FORMALIZED MATHEMATICS
Vol. 25, No. 2, Pages 141–147, 2017
DOI: 10.1515/forma-2017-0013

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Basel Problem – Preliminaries

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Summary. In the article we formalize in the Mizar system [4] preliminary facts needed to prove the Basel problem [7, 1]. Facts that are independent from the notion of structure are included here.

MSC: 11M06 03B35

Keywords: Basel problem

MML identifier: `BASEL_1`, version: `8.1.06 5.43.1297`

1. Preliminaries

From now on X denotes a set, k, m, n denote natural numbers, i denotes an integer, a, b, c, d, e, g, p, r, x, y denote real numbers, and z denotes a complex.

Now we state the proposition:

(1) If $0 < a$, then there exists m such that $0 < a \cdot m + b$.

Let f be a real-valued finite sequence. Let us consider n. Observe that $f \restriction n$ is $\mathbb{R}$-valued.

Let f be a complex-valued finite sequence. Let us observe that $f^{\mathbf{2}}$ is $(\operatorname{len} f)$-element and f^{-1} is $(\operatorname{len} f)$-element.

Let c be a complex. Note that $c + f$ is $(\operatorname{len} f)$-element.

Now we state the propositions:

(2) Let us consider complexes c, z. Then $c + \langle z \rangle = \langle c + z \rangle$.

(3) Let us consider complex-valued finite sequences f, g, and a complex c. Then $c + f \frown g = (c + f) \frown (c + g)$.

ISSN 1426–2630(Print), 1898-9934(Online)

(4) Let us consider a complex-valued finite sequence f, and a complex c. Then $\sum(c+f) = c \cdot \operatorname{len} f + \sum f$.
PROOF: Define $\mathcal{P}$[complex-valued finite sequence] $\equiv \sum(c+\$_1) = c \cdot \operatorname{len} \$_1 + \sum \$_1$. For every finite sequence p of elements of $\mathbb{C}$ and for every element x of $\mathbb{C}$ such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle x \rangle]$ by [3, (39), (22)], (2), [17, (32)]. For every finite sequence p of elements of $\mathbb{C}$, $\mathcal{P}[p]$ from [5, Sch. 2]. □

2. Limits of Sequences $\frac{an+b}{cn+d}$

Let a, b, c, d be complexes. The functor Rat-Exp-Seq(a, b, c, d) yielding a complex sequence is defined by

(Def. 1) $it(n) = \frac{\text{Polynom}(a,b,n)}{\text{Polynom}(c,d,n)}$.

Let us consider a, b, c, and d. The functor rseq(a, b, c, d) yielding a sequence of real numbers is defined by the term

(Def. 2) $\Re$(Rat-Exp-Seq(a, b, c, d)).

Now we state the propositions:

(5) $(\text{rseq}(a, b, c, d))(n) = \frac{a \cdot n+b}{c \cdot n+d}$.

(6) $(\text{rseq}(0, b, 0, d))(n) = \frac{b}{d}$. The theorem is a consequence of (5).

Let us consider a and b. Let us note that rseq$(a, b, 0, 0)$ is constant.

Let us consider d. One can verify that rseq$(0, b, 0, d)$ is constant.

Now we state the propositions:

(7) (i) $\text{rseq}(0, b, c, d) = b \cdot \text{rseq}(0, 1, c, d)$, and

(ii) $\text{rseq}(0, b, c, d) = (-b) \cdot \text{rseq}(0, 1, -c, -d)$.

The theorem is a consequence of (5).

(8) (i) $\text{rseq}(a, 0, c, d) = a \cdot \text{rseq}(1, 0, c, d)$, and

(ii) $\text{rseq}(a, 0, c, d) = (-a) \cdot \text{rseq}(1, 0, -c, -d)$.

The theorem is a consequence of (5).

Let us consider b, c, and d. Let us observe that rseq$(0, b, c, d)$ is convergent.

Now we state the propositions:

(9) $\lim \text{rseq}(0, b, 0, d) = \frac{b}{d}$. The theorem is a consequence of (6).

(10) Let us consider a non zero real number c. Then $\lim \text{rseq}(0, b, c, d) = 0$. The theorem is a consequence of (5).

Let c be a non zero real number. Let us consider a, b, and d. Note that rseq(a, b, c, d) is convergent.

Let a, d be positive real numbers and b be a real number. Let us observe that rseq$(a, b, 0, d)$ is non upper bounded.

Let a, d be negative real numbers. Let us consider b. One can check that $\mathrm{rseq}(a, b, 0, d)$ is non upper bounded.

Let a be a positive real number and d be a negative real number. Note that $\mathrm{rseq}(a, b, 0, d)$ is non lower bounded.

Let a be a negative real number and d be a positive real number. Let us note that $\mathrm{rseq}(a, b, 0, d)$ is non lower bounded.

Let a, d be non zero real numbers. One can check that $\mathrm{rseq}(a, b, 0, d)$ is non bounded and $\mathrm{rseq}(a, b, 0, d)$ is non convergent.

Now we state the propositions:

(11) Let us consider a non zero real number c. Then $\lim \mathrm{rseq}(a, b, c, d) = \frac{a}{c}$.
The theorem is a consequence of (5) and (10).

(12) Let us consider a non zero real number a. Then $\lim \mathrm{rseq}(a, b, a, d) = 1$.
The theorem is a consequence of (11).

3. Trigonometry

Now we state the propositions:

(13) $\sin(\pi \cdot i) = 0$.

(14) $\cos(\frac{\pi}{2} + (\pi \cdot i)) = 0$.

(15) (i) $\tan r = (\cot r)^{-1}$, and

(ii) $\cot r = (\tan r)^{-1}$.

(16) dom(the function tan) $= \bigcup$ the set of all $]-\frac{\pi}{2} + (\pi \cdot i), \frac{\pi}{2} + (\pi \cdot i)[$ where i is an integer.

Proof: Set $S =$ the set of all $]-\frac{\pi}{2} + (\pi \cdot i), \frac{\pi}{2} + (\pi \cdot i)[$ where i is an integer. Set $T =$ dom(the function tan). $T \subseteq \bigcup S$ by (14), [24, (29)]. For every set X such that $X \in S$ holds $X \subseteq T$ by [16, (11)], [8, (9)], [21, (1)], [16, (13)]. $\square$

Observe that dom(the function tan) is open as a subset of $\mathbb{R}$.

Now we state the propositions:

(17) If $0 \leqslant r$, then (the function sin)$(r) \leqslant r$.

Proof: Reconsider $A = [0, r]$ as a non empty, closed interval subset of $\mathbb{R}$. Reconsider $c =$ (the function cos) $\restriction A$ as a function from A into $\mathbb{R}$. $c\restriction A$ is bounded and c is integrable by [11, (11), (10)]. integral $c =$ (the function sin)(r) by [11, (19)], [22, (24)], [26, (30)]. Set $Z_0 = \square^0$. Reconsider $Z_3 = Z_0 \restriction A$ as a function from A into $\mathbb{R}$. $Z_3\restriction A$ is bounded and Z_3 is integrable by [11, (11), (10)]. integral $Z_3 = r$ by [14, (21)], [19, (35)], [11, (19)], [22, (30)]. For every r such that $r \in A$ holds $c(r) \leqslant Z_3(r)$ by [6, (49)], [19, (34)], [13, (6)]. $\square$

(18) If $0 \leqslant r < \frac{\pi}{2}$, then $r \leqslant$ (the function tan)(r).
PROOF: Reconsider $A = [0, r]$ as a non empty, closed interval subset of $\mathbb{R}$. Set $Z_0 = \square^0$. Reconsider $Z_3 = Z_0 \restriction A$ as a function from A into $\mathbb{R}$. $Z_3 \restriction A$ is bounded and Z_3 is integrable by [11, (11), (10)]. integral $Z_3 = r$ by [14, (21)], [19, (35)], [11, (19)], [22, (30)]. Set $T =$ dom(the function tan). Set $c_2 =$ (the function cos) $\cdot$ (the function cos). Set $c_3 = c_2 \restriction T$. Set $Z_1 = \frac{Z_0}{c_3}$. ${c_3}^{-1}(\{0\}) = \emptyset$ by [6, (47)]. Reconsider $Z_2 = Z_1 \restriction A$ as a function from A into $\mathbb{R}$. $Z_1 \restriction A$ is continuous and $Z_2 \restriction A$ is bounded and Z_2 is integrable by [20, (24)], [11, (11), (10)]. For every real number s such that $s \in T$ holds $Z_1(s) = \frac{1}{(\text{the function cos})(s)^2}$ and (the function cos)$(s) \neq 0$ by [19, (34)], [6, (47)]. integral $Z_2 =$ (the function tan)(r) by [12, (19)], [18, (59)], [15, (41)]. For every r such that $r \in A$ holds $Z_3(r) \leqslant Z_2(r)$ by [6, (49)], [19, (34)], [16, (11)], [13, (6)]. $\square$

4. Some Special Functions and Sequences

Let f be a real-valued function. The functors: $\cot f$ and $\operatorname{cosec} f$ yielding functions are defined by conditions

(Def. 3) $\operatorname{dom} \cot f = \operatorname{dom} f$ and for every object x such that $x \in \operatorname{dom} f$ holds $\cot f(x) = \cot(f(x))$,

(Def. 4) $\operatorname{dom} \operatorname{cosec} f = \operatorname{dom} f$ and for every object x such that $x \in \operatorname{dom} f$ holds $\operatorname{cosec} f(x) = \operatorname{cosec}(f(x))$,

respectively. Note that $\cot f$ is $\mathbb{R}$-valued and $\operatorname{cosec} f$ is $\mathbb{R}$-valued.

Let f be a real-valued finite sequence. Let us observe that $\cot f$ is finite sequence-like and $\operatorname{cosec} f$ is finite sequence-like.

Let us consider a real-valued finite sequence f. Now we state the propositions:

(19) $\operatorname{len} \cot f = \operatorname{len} f$.

(20) $\operatorname{len} \operatorname{cosec} f = \operatorname{len} f$.

Let f be a real-valued finite sequence. Note that $\cot f$ is $(\operatorname{len} f)$-element and $\operatorname{cosec} f$ is $(\operatorname{len} f)$-element.

Let us consider m. The functor x-r-seq(m) yielding a finite sequence is defined by the term

(Def. 5) $\frac{\pi}{2 \cdot m+1} \cdot \operatorname{idseq}(m)$.

Now we state the propositions:

(21) (i) len x-r-seq$(m) = m$, and

(ii) for every k such that $1 \leqslant k \leqslant m$ holds (x-r-seq$(m))(k) = \frac{k \cdot \pi}{2 \cdot m+1}$.

(22) rng x-r-seq$(m) \subseteq]0, \frac{\pi}{2}[$. The theorem is a consequence of (21).

Let us consider m. Let us note that x-r-seq(m) is $\mathbb{R}$-valued.

Now we state the proposition:

(23) If $1 \leqslant k \leqslant m$, then $0 < (\text{x-r-seq}(m))(k) < \frac{\pi}{2}$. The theorem is a consequence of (22) and (21).

Note that x-r-seq(0) is empty.

(24) If $1 \leqslant k \leqslant m$, then $\frac{2}{k \cdot \pi} + (\text{x-r-seq}(m))^{-1}(k) = (\text{x-r-seq}(m+1))^{-1}(k)$. The theorem is a consequence of (21).

(25) If $1 \leqslant k \leqslant m$, then $2 \cdot m + 1 \cdot (\text{x-r-seq}(m))(k) = k \cdot \pi$. The theorem is a consequence of (21).

(26) ${}^{2}\text{cosec}\, \text{x-r-seq}(m) = 1 + {}^{2}\text{cot}\, \text{x-r-seq}(m)$. The theorem is a consequence of (21) and (23).

(27) x-r-seq(n) is one-to-one. The theorem is a consequence of (21).

(28) ${}^{2}\text{cot}\, \text{x-r-seq}(n)$ is one-to-one.

PROOF: Set $f = \text{x-r-seq}(n)$. f is one-to-one. $0 < f(x_1) < \frac{\pi}{2}$ and $0 < f(x_2) < \frac{\pi}{2}$ and $\frac{\pi}{2} < \pi$. $\cot(f(x_1)) = \cot(f(x_2))$ by [23, (40)]. $f(x_1) = f(x_2)$ by [15, (2)], [25, (57)], [6, (47)], [15, (10)]. $\square$

(29) $\sum({}^{2}\text{cot}\, \text{x-r-seq}(m)) \leqslant \sum({}^{2}\text{x-r-seq}(m))^{-1}$. The theorem is a consequence of (21), (19), (15), (23), (16), and (18).

(30) $\sum({}^{2}\text{x-r-seq}(m))^{-1} \leqslant \sum({}^{2}\text{cosec}\, \text{x-r-seq}(m))$. The theorem is a consequence of (21), (20), (23), and (17).

The functors: Basel-seq, Basel-seq^1, and Basel-seq^2 yielding sequences of real numbers are defined by terms

(Def. 6) $\text{rseq}(0, 1, 1, 0) \cdot \text{rseq}(0, 1, 1, 0)$,

(Def. 7) $(\frac{\pi^{\mathbf{2}}}{6} \cdot \text{rseq}(2, 0, 2, 1)) \cdot \text{rseq}(2, -1, 2, 1)$,

(Def. 8) $(\frac{\pi^{\mathbf{2}}}{6} \cdot \text{rseq}(2, 0, 2, 1)) \cdot \text{rseq}(2, 2, 2, 1)$,

respectively. Now we state the propositions:

(31) $(\text{Basel-seq})(n) = \frac{1}{n^{\mathbf{2}}}$.

(32) $(\text{Basel-seq}^1)(n) = \frac{\pi^{\mathbf{2}}}{6} \cdot \frac{2 \cdot n}{2 \cdot n+1} \cdot \frac{2 \cdot n-1}{2 \cdot n+1}$. The theorem is a consequence of (5).

(33) $(\text{Basel-seq}^2)(n) = \frac{\pi^{\mathbf{2}}}{6} \cdot \frac{2 \cdot n}{2 \cdot n+1} \cdot \frac{2 \cdot n+2}{2 \cdot n+1}$. The theorem is a consequence of (5).

Let us observe that Basel-seq is convergent and Basel-seq^1 is convergent and Basel-seq^2 is convergent.

(34) $\lim \text{Basel-seq}^1 = \frac{\pi^{\mathbf{2}}}{6} = \lim \text{Basel-seq}^2$.

(35) $\sum({}^{2}\text{x-r-seq}(m))^{-1} = \frac{(2 \cdot m+1)^{\mathbf{2}}}{\pi^{\mathbf{2}}} \cdot \sum_{\kappa=0}^{m} \text{Basel-seq}(\kappa)$.

PROOF: Set $a = \pi^{\mathbf{2}}$. Set $b = (2 \cdot m + 1)^{\mathbf{2}}$. Set $B = \text{Basel-seq}$. Set $S = \text{Shift}(B \restriction \mathbb{Z}_{m+1}, 1)$. Set $M = \text{x-r-seq}(m)$. Set $G = ({}^{2}M)^{-1}$. Set $F = \langle 0 \rangle \frown G$. $B(0) = \frac{1}{0^{\mathbf{2}}}$. $F = \frac{b}{a} \cdot S$ by [9, (3)], [2, (11)], [10, (47)], (31). $\square$

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Received June 27, 2017

The English version of this volume of Formalized Mathematics was financed under agreement 548/P-DUN/2016 with the funds from the Polish Minister of Science and Higher Education for the dissemination of science.