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Pascal's Theorem in Real Projective Plane

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Summary. In this article we check, with the Mizar system [2], Pascal's theorem in the real projective plane (in projective geometry Pascal's theorem is also known as the Hexagrammum Mysticum Theorem)[1]. Pappus' theorem is a special case of a degenerate conic of two lines.

For proving Pascal's theorem, we use the techniques developed in the section "Projective Proofs of Pappus' Theorem" in the chapter "Pappus' Theorem: Nine proofs and three variations" [11]. We also follow some ideas from Harrison's work. With HOL Light, he has the proof of Pascal's theorem[2]. For a lemma, we use `PROVER9`[3] and `OTT2MIZ` by Josef Urban[4] [12, 6, 7]. We note, that we don't use Skolem/Herbrand functions (see "Skolemization" in [1]).

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1. Preliminaries

From now on n denotes a natural number, K denotes a field, a, b, c, d, e, f, g, h, i, a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1, i_1 denote elements of K, M, N denote square matrices over K of dimension 3, and p denotes a finite sequence of elements of $\mathbb{R}$.

Now we state the propositions:

(1) Let us consider points p, q, r of $\mathcal{E}^3_{\mathrm{T}}$. Then

[1] `https://en.wikipedia.org/wiki/Pascal's_theorem`
[2] `https://github.com/jrh13/hol-light/tree/master/100/pascal.ml`
[3] `https://www.cs.unm.edu/~mccune/prover9/`
[4] `https://github.com/JUrban/ott2miz`

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(i) $\langle |p, q, r| \rangle = \langle |r, p, q| \rangle$, and

(ii) $\langle |p, q, r| \rangle = \langle |q, r, p| \rangle$.

(2) Suppose $\langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle = \langle \langle a_1, b_1, c_1 \rangle, \langle d_1, e_1, f_1 \rangle, \langle g_1, h_1, i_1 \rangle \rangle$. Then

(i) $a = a_1$, and

(ii) $b = b_1$, and

(iii) $c = c_1$, and

(iv) $d = d_1$, and

(v) $e = e_1$, and

(vi) $f = f_1$, and

(vii) $g = g_1$, and

(viii) $h = h_1$, and

(ix) $i = i_1$.

(3) There exists a and there exists b and there exists c and there exists d and there exists e and there exists f and there exists g and there exists h and there exists i such that $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$.

(4) Suppose $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$. Then

(i) $a = M_{1,1}$, and

(ii) $b = M_{1,2}$, and

(iii) $c = M_{1,3}$, and

(iv) $d = M_{2,1}$, and

(v) $e = M_{2,2}$, and

(vi) $f = M_{2,3}$, and

(vii) $g = M_{3,1}$, and

(viii) $h = M_{3,2}$, and

(ix) $i = M_{3,3}$.

(5) Suppose $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$. Then $M^{\mathrm{T}} = \langle \langle a, d, g \rangle, \langle b, e, h \rangle, \langle c, f, i \rangle \rangle$. The theorem is a consequence of (4) and (3).

(6) Suppose $M = \langle \langle a, b, c \rangle, \langle d, e, f \rangle, \langle g, h, i \rangle \rangle$ and M is symmetric. Then

(i) $b = d$, and

(ii) $c = g$, and

(iii) $h = f$.

The theorem is a consequence of (5) and (2).

(7) Let us consider square matrices M, N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. If N is symmetric, then $M^{\mathrm{T}} \cdot N \cdot M$ is symmetric.

(8) Let us consider a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements a, b, c, d, e, f, g, h, i, x, y, z of $\mathbb{R}_{\mathrm{F}}$, an element v of $\mathcal{E}^3_{\mathrm{T}}$, a finite sequence u_{10} of elements of $\mathbb{R}_{\mathrm{F}}$, and a finite sequence p of elements of $\mathbb{R}^1$. Suppose $p = M \cdot u_{10}$ and $v = \mathrm{M2F}(p)$ and $M = \langle\langle a,b,c\rangle,\langle d,e,f\rangle,\langle g,h,i\rangle\rangle$ and $u_{10} = \langle x,y,z\rangle$. Then

(i) $p = \langle\langle a\cdot x+(b\cdot y)+(c\cdot z)\rangle,\langle d\cdot x+(e\cdot y)+(f\cdot z)\rangle,\langle g\cdot x+(h\cdot y)+(i\cdot z)\rangle\rangle$, and

(ii) $v = \langle a\cdot x+(b\cdot y)+(c\cdot z), d\cdot x+(e\cdot y)+(f\cdot z), g\cdot x+(h\cdot y)+(i\cdot z)\rangle$.

(9) Let us consider a square matrix M over $\mathbb{R}$ of dimension 3, and elements a, b, c, d, e, f, g, h, i, p_1, p_2, p_3 of $\mathbb{R}$. Suppose $M = \langle\langle a,b,c\rangle,\langle d,e,f\rangle,\langle g,h,i\rangle\rangle$ and $p = \langle p_1,p_2,p_3\rangle$. Then $M\cdot p = \langle a\cdot p_1+(b\cdot p_2)+(c\cdot p_3), d\cdot p_1+(e\cdot p_2)+(f\cdot p_3), g\cdot p_1+(h\cdot p_2)+(i\cdot p_3)\rangle$.

2. Conic in Real Projective Plane

Let a, b, c, d, e, f be real numbers and u be an element of $\mathcal{E}^3_{\mathrm{T}}$. The functor $\mathrm{qfconic}(a,b,c,d,e,f,u)$ yielding a real number is defined by the term

(Def. 1) $a\cdot u(1)\cdot u(1)+(b\cdot u(2)\cdot u(2))+(c\cdot u(3)\cdot u(3))+(d\cdot u(1)\cdot u(2))+(e\cdot u(1)\cdot u(3))+(f\cdot u(2)\cdot u(3))$.

The functor $\mathrm{conic}(a,b,c,d,e,f)$ yielding a subset of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 2) $\{P$, where P is a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$: for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $P =$ the direction of u holds $\mathrm{qfconic}(a,b,c,d,e,f,u) = 0\}$.

In the sequel a, b, c, d, e, f denote real numbers, u, u_1, u_2 denote non zero elements of $\mathcal{E}^3_{\mathrm{T}}$, and P denotes an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$.

Now we state the propositions:

(10) Suppose the direction of $u_1 =$ the direction of u_2 and $\mathrm{qfconic}(a,b,c,d,e,f,u_1) = 0$. Then $\mathrm{qfconic}(a,b,c,d,e,f,u_2) = 0$.

(11) If $P =$ the direction of u and $\mathrm{qfconic}(a,b,c,d,e,f,u) = 0$, then $P \in \mathrm{conic}(a,b,c,d,e,f)$. The theorem is a consequence of (10).

Let a, b, c, d, e, f be real numbers. The functor $\mathrm{symmetric3}(a,b,c,d,e,f)$ yielding a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 is defined by the term

(Def. 3) $\langle\langle a,d,e\rangle,\langle d,b,f\rangle,\langle e,f,c\rangle\rangle$.

Now we state the propositions:

(12) symmetric3(a, b, c, d, e, f) is symmetric. The theorem is a consequence of (5).

(13) Let us consider real numbers a, b, c, d, e, f, a point u of $\mathcal{E}^3_{\mathrm{T}}$, and a square matrix M over $\mathbb{R}$ of dimension 3. Suppose $p = u$ and $M =$ symmetric3(a, b, c, d, e, f).
Then SumAll QuadraticForm$(p, M, p) =$ qfconic$(a, b, c, 2 \cdot d, 2 \cdot e, 2 \cdot f, u)$.

(14) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, square matrices N_1, M_1, M_2 over $\mathbb{R}$ of dimension 3, and real numbers a, b, c, d, e, f. Suppose $N_1 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})N$ and $M_1 =$ symmetric3$(a, b, c, \frac{d}{2}, \frac{f}{2}, \frac{e}{2})$ and $M_2 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}}){N_1}^{\mathrm{T}})^{\smile} \cdot M_1 \cdot (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}})N_1)^{\smile}$.
Then $(\mathbb{R} \to \mathbb{R}_{\mathrm{F}})M_2$ is symmetric.
Proof: $((\mathbb{R} \to \mathbb{R}_{\mathrm{F}}){N_1}^{\mathrm{T}})^{\mathrm{T}} = (\mathbb{R} \to \mathbb{R}_{\mathrm{F}})N_1$ by [3, (16)]. $(\mathbb{R} \to \mathbb{R}_{\mathrm{F}})M_2$ is symmetric by [3, (16)], (12), (7). □

(15) Let us consider real numbers a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6. Suppose symmetric3$(a_1, a_2, a_3, a_4, a_5, a_6) =$ symmetric3$(b_1, b_2, b_3, b_4, b_5, b_6)$. Then

(i) $a_1 = b_1$, and

(ii) $a_2 = b_2$, and

(iii) $a_3 = b_3$, and

(iv) $a_4 = b_4$, and

(v) $a_5 = b_5$, and

(vi) $a_6 = b_6$.

The theorem is a consequence of (2).

(16) Let us consider real numbers a, b, c, d, e, f, a point P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that $P \in$ conic(a, b, c, d, e, f). Let us consider real numbers f_5, f_{12}, f_{19}, f_{20}, f_{21}, f_{23}, f_{22}, square matrices M_1, M_2 over $\mathbb{R}$ of dimension 3, and a square matrix N_1 over $\mathbb{R}$ of dimension 3. Suppose $M_1 =$ symmetric3$(a, b, c, \frac{d}{2}, \frac{e}{2}, \frac{f}{2})$ and $N_1 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})N$ and $M_2 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}}){N_1}^{\mathrm{T}})^{\smile} \cdot M_1 \cdot (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}})N_1)^{\smile}$ and $M_2 =$ symmetric3$(f_5, f_{21}, f_{23}, f_{12}, f_{19}, f_{22})$. Then

(i) it is not true that $f_5 = 0$ and $f_{21} = 0$ and $f_{23} = 0$ and $f_{12} = 0$ and $f_{22} = 0$ and $f_{19} = 0$, and

(ii) (the homography of N)$(P) \in$ conic$(f_5, f_{21}, f_{23}, 2 \cdot f_{12}, 2 \cdot f_{19}, 2 \cdot f_{22})$.

Proof: Consider Q being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P = Q$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $Q =$ the direction of u holds $\mathrm{qfconic}(a, b, c, d, e, f, u) = 0$. Reconsider $M = \mathrm{symmetric3}(a, b, c, \frac{d}{2}, \frac{e}{2}, \frac{f}{2})$ as a square matrix over $\mathbb{R}$ of dimension 3. Consider u_{19}, v_3 being elements of $\mathcal{E}^3_{\mathrm{T}}$, u_{17} being a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$, p_{11} being a finite sequence of elements of $\mathbb{R}^1$ such that $P =$ the direction of u_{19} and u_{19} is not zero and $u_{19} = u_{17}$ and $p_{11} = N \cdot u_{17}$ and $v_3 = \mathrm{M2F}(p_{11})$ and v_3 is not zero and (the homography of N)$(P) =$ the direction of v_3. Reconsider $p_{10} = u_{19}$ as a finite sequence of elements of $\mathbb{R}$. $\mathrm{SumAll\,QuadraticForm}(p_{10}, M, p_{10}) = \mathrm{qfconic}(a, b, c, 2 \cdot \frac{d}{2}, 2 \cdot \frac{e}{2}, 2 \cdot \frac{f}{2}, u_{19})$. Consider a_8, b_8, c_{11}, d_4, e_5, f_{24}, g_2, h_2, i_2 being elements of $\mathbb{R}_{\mathrm{F}}$ such that $N = \langle\langle a_8, b_8, c_{11}\rangle, \langle d_4, e_5, f_{24}\rangle, \langle g_2, h_2, i_2\rangle\rangle$. Reconsider $u_{10} = u_{17}$ as a finite sequence of elements of $\mathbb{R}$. Reconsider $N_1 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})N$ as a square matrix over $\mathbb{R}$ of dimension 3. Reconsider $M_2 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}})N_1{}^{\mathrm{T}})^{\smile} \cdot M \cdot (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}})N_1)^{\smile}$ as a square matrix over $\mathbb{R}$ of dimension 3. $((\mathbb{R} \to \mathbb{R}_{\mathrm{F}})N_1{}^{\mathrm{T}})^{\mathrm{T}} = (\mathbb{R} \to \mathbb{R}_{\mathrm{F}})N_1$ by [3, (16)]. $(\mathbb{R} \to \mathbb{R}_{\mathrm{F}})M_2$ is symmetric by [3, (16)], (12), (7). Consider m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9 being elements of $\mathbb{R}_{\mathrm{F}}$ such that $M_2 = \langle\langle m_1, m_2, m_3\rangle, \langle m_4, m_5, m_6\rangle, \langle m_7, m_8, m_9\rangle\rangle$. $m_2 = m_4$ and $m_3 = m_7$ and $m_8 = m_6$. Reconsider $u_3 = N_1 \cdot u_{10}$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. u_3 is not zero by [5, (24)], [14, (59), (86)]. Reconsider $u_2 = N_1 \cdot u_{10}$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $f_5 = m_1$, $f_{12} = m_2$, $f_{19} = m_3$, $f_{21} = m_5$, $f_{22} = m_6$, $f_{23} = m_9$ as a real number. $\mathrm{qfconic}(f_5, f_{21}, f_{23}, 2 \cdot f_{12}, 2 \cdot f_{19}, 2 \cdot f_{22}, u_2) = 0$. It is not true that $f_5 = 0$ and $f_{21} = 0$ and $f_{23} = 0$ and $2 \cdot f_{12} = 0$ and $2 \cdot f_{22} = 0$ and $2 \cdot f_{19} = 0$. $u_2 = v_3$. For every real numbers u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{18}, u_{16} and for every square matrices U_1, U_2 over $\mathbb{R}$ of dimension 3 and for every square matrix U_3 over $\mathbb{R}$ of dimension 3 such that $U_1 = \mathrm{symmetric3}(a, b, c, \frac{d}{2}, \frac{e}{2}, \frac{f}{2})$ and $U_3 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})N$ and $U_2 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}})U_3{}^{\mathrm{T}})^{\smile} \cdot U_1 \cdot (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}})U_3)^{\smile}$ and $U_2 = \mathrm{symmetric3}(u_{11}, u_{15}, u_{18}, u_{12}, u_{13}, u_{16})$ holds it is not true that $u_{11} = 0$ and $u_{15} = 0$ and $u_{18} = 0$ and $u_{12} = 0$ and $u_{16} = 0$ and $u_{13} = 0$. (the homography of N)$(P) \in \mathrm{conic}(u_{11}, u_{15}, u_{18}, 2 \cdot u_{12}, 2 \cdot u_{13}, 2 \cdot u_{16})$. $\square$

(17) Let us consider real numbers a, b, c, d, e, f, points P_1, P_2, P_3, P_4, P_5, P_6 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that P_1, P_2, P_3, P_4, P_5, $P_6 \in \mathrm{conic}(a, b, c, d, e, f)$. Then there exist real numbers a_2, b_2, c_2, d_2, e_2, f_2 such that

(i) it is not true that $a_2 = 0$ and $b_2 = 0$ and $c_2 = 0$ and $d_2 = 0$ and $e_2 = 0$ and $f_2 = 0$, and

(ii) (the homography of N)(P_1), (the homography of N)(P_2), (the homography of N)(P_3), (the homography of N)(P_4), (the homography of N)(P_5), (the homography of N)(P_6) $\in$ conic($a_2, b_2, c_2, d_2, e_2, f_2$).

The theorem is a consequence of (3), (14), (6), and (16).

From now on a, b, c, d, e, f, g, h, i denote elements of $\mathbb{R}_{\mathrm{F}}$.

Now we state the proposition:

(18) (i) if qfconic($a, b, c, d, e, f, [1, 0, 0]$) $= 0$, then $a = 0$, and

(ii) if qfconic($a, b, c, d, e, f, [0, 1, 0]$) $= 0$, then $b = 0$, and

(iii) if qfconic($a, b, c, d, e, f, [0, 0, 1]$) $= 0$, then $c = 0$, and

(iv) if qfconic($0, 0, 0, d, e, f, [1, 1, 1]$) $= 0$, then $d + e + f = 0$.

3. Pascal's Theorem

In the sequel M denotes a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, e_1, e_2, e_3, f_1, f_2, f_3 denote elements of $\mathbb{R}_{\mathrm{F}}$, M_8, M_{14}, M_{20}, M_{21}, M_{22}, M_{19}, M_{13}, M_{10}, M_9, M_{12}, M_{16}, M_{17}, M_{11}, M_{15}, M_{18} denote square matrices over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and r_1, r_2 denote real numbers.

Now we state the proposition:

(19) Suppose $M_9 = \langle\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle e_1, e_2, e_3\rangle\rangle$ and $M_{12} = \langle\langle 1, 0, 0\rangle, \langle 0, 0, 1\rangle, \langle f_1, f_2, f_3\rangle\rangle$ and $M_{16} = \langle\langle 0, 1, 0\rangle, \langle 1, 1, 1\rangle, \langle f_1, f_2, f_3\rangle\rangle$ and $M_{17} = \langle\langle 0, 0, 1\rangle, \langle 1, 1, 1\rangle, \langle e_1, e_2, e_3\rangle\rangle$ and $M_{10} = \langle\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle f_1, f_2, f_3\rangle\rangle$ and $M_{11} = \langle\langle 1, 0, 0\rangle, \langle 0, 0, 1\rangle, \langle e_1, e_2, e_3\rangle\rangle$ and $M_{15} = \langle\langle 0, 1, 0\rangle, \langle 1, 1, 1\rangle, \langle e_1, e_2, e_3\rangle\rangle$ and $M_{18} = \langle\langle 0, 0, 1\rangle, \langle 1, 1, 1\rangle, \langle f_1, f_2, f_3\rangle\rangle$ and ($r_1 \neq 0$ or $r_2 \neq 0$) and $r_1 \cdot e_1 \cdot e_2 + (r_2 \cdot e_1 \cdot e_3) = r_1 + r_2 \cdot e_2 \cdot e_3$ and $r_1 \cdot f_1 \cdot f_2 + (r_2 \cdot f_1 \cdot f_3) = r_1 + r_2 \cdot f_2 \cdot f_3$. Then $\mathrm{Det}\, M_9 \cdot \mathrm{Det}\, M_{12} \cdot \mathrm{Det}\, M_{16} \cdot \mathrm{Det}\, M_{17} = \mathrm{Det}\, M_{10} \cdot \mathrm{Det}\, M_{11} \cdot \mathrm{Det}\, M_{15} \cdot \mathrm{Det}\, M_{18}$.

In the sequel p_1, p_2, p_3, p_4, p_5, p_6 denote points of $\mathcal{E}^3_{\mathrm{T}}$.

(20) Suppose $M_9 = \langle p_1, p_2, p_5\rangle$ and $M_{12} = \langle p_1, p_3, p_6\rangle$ and $M_{16} = \langle p_2, p_4, p_6\rangle$ and $M_{17} = \langle p_3, p_4, p_5\rangle$ and $M_{10} = \langle p_1, p_2, p_6\rangle$ and $M_{11} = \langle p_1, p_3, p_5\rangle$ and $M_{15} = \langle p_2, p_4, p_5\rangle$ and $M_{18} = \langle p_3, p_4, p_6\rangle$. Then

(i) $\mathrm{Det}\, M_9 = \langle |p_1, p_2, p_5| \rangle$, and

(ii) $\mathrm{Det}\, M_{12} = \langle |p_1, p_3, p_6| \rangle$, and

(iii) $\mathrm{Det}\, M_{16} = \langle |p_2, p_4, p_6| \rangle$, and

(iv) $\mathrm{Det}\, M_{17} = \langle |p_3, p_4, p_5| \rangle$, and

(v) $\mathrm{Det}\, M_{10} = \langle |p_1, p_2, p_6| \rangle$, and

(vi) $\mathrm{Det}\, M_{11} = \langle |p_1, p_3, p_5| \rangle$, and

(vii) $\mathrm{Det}\, M_{15} = \langle |p_2, p_4, p_5| \rangle$, and

(viii) $\mathrm{Det}\, M_{18} = \langle|p_3, p_4, p_6|\rangle$.

From now on p_7, p_8, p_9 denote points of $\mathcal{E}^3_{\mathrm{T}}$.

(21) Suppose $\langle|p_1, p_5, p_9|\rangle = 0$. Then $\langle|p_1, p_5, p_7|\rangle \cdot \langle|p_2, p_5, p_9|\rangle = -(\langle|p_1, p_2, p_5|\rangle \cdot \langle|p_5, p_9, p_7|\rangle)$. The theorem is a consequence of (1).

(22) Suppose $\langle|p_1, p_6, p_8|\rangle = 0$. Then $\langle|p_1, p_2, p_6|\rangle \cdot \langle|p_3, p_6, p_8|\rangle = \langle|p_1, p_3, p_6|\rangle \cdot \langle|p_2, p_6, p_8|\rangle$. The theorem is a consequence of (1).

(23) Suppose $\langle|p_2, p_4, p_9|\rangle = 0$. Then $\langle|p_2, p_4, p_5|\rangle \cdot \langle|p_2, p_9, p_7|\rangle = -(\langle|p_2, p_4, p_7|\rangle \cdot \langle|p_2, p_5, p_9|\rangle)$.

(24) Suppose $\langle|p_2, p_6, p_7|\rangle = 0$. Then $\langle|p_2, p_4, p_7|\rangle \cdot \langle|p_2, p_6, p_8|\rangle = -(\langle|p_2, p_4, p_6|\rangle \cdot \langle|p_2, p_8, p_7|\rangle)$.

(25) Suppose $\langle|p_3, p_4, p_8|\rangle = 0$. Then $\langle|p_3, p_4, p_6|\rangle \cdot \langle|p_3, p_5, p_8|\rangle = \langle|p_3, p_4, p_5|\rangle \cdot \langle|p_3, p_6, p_8|\rangle$.

(26) Suppose $\langle|p_3, p_5, p_7|\rangle = 0$. Then $\langle|p_1, p_3, p_5|\rangle \cdot \langle|p_5, p_8, p_7|\rangle = -(\langle|p_1, p_5, p_7|\rangle \cdot \langle|p_3, p_5, p_8|\rangle)$. The theorem is a consequence of (1).

(27) Let us consider non zero real numbers r_{125}, r_{136}, r_{246}, r_{345}, r_{126}, r_{135}, r_{245}, r_{346}, r_{157}, r_{259}, r_{597}, r_{368}, r_{268}, r_{297}, r_{247}, r_{287}, r_{358}, r_{587}. Suppose $r_{125} \cdot r_{136} \cdot r_{246} \cdot r_{345} = r_{126} \cdot r_{135} \cdot r_{245} \cdot r_{346}$ and $r_{157} \cdot r_{259} = -(r_{125} \cdot r_{597})$ and $r_{126} \cdot r_{368} = r_{136} \cdot r_{268}$ and $r_{245} \cdot r_{297} = -(r_{247} \cdot r_{259})$ and $r_{247} \cdot r_{268} = -(r_{246} \cdot r_{287})$ and $r_{346} \cdot r_{358} = r_{345} \cdot r_{368}$ and $r_{135} \cdot r_{587} = -(r_{157} \cdot r_{358})$. Then $r_{287} \cdot r_{597} = r_{297} \cdot r_{587}$.

(28) Suppose $p_1 = \langle 1, 0, 0\rangle$ and $p_2 = \langle 0, 1, 0\rangle$ and $p_3 = \langle 0, 0, 1\rangle$ and $p_4 = \langle 1, 1, 1\rangle$ and $p_5 = \langle e_1, e_2, e_3\rangle$ and $p_6 = \langle f_1, f_2, f_3\rangle$ and qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_5) = 0$ and qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_6) = 0$. Then

(i) qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_1) = 0$, and

(ii) qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_2) = 0$, and

(iii) qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_3) = 0$, and

(iv) qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_4) = 0$, and

(v) $r_1 \cdot e_1 \cdot e_2 + (r_2 \cdot e_1 \cdot e_3) = r_1 + r_2 \cdot e_2 \cdot e_3$, and

(vi) $r_1 \cdot f_1 \cdot f_2 + (r_2 \cdot f_1 \cdot f_3) = r_1 + r_2 \cdot f_2 \cdot f_3$.

(29) Suppose $p_1 = \langle 1, 0, 0\rangle$ and $p_2 = \langle 0, 1, 0\rangle$ and $p_3 = \langle 0, 0, 1\rangle$ and $p_4 = \langle 1, 1, 1\rangle$ and $p_5 = \langle e_1, e_2, e_3\rangle$ and $p_6 = \langle f_1, f_2, f_3\rangle$ and $\langle|p_1, p_2, p_5|\rangle \neq 0$ and $\langle|p_1, p_3, p_6|\rangle \neq 0$ and $\langle|p_2, p_4, p_6|\rangle \neq 0$ and $\langle|p_3, p_4, p_5|\rangle \neq 0$ and $\langle|p_1, p_2, p_6|\rangle \neq 0$ and $\langle|p_1, p_3, p_5|\rangle \neq 0$ and $\langle|p_2, p_4, p_5|\rangle \neq 0$ and $\langle|p_3, p_4, p_6|\rangle \neq 0$ and $\langle|p_1, p_5, p_7|\rangle \neq 0$ and $\langle|p_2, p_5, p_9|\rangle \neq 0$ and $\langle|p_5, p_9, p_7|\rangle \neq 0$ and $\langle|p_3, p_6, p_8|\rangle \neq 0$ and $\langle|p_2, p_6, p_8|\rangle \neq 0$ and $\langle|p_2, p_9, p_7|\rangle \neq 0$ and $\langle|p_2, p_4, p_7|\rangle \neq 0$ and $\langle|p_2, p_8, p_7|\rangle \neq 0$ and $\langle|p_3, p_5, p_8|\rangle \neq 0$ and $\langle|p_5, p_8, p_7|\rangle$

$\neq 0$ and ($r_1 \neq 0$ or $r_2 \neq 0$) and $\mathrm{qfconic}(0,0,0,r_1,r_2,-(r_1+r_2),p_5) = 0$ and $\mathrm{qfconic}(0,0,0,r_1,r_2,-(r_1+r_2),p_6) = 0$ and $\langle|p_1,p_5,p_9|\rangle = 0$ and $\langle|p_1,p_6,p_8|\rangle = 0$ and $\langle|p_2,p_4,p_9|\rangle = 0$ and $\langle|p_2,p_6,p_7|\rangle = 0$ and $\langle|p_3,p_4,p_8|\rangle = 0$ and $\langle|p_3,p_5,p_7|\rangle = 0$. Then $\langle|p_2,p_8,p_7|\rangle \cdot \langle|p_5,p_9,p_7|\rangle = \langle|p_2,p_9,p_7|\rangle \cdot \langle|p_5,p_8,p_7|\rangle$. The theorem is a consequence of (20), (28), (19), (21), (22), (23), (24), (25), (26), and (27).

(30) Suppose $\langle|p_2,p_8,p_7|\rangle \cdot \langle|p_5,p_9,p_7|\rangle = \langle|p_2,p_9,p_7|\rangle \cdot \langle|p_5,p_8,p_7|\rangle$. Then $\langle|p_7,p_2,p_5|\rangle \cdot \langle|p_7,p_8,p_9|\rangle = 0$. The theorem is a consequence of (1).

(31) Let us consider a projective space P_{10} defined in terms of collinearity, and elements c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 of P_{10}. Suppose c_1, c_2 and c_4 are not collinear and c_1, c_2 and c_5 are not collinear and c_1, c_6 and c_4 are not collinear and c_1, c_6 and c_5 are not collinear and c_2, c_6 and c_4 are not collinear and c_3, c_4 and c_2 are not collinear and c_3, c_4 and c_6 are not collinear and c_3, c_5 and c_2 are not collinear and c_3, c_5 and c_6 are not collinear and c_4, c_5 and c_2 are not collinear and c_1, c_4 and c_7 are collinear and c_1, c_5 and c_8 are collinear and c_2, c_3 and c_7 are collinear and c_2, c_5 and c_9 are collinear and c_6, c_3 and c_8 are collinear and c_6, c_4 and c_9 are collinear. Then

(i) c_9, c_2 and c_4 are not collinear, and

(ii) c_1, c_4 and c_9 are not collinear, and

(iii) c_2, c_3 and c_9 are not collinear, and

(iv) c_2, c_4 and c_7 are not collinear, and

(v) c_2, c_5 and c_8 are not collinear, and

(vi) c_2, c_9 and c_8 are not collinear, and

(vii) c_2, c_9 and c_7 are not collinear, and

(viii) c_6, c_4 and c_8 are not collinear, and

(ix) c_6, c_5 and c_8 are not collinear, and

(x) c_4, c_9 and c_8 are not collinear, and

(xi) c_4, c_9 and c_7 are not collinear.

Proof: For every elements v_{102}, v_{103}, v_{100}, v_{104} of P_{10}, $v_{100} = v_{104}$ or v_{104}, v_{100} and v_{102} are not collinear or v_{104}, v_{100} and v_{103} are not collinear or v_{102}, v_{103} and v_{104} are collinear by [13, (5), (3)]. For every elements v_{102}, v_{104}, v_{100}, v_{103} of P_{10}, $v_{100} = v_{103}$ or v_{103}, v_{100} and v_{102} are not collinear or v_{103}, v_{100} and v_{104} are not collinear or v_{102}, v_{103} and v_{104} are collinear by [13, (5), (3)]. For every elements v_{102}, v_{103}, v_{104}, v_{101} of P_{10}, $v_{104} = v_{101}$ or v_{101}, v_{104} and v_{102} are not collinear or v_{101}, v_{104} and v_{103} are not collinear or v_{102}, v_{103} and v_{104} are collinear by [13, (2), (3)]. For every elements v_{103},

v_{104}, v_{102}, v_{101} of P_{10}, $v_{102} = v_{101}$ or v_{101}, v_{102} and v_{103} are not collinear or v_{101}, v_{102} and v_{104} are not collinear or v_{102}, v_{103} and v_{104} are collinear by [13, (2), (3)]. For every elements v_2, v_{101}, v_{100} of P_{10}, $v_{101} = v_{100}$ or v_{100}, v_{101} and v_2 are not collinear or v_2, v_{101} and v_{100} are collinear by [13, (2)]. □

In the sequel P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 denote points of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ and a, b, c, d, e, f denote real numbers.

Let P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 be points of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. We say that P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 form the Pascal configuration if and only if

(Def. 4) P_1, P_2 and P_4 are not collinear and P_1, P_3 and P_4 are not collinear and P_2, P_3 and P_4 are not collinear and P_1, P_2 and P_5 are not collinear and P_1, P_2 and P_6 are not collinear and P_1, P_3 and P_5 are not collinear and P_1, P_3 and P_6 are not collinear and P_2, P_4 and P_5 are not collinear and P_2, P_4 and P_6 are not collinear and P_3, P_4 and P_5 are not collinear and P_3, P_4 and P_6 are not collinear and P_2, P_3 and P_5 are not collinear and P_2, P_3 and P_6 are not collinear and P_4, P_5 and P_1 are not collinear and P_4, P_6 and P_1 are not collinear and P_5, P_6 and P_1 are not collinear and P_5, P_6 and P_2 are not collinear and P_1, P_5 and P_9 are collinear and P_1, P_6 and P_8 are collinear and P_2, P_4 and P_9 are collinear and P_2, P_6 and P_7 are collinear and P_3, P_4 and P_8 are collinear and P_3, P_5 and P_7 are collinear.

Now we state the propositions:

(32) Suppose P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 form the Pascal configuration. Then

(i) P_7, P_2 and P_5 are not collinear, and

(ii) P_1, P_5 and P_7 are not collinear, and

(iii) P_2, P_4 and P_7 are not collinear, and

(iv) P_2, P_5 and P_9 are not collinear, and

(v) P_2, P_6 and P_8 are not collinear, and

(vi) P_2, P_7 and P_8 are not collinear, and

(vii) P_2, P_7 and P_9 are not collinear, and

(viii) P_3, P_5 and P_8 are not collinear, and

(ix) P_3, P_6 and P_8 are not collinear, and

(x) P_5, P_7 and P_8 are not collinear, and

(xi) P_5, P_7 and P_9 are not collinear.

The theorem is a consequence of (31).

(33) Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that $\{P_1, P_2, P_3, P_4, P_5, P_6\} \subseteq \mathrm{conic}(a, b, c, d, e, f)$ and P_1, P_2 and P_3 are not collinear and P_1, P_2 and P_4 are not collinear and P_1, P_3 and P_4 are not collinear and P_2, P_3 and P_4 are not collinear and P_7, P_2 and P_5 are not collinear and P_1, P_2 and P_5 are not collinear and P_1, P_2 and P_6 are not collinear and P_1, P_3 and P_5 are not collinear and P_1, P_3 and P_6 are not collinear and P_1, P_5 and P_7 are not collinear and P_2, P_4 and P_5 are not collinear and P_2, P_4 and P_6 are not collinear and P_2, P_4 and P_7 are not collinear and P_2, P_5 and P_9 are not collinear and P_2, P_6 and P_8 are not collinear and P_2, P_7 and P_8 are not collinear and P_2, P_7 and P_9 are not collinear and P_3, P_4 and P_5 are not collinear and P_3, P_4 and P_6 are not collinear and P_3, P_5 and P_8 are not collinear and P_3, P_6 and P_8 are not collinear and P_5, P_7 and P_8 are not collinear and P_5, P_7 and P_9 are not collinear and P_1, P_5 and P_9 are collinear and P_1, P_6 and P_8 are collinear and P_2, P_4 and P_9 are collinear and P_2, P_6 and P_7 are collinear and P_3, P_4 and P_8 are collinear and P_3, P_5 and P_7 are collinear. Then P_7, P_8 and P_9 are collinear.

PROOF: Consider N being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of N)$(P_1) = \mathrm{Dir100}$ and (the homography of N)$(P_2) = \mathrm{Dir010}$ and (the homography of N)$(P_3) = \mathrm{Dir001}$ and (the homography of N)$(P_4) = \mathrm{Dir111}$. Consider u_5 being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that u_5 is not zero and (the homography of N)$(P_5) =$ the direction of u_5. Reconsider $p_{51} = u_5(1)$, $p_{52} = u_5(2)$, $p_{53} = u_5(3)$ as a real number. Consider u_6 being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that u_6 is not zero and (the homography of N)$(P_6) =$ the direction of u_6. Reconsider $p_{61} = u_6(1)$, $p_{62} = u_6(2)$, $p_{63} = u_6(3)$ as a real number. Consider u_7 being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that u_7 is not zero and (the homography of N)$(P_7) =$ the direction of u_7. Reconsider $p_{71} = u_7(1)$, $p_{72} = u_7(2)$, $p_{73} = u_7(3)$ as a real number. Consider u_8 being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that u_8 is not zero and (the homography of N)$(P_8) =$ the direction of u_8. Reconsider $p_{81} = u_8(1)$, $p_{82} = u_8(2)$, $p_{83} = u_8(3)$ as a real number. Consider u_9 being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that u_9 is not zero and (the homography of N)$(P_9) =$ the direction of u_9. Reconsider $p_{91} = u_9(1)$, $p_{92} = u_9(2)$, $p_{93} = u_9(3)$ as a real number. Consider a_2, b_2, c_2, d_2, e_2, f_2 being real numbers such that it is not true that $a_2 = 0$ and $b_2 = 0$ and $c_2 = 0$ and $d_2 = 0$ and $e_2 = 0$ and $f_2 = 0$. (the homography of N)$(P_1) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of N)$(P_2) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of N)$(P_3) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of N)$(P_4) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of N)$(P_5) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of N)$(P_6) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$. Consider P being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $[1,0,0] = P$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $P =$ the direction of u holds qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, u) = 0$. qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [1,0,0]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [0,1,0]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [0,0,1]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [1,1,1]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [p_{51}, p_{52}, p_{53}]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [p_{61}, p_{62}, p_{63}]) = 0$ by [4, (10)], [8, (3)]. Reconsider $a_7 = a_2$, $b_7 = b_2$, $c_{10} = c_2$, $d_3 = d_2$, $e_4 = e_2$, $f_4 = f_2$ as an element of $\mathbb{R}_{\mathrm{F}}$. $a_7 = 0$ and $b_7 = 0$ and $c_{10} = 0$. $a_7 = 0$ and $b_7 = 0$ and $c_{10} = 0$ and $d_3 + e_4 + f_4 = 0$. Reconsider $p_2 = \langle 0,1,0 \rangle$, $p_5 = \langle p_{51}, p_{52}, p_{53} \rangle$, $p_7 = \langle p_{71}, p_{72}, p_{73} \rangle$, $p_8 = \langle p_{81}, p_{82}, p_{83} \rangle$, $p_9 = \langle p_{91}, p_{92}, p_{93} \rangle$ as a point of $\mathcal{E}^3_{\mathrm{T}}$. $\langle |p_7, p_2, p_5| \rangle \neq 0$ by [3, (102)], [8, (3)], [3, (43)], [4, (10)]. $\langle |p_2, p_8, p_7| \rangle \cdot \langle |p_5, p_9, p_7| \rangle = \langle |p_2, p_9, p_7| \rangle \cdot \langle |p_5, p_8, p_7| \rangle$. $\langle |p_7, p_2, p_5| \rangle \cdot \langle |p_7, p_8, p_9| \rangle = 0$. □

(34) Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that $\{P_1, P_2, P_3, P_4, P_5, P_6\} \subseteq$ conic(a, b, c, d, e, f) and P_1, P_2 and P_3 are not collinear and P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 form the Pascal configuration. Then P_7, P_8 and P_9 are collinear. The theorem is a consequence of (32) and (33).

Note that $\mathcal{E}^3_{\mathrm{T}}$ is up 3-dimensional.

(35) Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that $\{P_1, P_2, P_3, P_4, P_5, P_6\} \subseteq$ conic(a, b, c, d, e, f) and P_1, P_2 and P_3 are collinear and P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 form the Pascal configuration. Then P_7, P_8 and P_9 are collinear.

PROOF: Consider N being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of N)$(P_1) =$ Dir100 and (the homography of N)$(P_2) =$ Dir010 and (the homography of N)$(P_4) =$ Dir001 and (the homography of N)$(P_5) =$ Dir111. Consider u_3 being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that u_3 is not zero and (the homography of N)$(P_3) =$ the direction of u_3. Reconsider $p_{31} = u_3(1)$, $p_{32} = u_3(2)$, $p_{33} = u_3(3)$ as a real number. Consider u_6 being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that u_6 is not zero and (the homography of N)$(P_6) =$ the direction of u_6. Reconsider $p_{61} = u_6(1)$, $p_{62} = u_6(2)$, $p_{63} = u_6(3)$ as a real number. Consider a_2, b_2, c_2, d_2, e_2, f_2 being real numbers such that it is not true that $a_2 = 0$ and $b_2 = 0$ and $c_2 = 0$ and $d_2 = 0$ and $e_2 = 0$ and $f_2 = 0$ and (the homography of N)$(P_1) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of N)$(P_2) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of N)$(P_3) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of N)$(P_4) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of N)$(P_5) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of N)$(P_6) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$. Consider P being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $[1,0,0] = P$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $P =$ the direction of u holds qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, u) = 0$. qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [1, 0, 0]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [0, 1, 0]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [0, 0, 1]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [1, 1, 1]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [p_{31}, p_{32}, p_{33}]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [p_{61}, p_{62}, p_{63}]) = 0$ by [4, (10)], [8, (3)]. Reconsider $a_7 = a_2$, $b_7 = b_2$, $c_{10} = c_2$, $d_3 = d_2$, $e_4 = e_2$, $f_4 = f_2$ as an element of $\mathbb{R}_{\mathrm{F}}$. $a_7 = 0$ and $b_7 = 0$ and $c_{10} = 0$. $a_7 = 0$ and $b_7 = 0$ and $c_{10} = 0$ and $d_3 + e_4 + f_4 = 0$. Reconsider $p_1 = \langle 1, 0, 0\rangle$, $p_2 = \langle 0, 1, 0\rangle$, $p_3 = \langle p_{31}, p_{32}, p_{33}\rangle$ as a point of $\mathcal{E}^3_{\mathrm{T}}$. $\langle | p_1, p_2, p_3 | \rangle = 0$ by [3, (102)], [10, (23)], [9, (25)], [4, (10)]. $p_{31} \neq 0$ and $p_{32} \neq 0$ by [8, (2), (8), (4)]. □

(36) PASCAL'S THEOREM:

Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that $\{P_1, P_2, P_3, P_4, P_5, P_6\} \subseteq$ conic(a, b, c, d, e, f) and P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 form the Pascal configuration. Then P_7, P_8 and P_9 are collinear. The theorem is a consequence of (35) and (34).

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