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On Roots of Polynomials and Algebraically Closed Fields

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Summary. In this article we further extend the algebraic theory of polynomial rings in Mizar [1, 2, 3]. We deal with roots and multiple roots of polynomials and show that both the real numbers and finite domains are not algebraically closed [5, 7]. We also prove the identity theorem for polynomials and that the number of multiple roots is bounded by the polynomial's degree [4, 6].

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1. Preliminaries

From now on n denotes a natural number.

Note that there exists a natural number which is non trivial and non prime.

Now we state the proposition:

(1) Let us consider an even natural number n, and an element x of $\mathbb{R}_{\mathrm{F}}$. Then $x^n \geqslant 0_{\mathbb{R}_{\mathrm{F}}}$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv x^{2 \cdot \$_1} \geqslant 0_{\mathbb{R}_{\mathrm{F}}}$. For every element x of $\mathbb{R}_{\mathrm{F}}$, $x^2 \geqslant 0_{\mathbb{R}_{\mathrm{F}}}$. For every natural number k, $\mathcal{P}[k]$. $\square$

Let us consider a ring R and an element a of R. Now we state the propositions:

(2) $2 \star a = a + a$.

(3) $a^2 = a \cdot a$.

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Let F be a field and a be an element of F. Note that $\frac{a}{1_F}$ reduces to a.

One can check that $\mathbb{Z}/2$ is non trivial and almost left invertible.

Let n be a non trivial, non prime natural number. Note that $\mathbb{Z}/n$ is non integral domain-like and $\mathbb{Z}/6$ is non degenerated.

2. Some More Properties of Polynomials

Let R be a non degenerated ring. Observe that every non zero polynomial over R is non-zero and every polynomial over R which is monic is also non zero.

Let p be a non zero polynomial over R. One can check that $\deg p$ is natural.

Let R be a ring, p be a zero polynomial over R, and q be a polynomial over R. Let us observe that $p * q$ is zero and $q * p$ is zero.

Let us observe that $p + q$ reduces to q and $q + p$ reduces to q.

Let p be a polynomial over R. One can check that $p * \mathbf{0}.R$ reduces to $\mathbf{0}.R$ and $p * \mathbf{1}.R$ reduces to p and $\mathbf{0}.R * p$ reduces to $\mathbf{0}.R$ and $\mathbf{1}.R * p$ reduces to p.

One can check that $1_R \cdot p$ reduces to p.

Now we state the propositions:

(4) Let us consider an integral domain R, a polynomial p over R, and a non zero element a of R. Then $\deg(a \cdot p) = \deg p$.

(5) Let us consider an integral domain R, a polynomial p over R, and an element a of R. Then $\mathrm{LC}(a \cdot p) = a \cdot \mathrm{LC}\, p$.

(6) Let us consider an integral domain R, and an element a of R. Then $\mathrm{LC}(a \restriction R) = a$. The theorem is a consequence of (5).

(7) Let us consider an integral domain R, a polynomial p over R, and elements v, x of R. Then $\mathrm{eval}(v \cdot p, x) = v \cdot \mathrm{eval}(p, x)$. The theorem is a consequence of (4).

(8) Let us consider a ring R, and elements a, b of R. Then $\mathrm{eval}(a \restriction R, b) = a$.

Let R be an integral domain and p, q be monic polynomials over R. Let us note that $p * q$ is monic.

Let a be an element of R and k be a natural number. One can check that $(\mathrm{rpoly}(1, a))^k$ is non zero and monic.

Now we state the propositions:

(9) Let us consider a non degenerated ring R, an element a of R, and a non zero element k of $\mathbb{N}$. Then $\mathrm{LC}\, \mathrm{rpoly}(k, a) = 1_R$.

(10) Let us consider a non degenerated, well unital, non empty double loop structure R, and an element a of R. Then $\langle -a, 1_R \rangle = \mathrm{rpoly}(1, a)$.

(11) Let us consider an integral domain R, a polynomial p over R, and an element x of R. Then $\mathrm{eval}(p, x) = 0_R$ if and only if $\mathrm{rpoly}(1, x) \mid p$.

(12) Let us consider an integral domain F, polynomials p, q over F, and an element a of F. Suppose $\mathrm{rpoly}(1, a) \mid p * q$. Then

(i) $\mathrm{rpoly}(1, a) \mid p$, or

(ii) $\mathrm{rpoly}(1, a) \mid q$.

The theorem is a consequence of (11).

(13) Let us consider an integral domain R, a polynomial p over R, and a non zero polynomial q over R. If $p \mid q$, then $\deg p \leqslant \deg q$.

(14) Let us consider a non degenerated commutative ring R, a polynomial q over R, a non zero polynomial p over R, and a non zero element b of R. If $q \mid p$, then $q \mid b \cdot p$.

(15) Let us consider a field F, a polynomial q over F, a non zero polynomial p over F, and a non zero element b of F. Then $q \mid p$ if and only if $q \mid b \cdot p$. The theorem is a consequence of (14).

Let us consider an integral domain R, a non zero polynomial p over R, an element a of R, and a non zero element b of R. Now we state the propositions:

(16) $\mathrm{rpoly}(1, a) \mid p$ if and only if $\mathrm{rpoly}(1, a) \mid b \cdot p$. The theorem is a consequence of (11), (7), and (14).

(17) $(\mathrm{rpoly}(1, a))^n \mid p$ if and only if $(\mathrm{rpoly}(1, a))^n \mid b \cdot p$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $(\mathrm{rpoly}(1, a))^{\$_1} \mid b \cdot p$, then $(\mathrm{rpoly}(1, a))^{\$_1} \mid p$. For every natural number k, $\mathcal{P}[k]$. $\square$

Let R be an integral domain, p be a non zero polynomial over R, and b be a non zero element of R. Let us note that $b \cdot p$ is non zero.

3. On Roots of Polynomials

Let R be a non degenerated ring. One can check that $\mathbf{1}.R$ and has not roots.

Let a be a non zero element of R. One can verify that $a \restriction R$ and has not roots and every polynomial over R which is non zero and has roots is also non constant and every polynomial over R which and has not roots is also non zero.

Let a be an element of R. One can check that $\mathrm{rpoly}(1, a)$ is non zero and has roots and there exists a polynomial over R which is non zero and has not roots and there exists a polynomial over R which is non zero and has roots.

Let R be an integral domain, p be a polynomial over R with non roots, and a be a non zero element of R. Let us note that $a \cdot p$ and has not roots.

Let p be a polynomial over R with roots and a be an element of R. Note that $a \cdot p$ has roots.

Let R be a non degenerated commutative ring and q be a polynomial over R. One can verify that $p * q$ has roots.

Let R be an integral domain and p, q be polynomials over R with non roots. One can check that $p * q$ and has not roots.

Let R be a non degenerated commutative ring, a be an element of R, and k be a non zero element of $\mathbb{N}$. Let us note that $\mathrm{rpoly}(k, a)$ is non constant and monic and has roots.

Let R be a non degenerated ring. Let us observe that there exists a polynomial over R which is non constant and monic.

Let R be an integral domain, a be an element of R, k be a non zero natural number, and n be a non zero element of $\mathbb{N}$. Note that $(\mathrm{rpoly}(n, a))^k$ is non constant and monic and has roots.

Let R be a ring and p be a polynomial over R with roots. Note that $\mathrm{Roots}(p)$ is non empty.

Let R be a non degenerated ring and p be a polynomial over R with non roots. Let us observe that $\mathrm{Roots}(p)$ is empty.

Let R be an integral domain. One can check that there exists a polynomial over R which is monic and has roots and there exists a polynomial over R which is monic and has not roots.

Now we state the propositions:

(18) Let us consider a non degenerated ring R, and an element a of R. Then $\mathrm{Roots}(\mathrm{rpoly}(1, a)) = \{a\}$.

(19) Let us consider an integral domain F, a polynomial p over F, and a non zero element b of F. Then $\mathrm{Roots}(b \cdot p) = \mathrm{Roots}(p)$. The theorem is a consequence of (7).

(20) There exist polynomials p, q over $\mathbb{Z}/6$ such that $\mathrm{Roots}(p*q) \not\subseteq \mathrm{Roots}(p) \cup \mathrm{Roots}(q)$.

(21) Let us consider an integral domain R, and elements a, b of R. Then $\mathrm{rpoly}(1, a) \mid \mathrm{rpoly}(1, b)$ if and only if $a = b$. The theorem is a consequence of (18).

(22) Let us consider an integral domain R, and a non zero polynomial p over R. Then $\overline{\overline{\mathrm{Roots}(p)}} \leqslant \deg p$.

4. More about Bags

Let X be a non empty set and B be a bag of X. We introduce the notation $\overline{\overline{B}}$ as a synonym of $\sum B$.

Observe that there exists a bag of X which is zero and there exists a bag of X which is non zero.

Let b_1 be a bag of X and b_2 be a bag of X. One can check that $b_1 + b_2$ is X-defined and $b_1 + b_2$ is total.

Let us consider a non empty set X and a bag b of X. Now we state the propositions:

(23) $\overline{\overline{b}} = 0$ if and only if $\operatorname{support} b = \emptyset$.

(24) b is zero if and only if $\operatorname{support} b = \emptyset$.

(25) b is zero if and only if $\operatorname{rng} b = \{0\}$.

Let X be a non empty set, b_1 be a non zero bag of X, and b_2 be a bag of X. One can check that $b_1 + b_2$ is non zero.

(26) Let us consider a non empty set X, a bag b of X, and an element x of X. Suppose $\operatorname{support} b = \{x\}$. Then $b = (\{x\}, b(x))$-bag.

(27) Let us consider a non empty set X, a non empty bag b of X, and an element x of X. Then $\operatorname{support} b = \{x\}$ if and only if $b = (\{x\}, b(x))$-bag and $b(x) \neq 0$. The theorem is a consequence of (26).

Let X be a set and S be a finite subset of X. The functor $\operatorname{Bag}(S)$ yielding a bag of X is defined by the term

(Def. 1) $(S, 1)$-bag.

Let X be a non empty set and S be a non empty, finite subset of X. Observe that $\operatorname{Bag}(S)$ is non zero.

Let b be a bag of X and a be an element of X. The functor $b \setminus a$ yielding a bag of X is defined by the term

(Def. 2) $b +\cdot (a, 0)$.

Let us consider a non empty set X, a bag b of X, and an element a of X. Now we state the propositions:

(28) $b \setminus a = b$ if and only if $a \notin \operatorname{support} b$.

(29) $\operatorname{support}(b \setminus a) = \operatorname{support} b \setminus \{a\}$.

(30) $(b \setminus a) + (\{a\}, b(a))$-bag $= b$.

(31) Let us consider a non empty set X, an element a of X, and an element n of $\mathbb{N}$. Then $\overline{\overline{(\{a\}, n)\text{-bag}}} = n$. The theorem is a consequence of (23).

5. On Multiple Roots of Polynomials

Let R be an integral domain and p be a non zero polynomial over R with roots. One can verify that $\operatorname{BRoots}(p)$ is non zero.

Now we state the propositions:

(32) Let us consider a non degenerated commutative ring R, a non zero polynomial p over R, and an element a of R. Then $\operatorname{multiplicity}(p, a) = 0$ if and only if $\operatorname{rpoly}(1, a) \nmid p$.

(33) Let us consider an integral domain R, a non zero polynomial p over R, and an element a of R. Then $\mathrm{multiplicity}(p, a) = n$ if and only if $(\mathrm{rpoly}(1, a))^n \mid p$ and $(\mathrm{rpoly}(1, a))^{n+1} \nmid p$. The theorem is a consequence of (10).

(34) Let us consider an integral domain R, and an element a of R. Then $\mathrm{multiplicity}(\mathrm{rpoly}(1, a), a) = 1$. The theorem is a consequence of (13) and (33).

(35) Let us consider an integral domain R, and elements a, b of R. If $b \neq a$, then $\mathrm{multiplicity}(\mathrm{rpoly}(1, a), b) = 0$. The theorem is a consequence of (21) and (32).

(36) Let us consider an integral domain R, a non zero polynomial p over R, a non zero element b of R, and an element a of R. Then $\mathrm{multiplicity}(p, a) = \mathrm{multiplicity}(b \cdot p, a)$. The theorem is a consequence of (33), (14), and (17).

(37) Let us consider an integral domain R, a non zero polynomial p over R, and a non zero element b of R. Then $\mathrm{BRoots}(b \cdot p) = \mathrm{BRoots}(p)$. The theorem is a consequence of (36).

(38) Let us consider an integral domain R, and a non zero polynomial p over R without roots. Then $\mathrm{BRoots}(p) = \mathrm{EmptyBag}(\text{the carrier of } R)$.

(39) Let us consider an integral domain R, and a non zero element a of R. Then $\overline{\overline{\mathrm{BRoots}(a \restriction R)}} = 0$. The theorem is a consequence of (23).

(40) Let us consider an integral domain R, and an element a of R. Then $\overline{\overline{\mathrm{BRoots}(\mathrm{rpoly}(1, a))}} = 1$. The theorem is a consequence of (10).

(41) Let us consider an integral domain R, and non zero polynomials p, q over R. Then $\overline{\overline{\mathrm{BRoots}(p * q)}} = \overline{\overline{\mathrm{BRoots}(p)}} + \overline{\overline{\mathrm{BRoots}(q)}}$.

(42) Let us consider an integral domain R, and a non zero polynomial p over R. Then $\overline{\overline{\mathrm{BRoots}(p)}} \leqslant \deg p$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non zero polynomial p over R such that $\deg p = \$_1$ holds $\overline{\overline{\mathrm{BRoots}(p)}} \leqslant \deg p$. $\mathcal{P}[0]$. For every natural number k, $\mathcal{P}[k]$. $\square$

6. The Polynomial $X^n + 1$

Let R be a unital, non empty double loop structure and n be a natural number. The functor npoly(R, n) yielding a sequence of R is defined by the term

(Def. 3) $\mathbf{0}.R+\cdot[0 \longmapsto 1_R, n \longmapsto 1_R]$.

One can check that npoly(R, n) is finite-Support and npoly(R, n) is non zero.

Let us consider a unital, non degenerated double loop structure R. Now we state the propositions:

(43) $\deg \mathrm{npoly}(R, n) = n$.

(44) $\mathrm{LC}\, \mathrm{npoly}(R, n) = 1_R$.

(45) Let us consider a non degenerated ring R, and an element x of R. Then $\mathrm{eval}(\mathrm{npoly}(R, 0), x) = 1_R$.

(46) Let us consider a non degenerated ring R, a non zero natural number n, and an element x of R. Then $\mathrm{eval}(\mathrm{npoly}(R, n), x) = x^n + 1_R$.

PROOF: Set $q = \mathrm{npoly}(R, n)$. Consider F being a finite sequence of elements of R such that $\mathrm{eval}(q, x) = \sum F$ and $\mathrm{len}\, F = \mathrm{len}\, q$ and for every element j of $\mathbb{N}$ such that $j \in \mathrm{dom}\, F$ holds $F(j) = q(j -' 1) \cdot \mathrm{power}_R(x, j -' 1)$. Consider f_1 being a sequence of the carrier of R such that $\sum F = f_1(\mathrm{len}\, F)$ and $f_1(0) = 0_R$ and for every natural number j and for every element v of R such that $j < \mathrm{len}\, F$ and $v = F(j + 1)$ holds $f_1(j + 1) = f_1(j) + v$. Define $\mathcal{P}[\text{element of } \mathbb{N}] \equiv \$_1 = 0$ and $f_1(\$_1) = 0_R$ or $0 < \$_1 < \mathrm{len}\, F$ and $f_1(\$_1) = 1_R$ or $\$_1 = \mathrm{len}\, F$ and $f_1(\$_1) = x^n + 1_R$. For every element j of $\mathbb{N}$ such that $0 \leqslant j \leqslant \mathrm{len}\, F$ holds $\mathcal{P}[j]$. $\square$

(47) Let us consider an even natural number n, and an element x of $\mathbb{R}_{\mathrm{F}}$. Then $\mathrm{eval}(\mathrm{npoly}(\mathbb{R}_{\mathrm{F}}, n), x) > 0_{\mathbb{R}_{\mathrm{F}}}$. The theorem is a consequence of (45), (1), and (46).

(48) Let us consider an odd natural number n. Then $\mathrm{eval}(\mathrm{npoly}(\mathbb{R}_{\mathrm{F}}, n), -1_{\mathbb{R}_{\mathrm{F}}}) = 0_{\mathbb{R}_{\mathrm{F}}}$. The theorem is a consequence of (46).

(49) $\mathrm{eval}(\mathrm{npoly}(\mathbb{Z}/2, 2), 1_{\mathbb{Z}/2}) = 0_{\mathbb{Z}/2}$. The theorem is a consequence of (46) and (2).

Let n be an even natural number. Let us note that npoly$(\mathbb{R}_{\mathrm{F}}, n)$ and has not roots.

Let n be an odd natural number. Observe that npoly$(\mathbb{R}_{\mathrm{F}}, n)$ has roots and npoly$(\mathbb{Z}/2, 2)$ has roots.

7. The Polynomials $(x - a_1) * (x - a_2) * \ldots * (x - a_n)$

Let R be a ring.

A product of linear polynomials of R is a polynomial over R and is defined by

(Def. 4) there exists a non empty finite sequence F of elements of PolyRing(R) such that $it = \prod F$ and for every natural number i such that $i \in \operatorname{dom} F$ there exists an element a of R such that $F(i) = \mathrm{rpoly}(1, a)$.

Let R be an integral domain. One can verify that every product of linear polynomials of R is non constant and monic and has roots.

Now we state the propositions:

(50) Let us consider an integral domain R, and a product of linear polynomials p of R. Then $\mathrm{LC}\, p = 1_R$.

(51) Let us consider an integral domain R, and an element a of R. Then $\mathrm{rpoly}(1, a)$ is a product of linear polynomials of R.

(52) Let us consider an integral domain R, and products of linear polynomials p, q of R. Then $p * q$ is a product of linear polynomials of R.

Let R be an integral domain and B be a non zero bag of the carrier of R.

A product of linear polynomials of R and B is a product of linear polynomials of R and is defined by

(Def. 5) $\deg it = \overline{\overline{B}}$ and for every element a of R, multiplicity$(it, a) = B(a)$.

Let us consider an integral domain R, a non zero bag B of the carrier of R, a product of linear polynomials p of R and B, and an element a of R. Now we state the propositions:

(53) If $a \in \operatorname{support} B$, then $\mathrm{eval}(p, a) = 0_R$. The theorem is a consequence of (11).

(54) (i) $(\mathrm{rpoly}(1, a))^{B(a)} \mid p$, and

(ii) $(\mathrm{rpoly}(1, a))^{B(a)+1} \nmid p$.

The theorem is a consequence of (33).

Let us consider an integral domain R, a non zero bag B of the carrier of R, and a product of linear polynomials p of R and B. Now we state the propositions:

(55) $\mathrm{BRoots}(p) = B$.

(56) $\deg p = \overline{\overline{\mathrm{BRoots}(p)}}$. The theorem is a consequence of (55).

(57) Let us consider an integral domain R, and an element a of R. Then $\mathrm{rpoly}(1, a)$ is a product of linear polynomials of R and $\mathrm{Bag}(\{a\})$. The theorem is a consequence of (51), (34), and (35).

(58) Let us consider an integral domain R, non zero bags B_1, B_2 of the carrier of R, a product of linear polynomials p of R and B_1, and a product of linear

polynomials q of R and B_2. Then $p * q$ is a product of linear polynomials of R and $B_1 + B_2$. The theorem is a consequence of (52), (56), and (55).

(59) Let us consider an integral domain R. Then every product of linear polynomials of R is a product of linear polynomials of R and BRoots(p).
Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every product of linear polynomials p of R such that $\deg p = \$_1$ holds p is a product of linear polynomials of R and BRoots(p). $\mathcal{P}[1]$. For every natural number k such that $k \geqslant 1$ holds $\mathcal{P}[k]$. □

Let R be an integral domain and S be a non empty, finite subset of R.

A product of linear polynomials of R and S is a product of linear polynomials of R and Bag(S). Now we state the proposition:

(60) Let us consider an integral domain R, a non empty, finite subset S of R, and a product of linear polynomials p of R and S. Then $\deg p = \overline{\overline{S}}$.

Let us consider an integral domain R, a non empty, finite subset S of R, a product of linear polynomials p of R and S, and an element a of R. Now we state the propositions:

(61) If $a \in S$, then $\mathrm{rpoly}(1, a) \mid p$ and $(\mathrm{rpoly}(1, a))^2 \nmid p$. The theorem is a consequence of (54).

(62) If $a \in S$, then $\mathrm{eval}(p, a) = 0_R$. The theorem is a consequence of (61).

(63) Let us consider an integral domain R, a non empty, finite subset S of R, and a product of linear polynomials p of R and S. Then Roots(p) $= S$. The theorem is a consequence of (62), (22), and (60).

8. Main Theorems

Now we state the proposition:

(64) Let us consider an integral domain R, and a non zero polynomial p over R with roots. Then there exists a product of linear polynomials q of R and BRoots(p) and there exists a polynomial r over R with non roots such that $p = q * r$ and Roots(q) $=$ Roots(p).
Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every non zero polynomial p over R with roots such that $\deg p = \$_1$ there exists a product of linear polynomials q of R and BRoots(p) and there exists a polynomial r over R with non roots such that $p = q * r$ and Roots(q) $=$ Roots(p). $\mathcal{P}[1]$ by (11), [9, (1)], (51), [8, (23), (27), (24)]. For every natural number k such that $1 \leqslant k$ holds $\mathcal{P}[k]$. Consider d being a natural number such that $\deg p = d$. □

Let us consider an integral domain R and a non zero polynomial p over R.

(65) $\overline{\overline{\mathrm{Roots}(p)}} \leqslant \overline{\overline{\mathrm{BRoots}(p)}}$. The theorem is a consequence of (64), (56), (55), (22), and (38).

(66) $\overline{\overline{\mathrm{BRoots}(p)}} = \deg p$ if and only if there exists an element a of R and there exists a product of linear polynomials q of R such that $p = a \cdot q$. The theorem is a consequence of (64), (56), (55), (59), (4), (37), and (38).

Now we state the proposition:

(67) Let us consider an integral domain R, and polynomials p, q over R. Suppose there exists a subset S of R such that $\overline{\overline{S}} = \max(\deg p, \deg q) + 1$ and for every element a of R such that $a \in S$ holds $\mathrm{eval}(p, a) = \mathrm{eval}(q, a)$. Then $p = q$. The theorem is a consequence of (22).

Let F be an algebraic closed field. Note that every non constant polynomial over F has roots and $\mathbb{R}_{\mathrm{F}}$ is non algebraic closed and every finite integral domain is non algebraic closed and every ring which is algebraic closed is also almost right invertible.

Now we state the propositions:

(68) Let us consider an algebraic closed field F, and a non constant polynomial p over F. Then there exists an element a of F and there exists a product of linear polynomials q of F and $\mathrm{BRoots}(p)$ such that $a \cdot q = p$. The theorem is a consequence of (64).

(69) Let us consider an algebraic closed field F. Then every non constant, monic polynomial over F is a product of linear polynomials of F and $\mathrm{BRoots}(p)$. The theorem is a consequence of (68).

(70) Let us consider a field F. Then F is algebraic closed if and only if every non constant, monic polynomial over F is a product of linear polynomials of F. The theorem is a consequence of (69).

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