FORMALIZED MATHEMATICS
Vol. 25, No. 2, Pages 87–92, 2017
DOI: 10.1515/forma-2017-0008

DE GRUYTER OPEN
degruyter.com/view/j/forma

Vieta's Formula about the Sum of Roots of Polynomials

Artur Korniłowicz
Institute of Informatics
University of Białystok
Poland

Karol Pąk
Institute of Informatics
University of Białystok
Poland

Summary. In the article we formalized in the Mizar system [2] the Vieta formula about the sum of roots of a polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ defined over an algebraically closed field. The formula says that $x_1 + x_2 + \cdots + x_{n-1} + x_n = -\frac{a_{n-1}}{a_n}$, where $x_1, x_2, \ldots, x_n$ are (not necessarily distinct) roots of the polynomial [12]. In the article the sum is denoted by `SumRoots`.

MSC: 12E05 03B35

Keywords: roots of polynomials; Vieta's formula

MML identifier: `POLYVIE1`, version: `8.1.06 5.43.1297`

Let F be a finite sequence and f be a function from $\operatorname{dom} F$ into $\operatorname{dom} F$. Observe that $F \cdot f$ is finite sequence-like.

Now we state the propositions:

(1) Let us consider objects a, b. Suppose $a \neq b$. Then

(i) $\operatorname{CFS}(\{a, b\}) = \langle a, b \rangle$, or

(ii) $\operatorname{CFS}(\{a, b\}) = \langle b, a \rangle$.

(2) Let us consider a finite set X. Then $\operatorname{CFS}(X)$ is an enumeration of X.

Let A be a set and X be a finite subset of A. Observe that $\operatorname{CFS}(X)$ is A-valued.

Now we state the proposition:

(3) Let us consider a right zeroed, non empty additive loop structure L, and an element a of L. Then $2 \cdot a = a + a$.

ISSN 1426–2630(Print), 1898-9934(Online)

Let L be an almost left invertible multiplicative loop with zero structure. Let us note that every element of L which is non zero is also left invertible.

Let L be an almost right invertible multiplicative loop with zero structure. Observe that every element of L which is non zero is also right invertible.

Let L be an almost left cancelable multiplicative loop with zero structure. Let us observe that every element of L which is non zero is also left mult-cancelable.

Let L be an almost right cancelable multiplicative loop with zero structure. One can verify that every element of L which is non zero is also right mult-cancelable.

Now we state the proposition:

(4) Let us consider a right unital, associative, non trivial double loop structure L, and elements a, b of L. Suppose b is left invertible and right mult-cancelable and $b \cdot \frac{1}{b} = \frac{1}{b} \cdot b$. Then $\frac{a \cdot b}{b} = a$.

Let L be a non degenerated zero-one structure, z_0 be an element of L, and z_1 be a non zero element of L. Note that $\langle z_0, z_1 \rangle$ is non-zero and $\langle z_1, z_0 \rangle$ is non-zero.

Let us consider a non trivial zero structure L and a polynomial p over L. Now we state the propositions:

(5) If $\operatorname{len} p = 1$, then there exists a non zero element a of L such that $p = \langle a \rangle$.

(6) If $\operatorname{len} p = 2$, then there exists an element a of L and there exists a non zero element b of L such that $p = \langle a, b \rangle$.

(7) If $\operatorname{len} p = 3$, then there exist elements a, b of L and there exists a non zero element c of L such that $p = \langle a, b, c \rangle$.

Now we state the propositions:

(8) Let us consider an add-associative, right zeroed, right complementable, associative, commutative, left distributive, well unital, almost left invertible, non empty double loop structure L, and elements a, b, x of L. If $b \neq 0_L$, then $\operatorname{eval}(\langle a, b \rangle, -\frac{a}{b}) = 0_L$.

(9) Let us consider a field L, elements a, x of L, and a non zero element b of L. Then x is a root of $\langle a, b \rangle$ if and only if $x = -\frac{a}{b}$. The theorem is a consequence of (4) and (8).

Let us consider a field L, an element a of L, and a non zero element b of L. Now we state the propositions:

(10) $\operatorname{Roots}(\langle a, b \rangle) = \{-\frac{a}{b}\}$. The theorem is a consequence of (9).

(11) $\operatorname{multiplicity}(\langle a, b \rangle, -\frac{a}{b}) = 1$. The theorem is a consequence of (9).

(12) $\operatorname{BRoots}(\langle a, b \rangle) = (\{-\frac{a}{b}\}, 1)$-bag. The theorem is a consequence of (10) and (11).

(13) Let us consider a field L, elements a, c of L, and non zero elements b, d of L. Then $\operatorname{Roots}(\langle a, b \rangle * \langle c, d \rangle) = \{-\frac{a}{b}, -\frac{c}{d}\}$. The theorem is a consequence

of (10).

(14) Let us consider a field L, elements a, x of L, and a non zero element b of L. If $x \neq -\frac{a}{b}$, then multiplicity$(\langle a, b\rangle, x) = 0$. The theorem is a consequence of (10).

Let us consider a field L, a non-zero polynomial p over L, an element a of L, and a non zero element b of L. Now we state the propositions:

(15) Suppose $-\frac{a}{b} \notin \mathrm{Roots}(p)$. Then $\overline{\overline{\mathrm{Roots}(\langle a, b\rangle * p)}} = 1 + \overline{\overline{\mathrm{Roots}(p)}}$. The theorem is a consequence of (10).

(16) Suppose $-\frac{a}{b} \notin \mathrm{Roots}(p)$. Then $\mathrm{CFS}(\mathrm{Roots}(p)) \frown \langle -\frac{a}{b}\rangle$ is an enumeration of $\mathrm{Roots}(\langle a, b\rangle * p)$. The theorem is a consequence of (10).

(17) Let us consider a field L, a non-zero polynomial p over L, an element a of L, a non zero element b of L, and an enumeration E of $\mathrm{Roots}(\langle a, b\rangle * p)$. Suppose $E = \mathrm{CFS}(\mathrm{Roots}(p)) \frown \langle -\frac{a}{b}\rangle$. Then

(i) $\mathrm{len}\, E = 1 + \overline{\overline{\mathrm{Roots}(p)}}$, and

(ii) $E(1 + \overline{\overline{\mathrm{Roots}(p)}}) = -\frac{a}{b}$, and

(iii) for every natural number n such that $1 \leqslant n \leqslant \overline{\overline{\mathrm{Roots}(p)}}$ holds $E(n) = (\mathrm{CFS}(\mathrm{Roots}(p)))(n)$.

Let L be a non empty double loop structure, B be a bag of the carrier of L, and E be a (the carrier of L)-valued finite sequence. The functor $B(++)E$ yielding a finite sequence of elements of L is defined by

(Def. 1) $\mathrm{len}\, it = \mathrm{len}\, E$ and for every natural number n such that $1 \leqslant n \leqslant \mathrm{len}\, it$ holds $it(n) = (B \cdot E)(n) \cdot E_n$.

Now we state the propositions:

(18) Let us consider an integral domain L, a non-zero polynomial p over L, a bag B of the carrier of L, and an enumeration E of $\mathrm{Roots}(p)$. If $\mathrm{Roots}(p) = \emptyset$, then $B(++)E = \emptyset$.

(19) Let us consider a left zeroed, add-associative, non empty double loop structure L, bags B_1, B_2 of the carrier of L, and a (the carrier of L)-valued finite sequence E. Then $B_1 + B_2(++)E = (B_1(++)E) + (B_2(++)E)$.

(20) Let us consider a left zeroed, add-associative, non empty double loop structure L, a bag B of the carrier of L, and (the carrier of L)-valued finite sequences E, F. Then $B(++)E \frown F = (B(++)E) \frown (B(++)F)$.

(21) Let us consider a left zeroed, add-associative, non empty double loop structure L, bags B_1, B_2 of the carrier of L, and (the carrier of L)-valued finite sequences E, F. Then $B_1 + B_2(++)E \frown F = (B_1(++)E) \frown (B_1(++)F) + (B_2(++)E) \frown (B_2(++)F)$. The theorem is a consequence of (19) and (20).

(22) Let us consider a field L, a non-zero polynomial p over L, an element a of L, a non zero element b of L, an enumeration E of $\mathrm{Roots}(\langle a, b\rangle * p)$, and a permutation P of $\mathrm{dom}\, E$. Then $(\mathrm{BRoots}(\langle a, b\rangle * p)(++)E) \cdot P = \mathrm{BRoots}(\langle a, b\rangle * p)(++)(E \cdot P)$.

Proof: Set $q = \langle a, b\rangle$. Set $B = \mathrm{BRoots}(q * p)$. Reconsider $P_1 = P$ as a permutation of $\mathrm{dom}(B(++)E)$. $(B(++)E) \cdot P_1 = B(++)(E \cdot P)$ by [13, (27)], [11, (29), (25)], [4, (13)]. □

Let us consider a field L, a non-zero polynomial p over L, an element a of L, a non zero element b of L, and an enumeration E of $\mathrm{Roots}(\langle a, b\rangle * p)$. Now we state the propositions:

(23) Suppose $-\frac{a}{b} \notin \mathrm{Roots}(p)$. Then suppose $E = \mathrm{CFS}(\mathrm{Roots}(p))^\frown\langle -\frac{a}{b}\rangle$. Then $(\mathrm{CFS}(\mathrm{Roots}(\langle a, b\rangle * p)))^{-1} \cdot E$ is a permutation of $\mathrm{dom}\, E$. The theorem is a consequence of (15) and (10).

(24) Suppose $-\frac{a}{b} \notin \mathrm{Roots}(p)$. Then suppose $E = \mathrm{CFS}(\mathrm{Roots}(p))^\frown\langle -\frac{a}{b}\rangle$. Then $\sum(\mathrm{BRoots}(\langle a, b\rangle * p)(++)E) = \sum(\mathrm{BRoots}(\langle a, b\rangle * p)(++)\,\mathrm{CFS}(\mathrm{Roots}(\langle a, b\rangle * p)))$.

Proof: Set $q = \langle a, b\rangle$. Set $B = \mathrm{BRoots}(q * p)$. Set $D = \mathrm{CFS}(\mathrm{Roots}(q * p))$. Reconsider $P = D^{-1} \cdot E$ as a permutation of $\mathrm{dom}\, E$. $E \cdot E^{-1} \cdot D = D$ by [4, (37)], [13, (27)], [4, (35), (12)]. $(B(++)E) \cdot P^{-1} = B(++)(E \cdot P^{-1})$. □

(25) $\sum(\mathrm{BRoots}(\langle a, b\rangle)(++)E) = -\frac{a}{b}$. The theorem is a consequence of (10), (11), and (14).

Let L be an integral domain and p be a non-zero polynomial over L. The functor $\mathrm{SumRoots}(p)$ yielding an element of L is defined by the term

(Def. 2) $\sum(\mathrm{BRoots}(p)(++)\,\mathrm{CFS}(\mathrm{Roots}(p)))$.

Now we state the propositions:

(26) Let us consider an integral domain L, and a non-zero polynomial p over L. If $\mathrm{Roots}(p) = \emptyset$, then $\mathrm{SumRoots}(p) = 0_L$. The theorem is a consequence of (2) and (18).

(27) Let us consider a field L, an element a of L, and a non zero element b of L. Then $\mathrm{SumRoots}(\langle a, b\rangle) = -\frac{a}{b}$. The theorem is a consequence of (10), (2), and (11).

(28) Let us consider a field L, a non-zero polynomial p over L, an element a of L, and a non zero element b of L. Then $\mathrm{SumRoots}(\langle a, b\rangle * p) = -\frac{a}{b} + \mathrm{SumRoots}(p)$. The theorem is a consequence of (16), (17), (24), (2), (10), (11), (25), and (19).

(29) Let us consider a field L, elements a, c of L, and non zero elements b, d of L. Then $\mathrm{SumRoots}(\langle a, b\rangle * \langle c, d\rangle) = -\frac{a}{b} + -\frac{c}{d}$. The theorem is a consequence of (27) and (28).

(30) Let us consider an algebraic closed field L, and non-zero polynomials p, q over L. Suppose $\operatorname{len} p \geqslant 2$. Then $\operatorname{SumRoots}(p * q) = \operatorname{SumRoots}(p) + \operatorname{SumRoots}(q)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non-zero polynomial f over L such that $\$_1 = \operatorname{len} f$ holds $\operatorname{SumRoots}(f * q) = \operatorname{SumRoots}(f) + \operatorname{SumRoots}(q)$. $\mathcal{P}[2]$. For every non trivial natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [6, (29)], [1, (11)], [8, (17), (50)]. For every non trivial natural number k, $\mathcal{P}[k]$ from [6, Sch. 2]. □

(31) Let us consider an algebraic closed integral domain L, a non-zero polynomial p over L, and a finite sequence r of elements of L. Suppose r is one-to-one and $\operatorname{len} r = \operatorname{len} p -' 1$ and $\operatorname{Roots}(p) = \operatorname{rng} r$. Then $\sum r = \operatorname{SumRoots}(p)$.

PROOF: Set $B = \operatorname{BRoots}(p)$. Set $s = \operatorname{support} B$. Set $L_1 = \operatorname{len} r \mapsto 1$. Consider f being a finite sequence of elements of $\mathbb{N}$ such that $\operatorname{degree}(B) = \sum f$ and $f = B \cdot \operatorname{CFS}(s)$. Reconsider $E = \operatorname{CFS}(s)$ as a finite sequence of elements of L. For every natural number j such that $j \in \operatorname{Seg} \operatorname{len} r$ holds $f(j) \geqslant L_1(j)$ by [8, (52)], [4, (12)], [3, (57)]. For every natural number j such that $1 \leqslant j \leqslant \operatorname{len} E$ holds $(B(++)E)(j) = E(j)$ by [5, (83)], [3, (57)], [9, (13)]. □

(32) VIETA'S FORMULA ABOUT THE SUM OF ROOTS:

Let us consider an algebraic closed field L, and a non-zero polynomial p over L. Suppose $\operatorname{len} p \geqslant 2$. Then $\operatorname{SumRoots}(p) = -\frac{p(\operatorname{len} p -' 2)}{p(\operatorname{len} p -' 1)}$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non-zero polynomial p over L such that $\$_1 = \operatorname{len} p$ holds $\operatorname{SumRoots}(p) = -\frac{p(\$_1 -' 2)}{p(\$_1 -' 1)}$. $\mathcal{P}[2]$ by (6), [7, (38)], (27). For every non trivial natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [6, (29)], [1, (11)], [8, (17)], [10, (5)]. For every non trivial natural number k, $\mathcal{P}[k]$ from [6, Sch. 2]. □

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[5] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[6] Robert Milewski. Natural numbers. *Formalized Mathematics*, 7(**1**):19–22, 1998.

[7] Robert Milewski. Fundamental theorem of algebra. *Formalized Mathematics*, 9(**3**):461–470, 2001.
[8] Piotr Rudnicki. Little Bezout theorem (factor theorem). *Formalized Mathematics*, 12(**1**): 49–58, 2004.
[9] Christoph Schwarzweller. The binomial theorem for algebraic structures. *Formalized Mathematics*, 9(**3**):559–564, 2001.
[10] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[11] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.
[12] E. B. Vinberg. *A Course in Algebra*. American Mathematical Society, 2003. ISBN 0821834134.
[13] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

Received May 25, 2017

The English version of this volume of Formalized Mathematics was financed under agreement 548/P-DUN/2016 with the funds from the Polish Minister of Science and Higher Education for the dissemination of science.