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Introduction to Stopping Time in Stochastic Finance Theory

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Summary. We start with the definition of stopping time according to [4], p.283. We prove, that different definitions for stopping time can coincide. We give examples of stopping time using constant-functions or functions defined with the operator max or min (defined in [6], pp.37–38). Finally we give an example with some given filtration. Stopping time is very important for stochastic finance. A stopping time is the moment, where a certain event occurs ([7], p.372) and can be used together with stochastic processes ([4], p.283). Look at the following example: we install a function ST: {1,2,3,4} $\rightarrow \{0, 1, 2\} \cup \{+\infty\}$, we define:

a. ST(1)=1, ST(2)=1, ST(3)=2, ST(4)=2.

b. The set {0,1,2} consists of time points: 0=now,1=tomorrow,2=the day after tomorrow.

We can prove:

c. {w, where w is Element of Ω: ST.w=0}=∅ & {w, where w is Element of Ω: ST.w=1}={1,2} & {w, where w is Element of Ω: ST.w=2}={3,4} and

ST is a stopping time.

We use a function Filt as Filtration of {0,1,2}, Σ where Filt(0)=Ω_{now}, Filt(1)=Ω_{fut1} and Filt(2)=Ω_{fut2}. From a.,b. and c. we know that:

d. {w, where w is Element of Ω: ST.w=0} in Ω_{now} and

{w, where w is Element of Ω: ST.w=1} in Ω_{fut1} and

{w, where w is Element of Ω: ST.w=2} in Ω_{fut2}.

The sets in d. are events, which occur at the time points 0(=now), 1(=tomorrow) or 2(=the day after tomorrow), see also [7], p.371. Suppose we have ST(1)=+∞, then this means that for 1 the corresponding event never occurs.

As an interpretation for our installed functions consider the given adapted stochastic process in the article [5].

ST(1)=1 means, that the given element 1 in {1,2,3,4} is stopped in 1 (=tomorrow). That tells us, that we have to look at the value $f_2(1)$ which is equal to 80. The same argumentation can be applied for the element 2 in {1,2,3,4}.

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ST(3)=2 means, that the given element 3 in {1,2,3,4} is stopped in 2 (=the day after tomorrow). That tells us, that we have to look at the value $f_3(3)$ which is equal to 100.

ST(4)=2 means, that the given element 4 in {1,2,3,4} is stopped in 2 (=the day after tomorrow). That tells us, that we have to look at the value $f_3(4)$ which is equal to 120.

In the real world, these functions can be used for questions like: when does the share price exceed a certain limit? (see [7], p.372).

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1. Preliminaries

From now on Ω denotes a non empty set, Σ denotes a σ-field of subsets of Ω, and T denotes a natural number.

Now we state the proposition:

(1) Let us consider a non empty set X, an object t, and a function f. Suppose $\operatorname{dom} f = X$. Then $\{w$, where w is an element of $X : f(w) = t\} = \operatorname{Coim}(f, t)$.

Proof: Set $A = \{w$, where w is an element of $X : f(w) = t\}$. $A \subseteq \operatorname{Coim}(f, t)$ by [2, (1)]. Consider y being an object such that $\langle x, y \rangle \in f$ and $y \in \{t\}$. $\square$

Let I be an extended real-membered set. The functor $I_{\{+\infty\}}$ yielding a subset of $\overline{\mathbb{R}}$ is defined by the term

(Def. 1) $I \cup \{+\infty\}$.

Let us note that $I_{\{+\infty\}}$ is non empty.

2. Definition of Stopping Time

Let T be a natural number. The functor $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T}\{t\}$ yielding a subset of $\mathbb{R}$ is defined by the term

(Def. 2) $\{t$, where t is an element of $\mathbb{N} : 0 \leqslant t \leqslant T\}$.

Let us note that $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T}\{t\}$ is non empty.

The functor $T_{\{+\infty\}}$ yielding a subset of $\overline{\mathbb{R}}$ is defined by the term

(Def. 3) $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T}\{t\} \cup \{+\infty\}$.

Let us note that $T_{\{+\infty\}}$ is non empty.

In the sequel T_1 denotes an element of $T_{\{+\infty\}}$, MF denotes a filtration of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T} \{t\}$ and Σ, and k, k_1, k_2 denote functions from Ω into $T_{\{+\infty\}}$.

Let T be a natural number, F be a function, and R be a binary relation. We say that R is StoppingTime(F,T) if and only if

(Def. 4) for every element t of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T} \{t\}$, $\mathrm{Coim}(R, t) \in F(t)$.

Let Ω be a non empty set, MF be a function, and k be a function from Ω into $T_{\{+\infty\}}$. Let us observe that k is StoppingTime(MF,T) if and only if the condition (Def. 5) is satisfied.

(Def. 5) for every element t of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T} \{t\}$, $\{w$, where w is an element of $\Omega : k(w) = t\} \in MF(t)$.

Now we state the proposition:

(2) k is StoppingTime(MF,T) if and only if for every element t of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T} \{t\}$, $\{w$, where w is an element of $\Omega : k(w) \leqslant t\} \in MF(t)$.

Proof: If k is StoppingTime(MF,T), then for every element t of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T} \{t\}$, $\{w$, where w is an element of $\Omega : k(w) \leqslant t\} \in MF(t)$ by [1, (8), (12), (13)], [8, (21)]. For every element t of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T} \{t\}$, $\{w$, where w is an element of $\Omega : k(w) = t\} \in MF(t)$ by [1, (13)], [8, (22), (24)], [1, (22)]. $\square$

3. Examples of Stopping Times

Now we state the proposition:

(3) $\Omega \longmapsto T_1$ is StoppingTime(MF,T).

Proof: Set $c = \Omega \longmapsto T_1$. For every element t of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant T} \{t\}$, $\{w$, where w is an element of $\Omega : c(w) = t\} \in MF(t)$ by [9, (7)], [8, (5), (4)]. $\square$

Let us consider Ω, T, k_1, and k_2. The functor $\max(k_1, k_2)$ yielding a function from Ω into $\overline{\mathbb{R}}$ is defined by

(Def. 6) for every element w of Ω, $it(w) = \max(k_1(w), k_2(w))$.

The functor $\min(k_1, k_2)$ yielding a function from Ω into $\overline{\mathbb{R}}$ is defined by

(Def. 7) for every element w of Ω, $it(w) = \min(k_1(w), k_2(w))$.

Now we state the propositions:

(4) Suppose k_1 is StoppingTime(MF,T) and k_2 is StoppingTime(MF,T). Then there exists a function k_3 from Ω into $T_{\{+\infty\}}$ such that

(i) $k_3 = \max(k_1, k_2)$, and

(ii) k_3 is StoppingTime(MF,T).

PROOF: Set $k_3 = \max(k_1, k_2)$. k_3 is a function from Ω into $T_{\{+\infty\}}$ by [2, (3)], [3, (2)]. k_3 is StoppingTime(MF,T) by (2), [8, (19)]. □

(5) Suppose k_1 is StoppingTime(MF,T) and k_2 is StoppingTime(MF,T). Then there exists a function k_3 from Ω into $T_{\{+\infty\}}$ such that

(i) $k_3 = \min(k_1, k_2)$, and

(ii) k_3 is StoppingTime(MF,T).

PROOF: Set $k_3 = \min(k_1, k_2)$. k_3 is a function from Ω into $T_{\{+\infty\}}$ by [2, (3)], [3, (2)]. k_3 is StoppingTime(MF,T) by (2), [8, (3)]. □

Let t be an object. The special element of $t_{\{+\infty\}}$ yielding an element of $2_{\{+\infty\}}$ is defined by the term

(Def. 8) IFIN($t, \{1,2\}, 1, 2$).

Now we state the proposition:

(6) Suppose $\Omega = \{1,2,3,4\}$. Let us consider a filtration MF of $\bigcup_{t \in \mathbb{N}: 0 \leqslant t \leqslant 2} \{t\}$ and Σ. Suppose $MF(0) = \Omega_{now}$ and $MF(1) = \Omega_{fut1}$ and $MF(2) = \Omega_{fut2}$. Then there exists a function S from Ω into $2_{\{+\infty\}}$ such that

(i) S is StoppingTime(MF,2), and

(ii) $S(1) = 1$, and

(iii) $S(2) = 1$, and

(iv) $S(3) = 2$, and

(v) $S(4) = 2$, and

(vi) $\{w$, where w is an element of $\Omega : S(w) = 0\} = \emptyset$, and

(vii) $\{w$, where w is an element of $\Omega : S(w) = 1\} = \{1,2\}$, and

(viii) $\{w$, where w is an element of $\Omega : S(w) = 2\} = \{3,4\}$.

PROOF: Define $\mathcal{U}$(element of Ω) = the special element of $\$_{1\{+\infty\}}$. Consider f being a function from Ω into $2_{\{+\infty\}}$ such that for every element d of Ω, $f(d) = \mathcal{U}(d)$ from [3, Sch. 4]. $f(1) = 1$ and $f(2) = 1$ and $f(3) = 2$ and $f(4) = 2$. f is StoppingTime(MF,2) and $\{w$, where w is an element of $\Omega : f(w) = 0\} = \emptyset$ and $\{w$, where w is an element of $\Omega : f(w) = 1\} = \{1,2\}$ and $\{w$, where w is an element of $\Omega : f(w) = 2\} = \{3,4\}$ by [1, (9)], [8, (4)], [5, (24)]. □

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