorname{Ball}(b, r_2)$ and $g(a) = b$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g(x)) = c$, and

(vii) for every partial functions g_1, g_2 from E to F such that $\operatorname{dom} g_1 = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g_1 \subseteq \operatorname{Ball}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g_1(x)) = c$ and $\operatorname{dom} g_2 = \operatorname{Ball}(a, r_1)$

and $\operatorname{rng} g_2 \subseteq \operatorname{Ball}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g_2(x)) = c$ holds $g_1 = g_2$.

Proof: Consider r_1, r_2 being real numbers such that $0 < r_1$ and $0 < r_2$ and $\operatorname{Ball}(a, r_1) \times \overline{\operatorname{Ball}}(b, r_2) \subseteq Z$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ there exists a point y of F such that $y \in \overline{\operatorname{Ball}}(b, r_2)$ and $f(x, y) = c$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ for every points y_1, y_2 of F such that y_1, $y_2 \in \overline{\operatorname{Ball}}(b, r_2)$ and $f(x, y_1) = c$ and $f(x, y_2) = c$ holds $y_1 = y_2$ and there exists a partial function g from E to F such that g is continuous on $\operatorname{Ball}(a, r_1)$ and $\operatorname{dom} g = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g \subseteq \overline{\operatorname{Ball}}(b, r_2)$ and $g(a) = b$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g(x)) = c$ and for every partial functions g_1, g_2 from E to F such that $\operatorname{dom} g_1 = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g_1 \subseteq \overline{\operatorname{Ball}}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g_1(x)) = c$ and $\operatorname{dom} g_2 = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g_2 \subseteq \overline{\operatorname{Ball}}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g_2(x)) = c$ holds $g_1 = g_2$.

Consider g being a partial function from E to F such that g is continuous on $\operatorname{Ball}(a, r_1)$ and $\operatorname{dom} g = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g \subseteq \overline{\operatorname{Ball}}(b, r_2)$ and $g(a) = b$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g(x)) = c$ and for every partial functions g_1, g_2 from E to F such that $\operatorname{dom} g_1 = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g_1 \subseteq \overline{\operatorname{Ball}}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g_1(x)) = c$ and $\operatorname{dom} g_2 = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g_2 \subseteq \overline{\operatorname{Ball}}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g_2(x)) = c$ holds $g_1 = g_2$. $a \in \operatorname{Ball}(a, r_1)$. Consider r_3 being a real number such that $0 < r_3$ and for every point x_1 of E such that $x_1 \in \operatorname{dom} g$ and $\|x_1 - a\| < r_3$ holds $\|g_{x_1} - g_a\| < r_2$. Reconsider $r_0 = \min(r_1, r_3)$ as a real number. $\operatorname{Ball}(a, r_0) \subseteq \operatorname{Ball}(a, r_1)$ and $\operatorname{Ball}(a, r_0) \subseteq \operatorname{Ball}(a, r_3)$. For every point x of E such that $x \in \operatorname{Ball}(a, r_0)$ there exists a point y of F such that $y \in \operatorname{Ball}(b, r_2)$ and $f(x, y) = c$.

For every point x of E such that $x \in \operatorname{Ball}(a, r_0)$ for every points y_1, y_2 of F such that y_1, $y_2 \in \operatorname{Ball}(b, r_2)$ and $f(x, y_1) = c$ and $f(x, y_2) = c$ holds $y_1 = y_2$. Reconsider $g_1 = g \restriction \operatorname{Ball}(a, r_0)$ as a partial function from E to F. $\operatorname{dom} g_1 = \operatorname{Ball}(a, r_0)$. For every object y such that $y \in \operatorname{rng} g_1$ holds $y \in \operatorname{Ball}(b, r_2)$. $g_1(a) = b$. For every point x of E such that $x \in \operatorname{Ball}(a, r_0)$ holds $f(x, g_1(x)) = c$. For every partial functions g_1, g_2 from E to F such that $\operatorname{dom} g_1 = \operatorname{Ball}(a, r_0)$ and $\operatorname{rng} g_1 \subseteq \operatorname{Ball}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_0)$ holds $f(x, g_1(x)) = c$ and $\operatorname{dom} g_2 = \operatorname{Ball}(a, r_0)$ and $\operatorname{rng} g_2 \subseteq \operatorname{Ball}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_0)$ holds $f(x, g_2(x)) = c$ holds $g_1 = g_2$. □

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