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Differentiability of Polynomials over Reals

Artur Korniłowicz
Institute of Informatics
University of Białystok
Poland

Summary. In this article, we formalize in the Mizar system [3] the notion of the derivative of polynomials over the field of real numbers [4]. To define it, we use the derivative of functions between reals and reals [9].

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1. Preliminaries

From now on c denotes a complex, r denotes a real number, m, n denote natural numbers, and f denotes a complex-valued function.

Now we state the propositions:

(1) $0 + f = f$.

(2) $f - 0 = f$.

Let f be a complex-valued function. Observe that $0 + f$ reduces to f and $f - 0$ reduces to f.

Now we state the propositions:

(3) $c + f = (\operatorname{dom} f \longmapsto c) + f$.

(4) $f - c = f - (\operatorname{dom} f \longmapsto c)$.

(5) $c \cdot f = (\operatorname{dom} f \longmapsto c) \cdot f$.

(6) $f + (\operatorname{dom} f \longmapsto 0) = f$. The theorem is a consequence of (3).

(7) $f - (\operatorname{dom} f \longmapsto 0) = f$. The theorem is a consequence of (4).

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(8) $\square^0 = \mathbb{R} \longmapsto 1$.
PROOF: Reconsider $s = 1$ as an element of $\mathbb{R}$. $\square^0 = \mathbb{R} \longmapsto s$ by [8, (34)], [10, (7)]. $\square$

2. Differentiability of Real Functions

One can check that every function from $\mathbb{R}$ into $\mathbb{R}$ which is differentiable is also continuous.

Let f be a differentiable function from $\mathbb{R}$ into $\mathbb{R}$. The functor f' yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by the term

(Def. 1) $f'_{\restriction \mathbb{R}}$.

Now we state the propositions:

(9) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$. Then f is differentiable if and only if for every r, f is differentiable in r.

(10) Let us consider a differentiable function f from $\mathbb{R}$ into $\mathbb{R}$. Then $f'(r) = f'(r)$[1].

Let f be a function from $\mathbb{R}$ into $\mathbb{R}$. Observe that f is differentiable if and only if the condition (Def. 2) is satisfied.

(Def. 2) for every r, f is differentiable in r.

Let us note that every function from $\mathbb{R}$ into $\mathbb{R}$ which is constant is also differentiable.

Now we state the proposition:

(11) Let us consider a constant function f from $\mathbb{R}$ into $\mathbb{R}$. Then $f' = \mathbb{R} \longmapsto 0$.
PROOF: Reconsider $z = 0$ as an element of $\mathbb{R}$. $f' = \mathbb{R} \longmapsto z$ by [9, (22)], [10, (7)]. $\square$

One can verify that $\mathrm{id}_{\mathbb{R}}$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$.

Now we state the proposition:

(12) $\mathrm{id}'_{\mathbb{R}} = \mathbb{R} \longmapsto 1$.
PROOF: Set $f = \mathrm{id}_{\mathbb{R}}$. Reconsider $z = 1$ as an element of $\mathbb{R}$. $f' = \mathbb{R} \longmapsto z$ by [9, (17)], [10, (7)]. $\square$

Let us consider n. One can verify that $\square^n$ is differentiable.

Now we state the proposition:

(13) $(\square^n)' = n \cdot (\square^{n-1})$.

From now on f, g denote differentiable functions from $\mathbb{R}$ into $\mathbb{R}$.

[1]Left-side $f'(r)$ is the value of the derivative defined in this article for differentiable functions $f : \mathbb{R} \mapsto \mathbb{R}$, and right-side $f'(r)$ is the value of the derivative defined for partial functions in [9].

Let us consider f and g. Let us observe that $f+g$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $f-g$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $f \cdot g$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$.

Let us consider r. One can verify that $r+f$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $r \cdot f$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $f-r$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $-f$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$ and $f^{\mathbf{2}}$ is differentiable as a function from $\mathbb{R}$ into $\mathbb{R}$.

Now we state the propositions:

(14) $(f+g)' = f' + g'$. The theorem is a consequence of (9) and (10).

(15) $(f-g)' = f' - g'$. The theorem is a consequence of (9) and (10).

(16) $(f \cdot g)' = g \cdot f' + f \cdot g'$. The theorem is a consequence of (9) and (10).

(17) $(r+f)' = f'$. The theorem is a consequence of (11), (3), (14), and (6).

(18) $(f-r)' = f'$. The theorem is a consequence of (11), (4), (15), and (7).

(19) $(r \cdot f)' = r \cdot f'$. The theorem is a consequence of (9) and (10).

(20) $(-f)' = -f'$.

3. Polynomials

In the sequel L denotes a non empty zero structure and x denotes an element of L.

Now we state the proposition:

(21) Let us consider a (the carrier of L)-valued function f, and an object a. Then $\mathrm{Support}(f +\cdot (a, x)) \subseteq \mathrm{Support}\, f \cup \{a\}$.
PROOF: $a = z$ or $z \in \mathrm{Support}\, f$ by [2, (32), (30)]. $\square$

Let us consider L and x. Let f be a finite-Support sequence of L and a be an object. Observe that $f +\cdot (a, x)$ is finite-Support as a sequence of L.

Now we state the proposition:

(22) Let us consider a polynomial p over L. If $p \neq \mathbf{0}.L$, then $\mathrm{len}\, p -' 1 = \mathrm{len}\, p - 1$.

Let L be a non empty zero structure and x be an element of L. Let us note that $\langle x \rangle$ is constant and $\langle x, 0_L \rangle$ is constant.

Now we state the proposition:

(23) Let us consider a non empty zero structure L, and a constant polynomial p over L. Then

(i) $p = \mathbf{0}.L$, or

(ii) $p = \langle p(0) \rangle$.

Let us consider L, x, and n. The functor $\operatorname{seq}(n,x)$ yielding a sequence of L is defined by the term

(Def. 3) $\mathbf{0}.L +\cdot (n,x)$.

Observe that $\operatorname{seq}(n,x)$ is finite-Support.

Now we state the propositions:

(24) $(\operatorname{seq}(n,x))(n) = x$.

(25) If $m \neq n$, then $(\operatorname{seq}(n,x))(m) = 0_L$.

(26) the length of $\operatorname{seq}(n,x)$ is at most $n+1$.

(27) If $x \neq 0_L$, then $\operatorname{len}\operatorname{seq}(n,x) = n+1$.

PROOF: Set $p = \operatorname{seq}(n,x)$. For every m such that the length of p is at most m holds $n+1 \leqslant m$ by (24), [1, (13)]. $\square$

(28) $\operatorname{seq}(n,0_L) = \mathbf{0}.L$. The theorem is a consequence of (24).

(29) Let us consider a right zeroed, non empty additive loop structure L, and elements x, y of L. Then $\operatorname{seq}(n,x)+\operatorname{seq}(n,y) = \operatorname{seq}(n,x+y)$. The theorem is a consequence of (24) and (25).

(30) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L, and an element x of L. Then $-\operatorname{seq}(n,x) = \operatorname{seq}(n,-x)$. The theorem is a consequence of (24) and (25).

(31) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L, and elements x, y of L. Then $\operatorname{seq}(n,x) - \operatorname{seq}(n,y) = \operatorname{seq}(n,x-y)$. The theorem is a consequence of (30) and (29).

Let L be a non empty zero structure and p be a sequence of L. Let us consider n. The functor $p \restriction n$ yielding a sequence of L is defined by the term

(Def. 4) $p +\cdot (n, 0_L)$.

Let p be a polynomial over L. Let us note that $p \restriction n$ is finite-Support.

Let us consider a non empty zero structure L and a sequence p of L. Now we state the propositions:

(32) $(p \restriction n)(n) = 0_L$.

(33) If $m \neq n$, then $(p \restriction n)(m) = p(m)$.

Now we state the proposition:

(34) Let us consider a non empty zero structure L. Then $\mathbf{0}.L \restriction n = \mathbf{0}.L$. The theorem is a consequence of (32).

Let L be a non empty zero structure. Let us consider n. One can verify that $\mathbf{0}.L \restriction n$ reduces to $\mathbf{0}.L$.

Let us consider a non empty zero structure L and a polynomial p over L. Now we state the propositions:

(35) If $n > \operatorname{len} p -' 1$, then $p \restriction n = p$. The theorem is a consequence of (32).

(36) If $p \neq \mathbf{0}.L$, then $\operatorname{len}(p \restriction (\operatorname{len} p -' 1)) < \operatorname{len} p$.
PROOF: Set $m = \operatorname{len} p -' 1$. $m = \operatorname{len} p - 1$. the length of $p \restriction m$ is at most $\operatorname{len} p$ by [2, (32)], [7, (8)]. $\square$

Now we state the proposition:

(37) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L, and a polynomial p over L. Then $p \restriction (\operatorname{len} p -' 1) + \text{Leading-Monomial}\, p = p$. The theorem is a consequence of (32).

Let L be a non empty zero structure and p be a constant polynomial over L. Observe that Leading-Monomial p is constant.

Now we state the proposition:

(38) Let us consider an add-associative, right zeroed, right complementable, distributive, unital, non empty double loop structure L, and elements x, y of L. Then $\operatorname{eval}(\operatorname{seq}(n, x), y) = (\operatorname{seq}(n, x))(n) \cdot \operatorname{power}(y, n)$. The theorem is a consequence of (28), (27), and (25).

4. Differentiability of Polynomials over Reals

In the sequel p, q denote polynomials over $\mathbb{R}_{\rm F}$.

Now we state the propositions:

(39) Let us consider an element r of $\mathbb{R}_{\rm F}$. Then $\operatorname{power}(r, n) = r^n$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \operatorname{power}(r, \$_1) = r^{\$_1}$. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. $\square$

(40) $\square^n = \operatorname{FPower}(1_{\mathbb{R}_{\rm F}}, n)$.
PROOF: Reconsider $f = \operatorname{FPower}(1_{\mathbb{R}_{\rm F}}, n)$ as a function from $\mathbb{R}$ into $\mathbb{R}$. $\square^n = f$ by [8, (36)], (39). $\square$

Let us consider an element r of $\mathbb{R}_{\rm F}$. Now we state the propositions:

(41) $\operatorname{FPower}(r, n+1) = \operatorname{FPower}(r, n) \cdot \mathrm{id}_{\mathbb{R}}$.

(42) $\operatorname{FPower}(r, n)$ is a differentiable function from $\mathbb{R}$ into $\mathbb{R}$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \operatorname{FPower}(r, \$_1)$ is a differentiable function from $\mathbb{R}$ into $\mathbb{R}$. $\mathcal{P}[0]$ by [6, (66)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. $\square$

(43) $\operatorname{power}(r, n) = (\square^n)(r)$. The theorem is a consequence of (40).

Let us consider p. The functor p' yielding a sequence of $\mathbb{R}_{\rm F}$ is defined by

(Def. 5) for every natural number n, $it(n) = p(n+1) \cdot (n+1)$.

Note that p' is finite-Support.

Now we state the propositions:

(44) If $p \neq \mathbf{0}.\mathbb{R}_{\rm F}$, then $\operatorname{len} p' = \operatorname{len} p - 1$.
PROOF: Set $x = \operatorname{len} p - 1$. Set $d = p'$. the length of d is at most x by [7, (8)]. For every n such that the length of d is at most n holds $x \leqslant n$ by [11, (7)], [7, (10)], [1, (21)]. □

(45) If $p \neq \mathbf{0}.\mathbb{R}_{\rm F}$, then $\operatorname{len} p = \operatorname{len} p' + 1$. The theorem is a consequence of (44).

(46) Let us consider a constant polynomial p over $\mathbb{R}_{\rm F}$. Then $p' = \mathbf{0}.\mathbb{R}_{\rm F}$. The theorem is a consequence of (45).

(47) $(p+q)' = p' + q'$.

(48) $(-p)' = -p'$.

(49) $(p-q)' = p' - q'$. The theorem is a consequence of (47) and (48).

(50) Leading-Monomial $p' = \mathbf{0}.\mathbb{R}_{\rm F} +\cdot (\operatorname{len} p -' 2, p(\operatorname{len} p -' 1) \cdot (\operatorname{len} p -' 1))$.
PROOF: Set $l =$ Leading-Monomial p. Set $m = \operatorname{len} p -' 1$. Set $k = \operatorname{len} p -' 2$. Reconsider $a = p(m) \cdot m$ as an element of $\mathbb{R}_{\rm F}$. Set $f = \mathbf{0}.F +\cdot (k, a)$. $l' = f$ by [1, (53)], [2, (31), (32)], [10, (7)]. □

(51) Let us consider elements r, s of $\mathbb{R}_{\rm F}$. Then $\langle r, s\rangle' = \langle s\rangle$.

Let us consider p. The functor Eval(p) yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by the term

(Def. 6) Polynomial-Function$(\mathbb{R}_{\rm F}, p)$.

Let us note that Eval(p) is differentiable.

Now we state the propositions:

(52) $\mathrm{Eval}(\mathbf{0}.\mathbb{R}_{\rm F}) = \mathbb{R} \longmapsto 0$.
PROOF: $\mathrm{Eval}(\mathbf{0}.F) = \mathbb{R} \longmapsto 0 (\in \mathbb{R})$ by [5, (17)], [10, (7)]. □

(53) Let us consider an element r of $\mathbb{R}_{\rm F}$. Then $\mathrm{Eval}(\langle r\rangle) = \mathbb{R} \longmapsto r$.
PROOF: $\mathrm{Eval}(\langle r\rangle) = \mathbb{R} \longmapsto r(\in \mathbb{R})$ by [6, (37)], [10, (7)]. □

(54) If p is constant, then $\mathrm{Eval}(p)' = \mathbb{R} \longmapsto 0$. The theorem is a consequence of (23), (52), and (11).

(55) $\mathrm{Eval}(p+q) = \mathrm{Eval}(p) + \mathrm{Eval}(q)$.

(56) $\mathrm{Eval}(-p) = -\mathrm{Eval}(p)$.

(57) $\mathrm{Eval}(p-q) = \mathrm{Eval}(p) - \mathrm{Eval}(q)$. The theorem is a consequence of (55) and (56).

(58) Eval(Leading-Monomial p) = FPower$(p(\operatorname{len} p -' 1), \operatorname{len} p -' 1)$.
PROOF: Set $l =$ Leading-Monomial p. Set $m = \operatorname{len} p -' 1$. Reconsider $f =$ FPower$(p(m), m)$ as a function from $\mathbb{R}$ into $\mathbb{R}$. Eval$(l) = f$ by [5, (22)]. □

(59) Eval(Leading-Monomial p) $= p(\operatorname{len} p -' 1) \cdot (\square^{\operatorname{len} p -' 1})$.
PROOF: Set $l =$ Leading-Monomial p. Set $m = \operatorname{len} p -' 1$. Set $f = p(m) \cdot (\square^m)$. Eval$(l) = f$ by (39), [8, (36)], [5, (22)]. □

(60) Let us consider an element r of $\mathbb{R}_{\rm F}$. Then $\mathrm{Eval}(\mathrm{seq}(n, r)) = r \cdot (\square^{n})$. The theorem is a consequence of (24), (43), and (38).

(61) $\mathrm{Eval}(p)' = \mathrm{Eval}(p')$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every p such that $\mathrm{len}\, p \leqslant \$_1$ holds $\mathrm{Eval}(p)' = \mathrm{Eval}(p')$. $\mathcal{P}[0]$ by [5, (5)], (46), (52), (54). If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by (36), [5, (3)], [1, (13)], (37). $\mathcal{P}[n]$ from [1, Sch. 2]. $\square$

Let us consider p. Let us observe that $\mathrm{Eval}(p)'$ is differentiable.

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