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Basic Formal Properties of Triangular Norms and Conorms

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Summary. In the article we present in the Mizar system [1], [8] the catalogue of triangular norms and conorms, used especially in the theory of fuzzy sets [13]. The name *triangular* emphasizes the fact that in the framework of probabilistic metric spaces they generalize triangle inequality [2].

After defining corresponding Mizar mode using four attributes, we introduced the following t-norms:

- minimum t-norm `minnorm` (Def. 6),
- product t-norm `prodnorm` (Def. 8),
- Łukasiewicz t-norm `Lukasiewicz_norm` (Def. 10),
- drastic t-norm `drastic_norm` (Def. 11),
- nilpotent minimum `nilmin_norm` (Def. 12),
- Hamacher product `Hamacher_norm` (Def. 13),

and corresponding t-conorms:

- maximum t-conorm `maxnorm` (Def. 7),
- probabilistic sum `probsum_conorm` (Def. 9),
- bounded sum `BoundedSum_conorm` (Def. 19),
- drastic t-conorm `drastic_conorm` (Def. 14),
- nilpotent maximum `nilmax_conorm` (Def. 18),
- Hamacher t-conorm `Hamacher_conorm` (Def. 17).

Their basic properties and duality are shown; we also proved the predicate of the ordering of norms [10], [9]. It was proven formally that drastic-norm is the pointwise smallest t-norm and minnorm is the pointwise largest t-norm (maxnorm is the pointwise smallest t-conorm and drastic-conorm is the pointwise largest t-conorm).

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This work is a continuation of the development of fuzzy sets in Mizar [6] started in [11] and [3]; it could be used to give a variety of more general operations on fuzzy sets. Our formalization is much closer to the set theory used within the Mizar Mathematical Library than the development of rough sets [4], the approach which was chosen allows however for merging both theories [5], [7].

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1. Preliminaries

One can verify that $[0,1]$ is non empty.

Let us consider elements a, b of $[0,1]$. Now we state the propositions:

(1) $\min(a,b) \in [0,1]$.

(2) $\max(a,b) \in [0,1]$.

(3) $a \cdot b \in [0,1]$.

(4) $\max(0, a+b-1) \in [0,1]$.

(5) $\min(a+b, 1) \in [0,1]$.

(6) Let us consider elements a, b, c of $[0,1]$. Then $\max(0, \max(0, a+b-1) + c-1) = \max(0, a + \max(0, b+c-1) - 1)$.

(7) Let us consider an element a of $[0,1]$. Then $1 - a \in [0,1]$.

Let us consider elements a, b of $[0,1]$. Now we state the propositions:

(8) $a + b - (a \cdot b) \in [0,1]$. The theorem is a consequence of (7) and (3).

(9) $\frac{a \cdot b}{a+b-(a \cdot b)} \in [0,1]$. The theorem is a consequence of (3) and (8).

(10) If $\max(a,b) \neq 1$, then $a \neq 1$ and $b \neq 1$.

(11) Let us consider elements x, y of $[0,1]$. If $x \cdot y = x + y$, then $x = 0$. The theorem is a consequence of (7).

Let us consider elements a, b of $[0,1]$. Now we state the propositions:

(12) $\max(a,b) = 1 - \min(1-a, 1-b)$.

(13) $\min(a+b, 1) = 1 - \max(0, 1-a+(1-b)-1)$.

(14) $\frac{a+b-(2 \cdot a \cdot b)}{1-(a \cdot b)} \in [0,1]$. The theorem is a consequence of (7) and (3).

Let f be a binary operation on $[0,1]$ and a, b be real numbers. Let us observe that $f(a,b)$ is real.

Now we state the propositions:

(15) Let us consider real numbers a, b, and a binary operation t on $[0,1]$. Then $t(a,b) \in [0,1]$.

(16) Let us consider a binary operation f on $[0,1]$, and real numbers a, b. Then $1 - f(1-a, 1-b) \in [0,1]$. The theorem is a consequence of (15) and (7).

(17) Let us consider real numbers x, y, k. Suppose $k \leqslant 0$. Then

(i) $k \cdot \min(x,y) = \max(k \cdot x, k \cdot y)$, and

(ii) $k \cdot \max(x,y) = \min(k \cdot x, k \cdot y)$.

2. Basic Example of a Triangular Norm and Conorm: min and max

Let A be a real-membered set and f be a binary operation on A. We say that f is monotonic if and only if

(Def. 1) for every elements a, b, c, d of A such that $a \leqslant c$ and $b \leqslant d$ holds $f(a,b) \leqslant f(c,d)$.

We say that f has 1-identity if and only if

(Def. 2) for every element a of A, $f(a,1) = a$.

We say that f has 1-annihilating if and only if

(Def. 3) for every element a of A, $f(a,1) = 1$.

We say that f has 0-identity if and only if

(Def. 4) for every element a of A, $f(a,0) = a$.

We say that f has 0-annihilating if and only if

(Def. 5) for every element a of A, $f(a,0) = 0$.

The scheme *ExBinOp* deals with a non empty, real-membered set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a set and states that

(Sch. 1) There exists a binary operation f on $\mathcal{A}$ such that for every elements a, b of $\mathcal{A}$, $f(a,b) = \mathcal{F}(a,b)$

provided

- for every elements a, b of $\mathcal{A}$, $\mathcal{F}(a,b) \in \mathcal{A}$.

The functor minnorm yielding a binary operation on $[0,1]$ is defined by

(Def. 6) for every elements a, b of $[0,1]$, $it(a,b) = \min(a,b)$.

Observe that minnorm is commutative, associative, and monotonic and has 1-identity and there exists a binary operation on $[0,1]$ which is commutative, associative, and monotonic and has 1-identity.

A t-norm is a commutative, associative, monotonic binary operation on $[0,1]$ with 1-identity. The functor maxnorm yielding a binary operation on $[0,1]$ is defined by

(Def. 7) for every elements a, b of $[0,1]$, $it(a,b) = \max(a,b)$.

One can verify that maxnorm is commutative, associative, and monotonic and has 0-identity and there exists a binary operation on $[0,1]$ which is commutative, associative, and monotonic and has 0-identity.

A t-conorm is a commutative, associative, monotonic binary operation on $[0,1]$ with 0-identity. Now we state the propositions:

(18) Let us consider a commutative, monotonic binary operation t on $[0,1]$ with 1-identity, and an element a of $[0,1]$. Then $t(a,0) = 0$. The theorem is a consequence of (15).

(19) Let us consider a commutative, monotonic binary operation t on $[0,1]$ with 0-identity, and an element a of $[0,1]$. Then $t(a,1) = 1$. The theorem is a consequence of (15).

Let us note that every commutative, monotonic binary operation on $[0,1]$ with 1-identity has 0-annihilating and every commutative, monotonic binary operation on $[0,1]$ with 0-identity has 1-annihilating.

3. Further Examples of Triangular Norms

The functor prodnorm yielding a binary operation on $[0,1]$ is defined by

(Def. 8) for every elements a, b of $[0,1]$, $it(a,b) = a \cdot b$.

Let us observe that prodnorm is commutative, associative, and monotonic and has 1-identity.

The functor probsum-conorm yielding a binary operation on $[0,1]$ is defined by

(Def. 9) for every elements a, b of $[0,1]$, $it(a,b) = a + b - (a \cdot b)$.

The functor Lukasiewicz-norm yielding a binary operation on $[0,1]$ is defined by

(Def. 10) for every elements a, b of $[0,1]$, $it(a,b) = \max(0, a + b - 1)$.

One can check that Lukasiewicz-norm is commutative, associative, and monotonic and has 1-identity.

The functor drastic-norm yielding a binary operation on $[0,1]$ is defined by

(Def. 11) for every elements a, b of $[0,1]$, if $\max(a,b) = 1$, then $it(a,b) = \min(a,b)$ and if $\max(a,b) \neq 1$, then $it(a,b) = 0$.

Now we state the proposition:

(20) Let us consider elements a, b of $[0,1]$. Then

(i) if $a = 1$, then $(\text{drastic-norm})(a,b) = b$, and

(ii) if $b = 1$, then $(\text{drastic-norm})(a,b) = a$, and

(iii) if $a \neq 1$ and $b \neq 1$, then (drastic-norm)$(a, b) = 0$.

Note that drastic-norm is commutative, associative, and monotonic and has 1-identity.

The functor nilmin-norm yielding a binary operation on $[0, 1]$ is defined by

(Def. 12) for every elements a, b of $[0, 1]$, if $a + b > 1$, then $it(a, b) = \min(a, b)$ and if $a + b \leqslant 1$, then $it(a, b) = 0$.

Observe that nilmin-norm is commutative, associative, and monotonic and has 1-identity.

The functor Hamacher-norm yielding a binary operation on $[0, 1]$ is defined by

(Def. 13) for every elements a, b of $[0, 1]$, $it(a, b) = \frac{a \cdot b}{a + b - (a \cdot b)}$.

One can verify that Hamacher-norm is commutative, associative, and monotonic and has 1-identity.

4. Basic Triangular Conorms

The functor drastic-conorm yielding a binary operation on $[0, 1]$ is defined by

(Def. 14) for every elements a, b of $[0, 1]$, if $\min(a, b) = 0$, then $it(a, b) = \max(a, b)$ and if $\min(a, b) \neq 0$, then $it(a, b) = 1$.

5. Translating between Triangular Norms and Conorms

Let t be a binary operation on $[0, 1]$. The functor conorm t yielding a binary operation on $[0, 1]$ is defined by

(Def. 15) for every elements a, b of $[0, 1]$, $it(a, b) = 1 - t(1 - a, 1 - b)$.

Let t be a t-norm. Let us observe that conorm t is monotonic, commutative, and associative and has 0-identity.

Now we state the propositions:

(21) maxnorm = conorm minnorm.

PROOF: For every elements a, b of $[0, 1]$, (maxnorm)$(a, b) = 1 -$ (minnorm) $(1 - a, 1 - b)$ by (7), (17), [12, (42)]. $\square$

(22) Let us consider a binary operation t on $[0, 1]$. Then conorm conorm $t = t$. The theorem is a consequence of (7).

6. The Ordering of Triangular Norms (and Conorms)

Let f_1, f_2 be binary operations on $[0,1]$. We say that $f_1 \leqslant f_2$ if and only if

(Def. 16) for every elements a, b of $[0,1]$, $f_1(a,b) \leqslant f_2(a,b)$.

Let us consider a t-norm t. Now we state the propositions:

(23) drastic-norm $\leqslant t$. The theorem is a consequence of (20).

(24) $t \leqslant$ minnorm.

Now we state the proposition:

(25) Let us consider t-norms t_1, t_2. If $t_1 \leqslant t_2$, then conorm $t_2 \leqslant$ conorm t_1. The theorem is a consequence of (7).

7. Triangular Conorms Generated from t-Norms

The functor Hamacher-conorm yielding a binary operation on $[0,1]$ is defined by

(Def. 17) for every elements a, b of $[0,1]$, if $a = b = 1$, then $it(a,b) = 1$ and if $a \neq 1$ or $b \neq 1$, then $it(a,b) = \frac{a+b-(2 \cdot a \cdot b)}{1-(a \cdot b)}$.

Now we state the proposition:

(26) conorm Hamacher-norm = Hamacher-conorm. The theorem is a consequence of (7).

Let us note that Hamacher-conorm is commutative, associative, and monotonic and has 0-identity.

Now we state the propositions:

(27) conorm drastic-norm = drastic-conorm. The theorem is a consequence of (7).

(28) conorm prodnorm = probsum-conorm. The theorem is a consequence of (7).

One can check that probsum-conorm is commutative, associative, and monotonic and has 0-identity.

The functor nilmax-conorm yielding a binary operation on $[0,1]$ is defined by

(Def. 18) for every elements a, b of $[0,1]$, if $a+b < 1$, then $it(a,b) = \max(a,b)$ and if $a + b \geqslant 1$, then $it(a,b) = 1$.

Now we state the proposition:

(29) conorm nilmin-norm = nilmax-conorm. The theorem is a consequence of (7) and (12).

Let us note that nilmax-conorm is commutative, associative, and monotonic and has 0-identity.

The functor BoundedSum-conorm yielding a binary operation on $[0,1]$ is defined by

(Def. 19) for every elements a, b of $[0,1]$, $it(a,b) = \min(a+b, 1)$.

Now we state the proposition:

(30) conorm Lukasiewicz-norm = BoundedSum-conorm. The theorem is a consequence of (7) and (13).

One can check that BoundedSum-conorm is commutative, associative, and monotonic and has 0-identity.

Let us consider a t-conorm t. Now we state the propositions:

(31) maxnorm $\leqslant t$.

(32) $t \leqslant$ drastic-conorm.

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