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# The Axiomatization of Propositional Logic[1]

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**Summary.** This article introduces propositional logic as a formal system ([14], [10], [11]). The formulae of the language are as follows $\phi ::= \bot \,|\, p \,|\, \phi \to \phi$. Other connectives are introduced as abbrevations. The notions of model and satisfaction in model are defined. The axioms are all the formulae of the following schemes

- $\alpha \Rightarrow (\beta \Rightarrow \alpha)$,
- $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$,
- $(\neg\beta \Rightarrow \neg\alpha) \Rightarrow ((\neg\beta \Rightarrow \alpha) \Rightarrow \beta)$.

Modus ponens is the only derivation rule. The soundness theorem and the strong completeness theorem are proved. The proof of the completeness theorem is carried out by a counter-model existence method. In order to prove the completeness theorem, Lindenbaum's Lemma is proved. Some most widely used tautologies are presented.

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## 1. Preliminaries

Now we state the propositions:

(1) Let us consider functions $f$, $g$. Suppose $\operatorname{dom} f \subseteq \operatorname{dom} g$ and for every set $x$ such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$. Then $\operatorname{rng} f \subseteq \operatorname{rng} g$.

(2) Let us consider Boolean objects $p$, $q$. Then $p \wedge q \Rightarrow p = \mathit{true}$.

(3) Let us consider a Boolean object $p$. Then $\neg\neg p \Leftrightarrow p = \mathit{true}$.

Let us consider Boolean objects $p$, $q$. Now we state the propositions:

(4) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q = \mathit{true}$.

(5) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q = \mathit{true}$.

(6) $p \Rightarrow q \Rightarrow (\neg q \Rightarrow \neg p) = \mathit{true}$.

Let us consider Boolean objects $p$, $q$, $r$. Now we state the propositions:

(7) $p \Rightarrow q \Rightarrow (p \Rightarrow r \Rightarrow (p \Rightarrow q \wedge r)) = \mathit{true}$.

(8) $p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r)) = \mathit{true}$.

Let us consider Boolean objects $p$, $q$. Now we state the propositions:

(9) $p \wedge q \Leftrightarrow q \wedge p = \mathit{true}$.

(10) $p \vee q \Leftrightarrow q \vee p = \mathit{true}$.

Let us consider Boolean objects $p$, $q$, $r$. Now we state the propositions:

(11) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r) = \mathit{true}$.

(12) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r) = \mathit{true}$.

(13) Let us consider Boolean objects $p$, $q$. Then $\neg q \Rightarrow \neg p \Rightarrow (\neg q \Rightarrow p \Rightarrow q) = \mathit{true}$.

Let us consider Boolean objects $p$, $q$, $r$. Now we state the propositions:

(14) $p \wedge (q \vee r) \Leftrightarrow p \wedge q \vee p \wedge r = \mathit{true}$.

(15) $p \vee q \wedge r \Leftrightarrow (p \vee q) \wedge (p \vee r) = \mathit{true}$.

(16) Let us consider a finite set $X$, and a set $Y$. Suppose $Y$ is $\subseteq$-linear and $X \subseteq \bigcup Y$ and $Y \neq \emptyset$. Then there exists a set $Z$ such that

  (i) $X \subseteq Z$, and

  (ii) $Z \in Y$.

## 2. The Syntax

Let $D$ be a set. We say that $D$ has propositional variables if and only if

(Def. 1) for every element $n$ of $\mathbb{N}$, $\langle 3 + n \rangle \in D$.

We say that $D$ is PL-closed if and only if

(Def. 2) $D \subseteq \mathbb{N}^*$ and $D$ has FALSUM, implication and propositional variables.

Let us note that every set which is PL-closed is also non empty and has also FALSUM, implication, and propositional variables and every subset of $\mathbb{N}^*$ which has FALSUM, implication, and propositional variables is also PL-closed.

The functor PL-WFF yielding a set is defined by

(Def. 3) *it* is PL-closed and for every set $D$ such that $D$ is PL-closed holds $it \subseteq D$.

Observe that PL-WFF is PL-closed and there exists a set which is PL-closed and non empty and PL-WFF is functional and every element of PL-WFF is finite sequence-like.

The functor $\perp_{\mathrm{PL}}$ yielding an element of PL-WFF is defined by the term

(Def. 4) $\langle 0 \rangle$.

Let $p$, $q$ be elements of PL-WFF. The functor $p \Rightarrow q$ yielding an element of PL-WFF is defined by the term

(Def. 5) $(\langle 1 \rangle \frown p) \frown q$.

Let $n$ be an element of $\mathbb{N}$. The functor $\mathrm{Prop}\, n$ yielding an element of PL-WFF is defined by the term

(Def. 6) $\langle 3 + n \rangle$.

The functor $AP$ yielding a subset of PL-WFF is defined by

(Def. 7) for every set $x$, $x \in it$ iff there exists an element $n$ of $\mathbb{N}$ such that $x = \mathrm{Prop}\, n$.

From now on $p$, $q$, $r$, $s$, $A$, $B$ denote elements of PL-WFF, $F$, $G$, $H$ denote subsets of PL-WFF, $k$, $n$ denote elements of $\mathbb{N}$, and $f$, $f_1$, $f_2$ denote finite sequences of elements of PL-WFF.

Let $D$ be a subset of PL-WFF. Observe that $D$ has implication if and only if the condition (Def. 8) is satisfied.

(Def. 8) for every $p$ and $q$ such that $p, q \in D$ holds $p \Rightarrow q \in D$.

The scheme *PLInd* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 1) For every $r$, $\mathcal{P}[r]$

provided

- $\mathcal{P}[\perp_{\mathrm{PL}}]$ and
- for every $n$, $\mathcal{P}[\mathrm{Prop}\, n]$ and
- for every $r$ and $s$ such that $\mathcal{P}[r]$ and $\mathcal{P}[s]$ holds $\mathcal{P}[r \Rightarrow s]$.

Now we state the proposition:

(17) PL-WFF $\subseteq$ HP-WFF.

PROOF: Define $\mathcal{P}[\text{element of PL-WFF}] \equiv \$_1 \in \text{HP-WFF}$. For every $n$, $\mathcal{P}[\mathrm{Prop}\, n]$. For every $r$ and $s$ such that $\mathcal{P}[r]$ and $\mathcal{P}[s]$ holds $\mathcal{P}[r \Rightarrow s]$. For every $A$, $\mathcal{P}[A]$ from *PLInd*. □

Let us consider $p$. The functor $\neg p$ yielding an element of PL-WFF is defined by the term

(Def. 9) $p \Rightarrow \bot_{\rm PL}$.

The functor $\top_{\rm PL}$ yielding an element of PL-WFF is defined by the term

(Def. 10) $\neg \bot_{\rm PL}$.

Let us consider $p$ and $q$. The functors: $p \wedge q$ and $p \vee q$ yielding elements of PL-WFF are defined by terms

(Def. 11) $\neg (p \Rightarrow \neg q)$,

(Def. 12) $\neg p \Rightarrow q$,

respectively. The functor $p \Leftrightarrow q$ yielding an element of PL-WFF is defined by the term

(Def. 13) $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

## 3. THE SEMANTICS

A PL-model is a subset of $AP$. From now on $M$ denotes a PL-model.

Let $M$ be a PL-model. The functor $\mathrm{SAT}_M$ yielding a function from PL-WFF into *Boolean* is defined by

(Def. 14) $it(\bot_{\rm PL}) = 0$ and for every $k$, $it(\mathrm{Prop}\, k) = 1$ iff $\mathrm{Prop}\, k \in M$ and for every $p$ and $q$, $it(p \Rightarrow q) = it(p) \Rightarrow it(q)$.

Now we state the propositions:

(18) $\mathrm{SAT}_M(A \Rightarrow B) = 1$ if and only if $\mathrm{SAT}_M(A) = 0$ or $\mathrm{SAT}_M(B) = 1$.

(19) $\mathrm{SAT}_M(\neg p) = \neg(\mathrm{SAT}_M(p))$.

(20) $\mathrm{SAT}_M(\neg A) = 1$ if and only if $\mathrm{SAT}_M(A) = 0$. The theorem is a consequence of (19).

(21) $\mathrm{SAT}_M(A \wedge B) = \mathrm{SAT}_M(A) \wedge \mathrm{SAT}_M(B)$. The theorem is a consequence of (19).

(22) $\mathrm{SAT}_M(A \wedge B) = 1$ if and only if $\mathrm{SAT}_M(A) = 1$ and $\mathrm{SAT}_M(B) = 1$. The theorem is a consequence of (21).

(23) $\mathrm{SAT}_M(A \vee B) = \mathrm{SAT}_M(A) \vee \mathrm{SAT}_M(B)$. The theorem is a consequence of (19).

(24) $\mathrm{SAT}_M(A \vee B) = 1$ if and only if $\mathrm{SAT}_M(A) = 1$ or $\mathrm{SAT}_M(B) = 1$. The theorem is a consequence of (23).

(25) $\mathrm{SAT}_M(A \Leftrightarrow B) = \mathrm{SAT}_M(A) \Leftrightarrow \mathrm{SAT}_M(B)$. The theorem is a consequence of (21).

(26) $\mathrm{SAT}_M(A \Leftrightarrow B) = 1$ if and only if $\mathrm{SAT}_M(A) = \mathrm{SAT}_M(B)$. The theorem is a consequence of (25).

Let us consider $M$ and $p$. We say that $M \models p$ if and only if

(Def. 15) $\mathrm{SAT}_M(p) = 1$.

Let us consider $F$. We say that $M \models F$ if and only if

(Def. 16) for every $p$ such that $p \in F$ holds $M \models p$.

Let us consider $p$. We say that $F \models p$ if and only if

(Def. 17) for every $M$ such that $M \models F$ holds $M \models p$.

Let us consider $A$. We say that $A$ is a tautology if and only if

(Def. 18) for every $M$, $\mathrm{SAT}_M(A) = 1$.

Now we state the propositions:

(27) $A$ is a tautology if and only if $\emptyset_{\text{PL-WFF}} \models A$.

(28) $p \Rightarrow (q \Rightarrow p)$ is a tautology.

(29) $p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$ is a tautology.

(30) $\neg q \Rightarrow \neg p \Rightarrow (\neg q \Rightarrow p \Rightarrow q)$ is a tautology. The theorem is a consequence of (19) and (13).

(31) $p \Rightarrow q \Rightarrow (\neg q \Rightarrow \neg p)$ is a tautology. The theorem is a consequence of (19) and (6).

(32) $p \wedge q \Rightarrow p$ is a tautology. The theorem is a consequence of (21) and (2).

(33) $p \wedge q \Rightarrow q$ is a tautology. The theorem is a consequence of (21) and (2).

(34) $p \Rightarrow p \vee q$ is a tautology. The theorem is a consequence of (23).

(35) $q \Rightarrow p \vee q$ is a tautology. The theorem is a consequence of (23).

(36) $p \wedge q \Leftrightarrow q \wedge p$ is a tautology. The theorem is a consequence of (25), (21), and (9).

(37) $p \vee q \Leftrightarrow q \vee p$ is a tautology. The theorem is a consequence of (25), (23), and (10).

(38) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ is a tautology. The theorem is a consequence of (25), (21), and (11).

(39) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ is a tautology. The theorem is a consequence of (25), (23), and (12).

(40) $p \wedge (q \vee r) \Leftrightarrow p \wedge q \vee p \wedge r$ is a tautology. The theorem is a consequence of (25), (21), (23), and (14).

(41) $p \vee q \wedge r \Leftrightarrow (p \vee q) \wedge (p \vee r)$ is a tautology. The theorem is a consequence of (25), (23), (21), and (15).

(42) $\neg\neg p \Leftrightarrow p$ is a tautology. The theorem is a consequence of (25), (19), and (3).

(43) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ is a tautology. The theorem is a consequence of (25), (19), (21), (23), and (4).

(44) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ is a tautology. The theorem is a consequence of (25), (19), (23), (21), and (5).

(45) $p \Rightarrow q \Rightarrow (p \Rightarrow r \Rightarrow (p \Rightarrow q \wedge r))$ is a tautology. The theorem is a consequence of (21) and (7).

(46) $p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r))$ is a tautology. The theorem is a consequence of (23) and (8).

(47) If $F \models A$ and $F \models A \Rightarrow B$, then $F \models B$.

## 4. The Axioms. Derivability.

Let $D$ be a set. We say that $D$ is with axioms of PL if and only if

(Def. 19) for every $p$, $q$, and $r$ holds $p \Rightarrow (q \Rightarrow p)$, $p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$, $\neg q \Rightarrow \neg p \Rightarrow (\neg q \Rightarrow p \Rightarrow q) \in D$.

The functor PL-axioms yielding a subset of PL-WFF is defined by

(Def. 20) *it* is with axioms of PL and for every subset $D$ of PL-WFF such that $D$ is with axioms of PL holds $it \subseteq D$.

One can check that PL-axioms is with axioms of PL.

Let us consider $p$, $q$, and $r$. We say that $\mathrm{MP}(p, q, r)$ if and only if

(Def. 21) $q = p \Rightarrow r$.

Observe that PL-axioms is non empty.

Let us consider $A$. We say that $A$ is the simplification axiom if and only if

(Def. 22) there exists $p$ and there exists $q$ such that $A = p \Rightarrow (q \Rightarrow p)$.

We say that $A$ is Frege axiom if and only if

(Def. 23) there exists $p$ and there exists $q$ and there exists $r$ such that $A = p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$.

We say that $A$ is the explosion axiom if and only if

(Def. 24) there exists $p$ and there exists $q$ such that $A = \neg q \Rightarrow \neg p \Rightarrow (\neg q \Rightarrow p \Rightarrow q)$.

Now we state the propositions:

(48) Every element of PL-axioms is the simplification axiom or Frege axiom or the explosion axiom.

(49) If $A$ is the simplification axiom or Frege axiom or the explosion axiom, then $F \models A$. The theorem is a consequence of (28), (29), and (30).

Let $i$ be a natural number. Let us consider $f$ and $F$. We say that $\mathrm{prc}(f, F, i)$ if and only if

(Def. 25) $f(i) \in$ PL-axioms or $f(i) \in F$ or there exist natural numbers $j$, $k$ such that $1 \leqslant j < i$ and $1 \leqslant k < i$ and $\mathrm{MP}(f_j, f_k, f_i)$.

Let us consider $p$. We say that $F \vdash p$ if and only if

(Def. 26) there exists $f$ such that $f(\operatorname{len} f) = p$ and $1 \leqslant \operatorname{len} f$ and for every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len} f$ holds $\operatorname{prc}(f, F, i)$.

Now we state the propositions:

(50) Let us consider natural numbers $i$, $n$. Suppose $n + \operatorname{len} f \leqslant \operatorname{len} f_2$ and for every natural number $k$ such that $1 \leqslant k \leqslant \operatorname{len} f$ holds $f(k) = f_2(k + n)$ and $1 \leqslant i \leqslant \operatorname{len} f$. If $\operatorname{prc}(f, F, i)$, then $\operatorname{prc}(f_2, F, i + n)$.

(51) Suppose $f_2 = f \frown f_1$ and $1 \leqslant \operatorname{len} f$ and $1 \leqslant \operatorname{len} f_1$ and for every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len} f$ holds $\operatorname{prc}(f, F, i)$ and for every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len} f_1$ holds $\operatorname{prc}(f_1, F, i)$. Let us consider a natural number $i$. If $1 \leqslant i \leqslant \operatorname{len} f_2$, then $\operatorname{prc}(f_2, F, i)$. The theorem is a consequence of (50).

(52) Suppose $f = f_1 \frown \langle p \rangle$ and $1 \leqslant \operatorname{len} f_1$ and for every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len} f_1$ holds $\operatorname{prc}(f_1, F, i)$ and $\operatorname{prc}(f, F, \operatorname{len} f)$. Then

(i) for every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len} f$ holds $\operatorname{prc}(f, F, i)$, and

(ii) $F \vdash p$.

The theorem is a consequence of (50).

(53) If $p \in$ PL-axioms or $p \in F$, then $F \vdash p$.
Proof: Define $\mathcal{P}[\text{set}, \text{set}] \equiv \$_2 = p$. Consider $f$ such that $\operatorname{dom} f = \operatorname{Seg} 1$ and for every natural number $k$ such that $k \in \operatorname{Seg} 1$ holds $\mathcal{P}[k, f(k)]$ from [3, Sch. 5]. For every natural number $j$ such that $1 \leqslant j \leqslant \operatorname{len} f$ holds $\operatorname{prc}(f, F, j)$. $\square$

(54) If $F \vdash p$ and $F \vdash p \Rightarrow q$, then $F \vdash q$.
Proof: Consider $f$ such that $f(\operatorname{len} f) = p$ and $1 \leqslant \operatorname{len} f$ and for every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len} f$ holds $\operatorname{prc}(f, F, i)$. Consider $f_1$ such that $f_1(\operatorname{len} f_1) = p \Rightarrow q$ and $1 \leqslant \operatorname{len} f_1$ and for every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len} f_1$ holds $\operatorname{prc}(f_1, F, i)$. Set $g = (f \frown f_1) \frown \langle q \rangle$. For every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len} f_1$ holds $g(\operatorname{len} f + i) = f_1(i)$ by [3, (22), (39)], [1, (12)], [3, (65), (64)]. For every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len}(f \frown f_1)$ holds $\operatorname{prc}(f \frown f_1, F, i)$. $\square$

(55) If $F \subseteq G$, then if $F \vdash p$, then $G \vdash p$.
Proof: Consider $f$ such that $f(\operatorname{len} f) = p$ and $1 \leqslant \operatorname{len} f$ and for every natural number $k$ such that $1 \leqslant k \leqslant \operatorname{len} f$ holds $\operatorname{prc}(f, F, k)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} f$, then $G \vdash f_{\$_1}$. For every natural number $k$, $\mathcal{P}[k]$ from [1, Sch. 4]. $\square$

## 5. Soundness Theorem. Deduction Theorem.

Now we state the propositions:

(56) If $F \vdash A$, then $F \models A$.

Proof: Consider $f$ such that $f(\operatorname{len} f) = A$ and $1 \leqslant \operatorname{len} f$ and for every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len} f$ holds $\operatorname{prc}(f, F, i)$. Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} f$, then $F \models f_{\$_1}$. For every natural number $i$ such that for every natural number $j$ such that $j < i$ holds $\mathcal{P}[j]$ holds $\mathcal{P}[i]$ by [1, (14)], [9, (1)], (48), (49). For every natural number $i$, $\mathcal{P}[i]$ from [1, Sch. 4]. □

(57) $F \vdash A \Rightarrow A$. The theorem is a consequence of (53) and (54).

(58) Deduction theorem:

If $F \cup \{A\} \vdash B$, then $F \vdash A \Rightarrow B$.

Proof: Consider $f$ such that $f(\operatorname{len} f) = B$ and $1 \leqslant \operatorname{len} f$ and for every natural number $i$ such that $1 \leqslant i \leqslant \operatorname{len} f$ holds $\operatorname{prc}(f, F \cup \{A\}, i)$. Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} f$, then $F \vdash A \Rightarrow f_{\$_1}$. For every natural number $i$ such that for every natural number $j$ such that $j < i$ holds $\mathcal{P}[j]$ holds $\mathcal{P}[i]$ by [1, (14)], (53), [9, (1)], (54). For every natural number $i$, $\mathcal{P}[i]$ from [1, Sch. 4]. □

(59) If $F \vdash A \Rightarrow B$, then $F \cup \{A\} \vdash B$. The theorem is a consequence of (53), (55), and (54).

(60) $F \vdash \neg A \Rightarrow (A \Rightarrow B)$. The theorem is a consequence of (53), (54), and (58).

(61) $F \vdash \neg A \Rightarrow A \Rightarrow A$. The theorem is a consequence of (53), (57), and (54).

## 6. Strong Completeness Theorem

Let us consider $F$. We say that $F$ is consistent if and only if

(Def. 27) there exists no $p$ such that $F \vdash p$ and $F \vdash \neg p$.

Now we state the propositions:

(62) $F$ is consistent if and only if there exists $A$ such that $F \nvdash A$. The theorem is a consequence of (60) and (54).

(63) If $F \nvdash A$, then $F \cup \{\neg A\}$ is consistent. The theorem is a consequence of (58), (62), (61), and (54).

(64) $F \vdash A$ if and only if there exists $G$ such that $G \subseteq F$ and $G$ is finite and $G \vdash A$. The theorem is a consequence of (55).

(65) If $F$ is not consistent, then there exists $G$ such that $G$ is finite and $G$ is not consistent and $G \subseteq F$. The theorem is a consequence of (64) and (55).

Let us consider $F$. We say that $F$ is maximal if and only if

(Def. 28) for every $p$ holds $p \in F$ or $\neg p \in F$.

Now we state the propositions:

(66) If $F \subseteq G$ and $F$ is not consistent, then $G$ is not consistent. The theorem is a consequence of (55).

(67) If $F$ is consistent and $F \cup \{A\}$ is not consistent, then $F \cup \{\neg A\}$ is consistent. The theorem is a consequence of (58), (62), (61), and (54).

In the sequel $x$, $y$ denote sets. Now we state the propositions:

(68) Lindenbaum's lemma:

If $F$ is consistent, then there exists $G$ such that $F \subseteq G$ and $G$ is consistent and maximal.

Proof: Set $L =$ PL-WFF. Consider $R$ being a binary relation such that $R$ well orders $L$. Reconsider $R_2 = R\,|^2\, L$ as a binary relation on $L$. Reconsider $R_1 = \langle L, R_2 \rangle$ as a non empty relational structure. Set $c =$ the carrier of $R_1$. Define $\mathcal{H}[\text{object}, \text{object}, \text{object}] \equiv$ for every $p$ for every partial function $f$ from $c$ to $2^L$ such that $\$_1 = p$ and $\$_2 = f$ holds if $(\bigcup \operatorname{rng}(f$ **qua** $(2^L)$-valued binary relation$) \cup F) \cup \{p\}$ is consistent, then $\$_3 = (\bigcup \operatorname{rng} f \cup F) \cup \{p\}$ and if $(\bigcup \operatorname{rng}(f$ **qua** $(2^L)$-valued binary relation$) \cup F) \cup \{p\}$ is not consistent, then $\$_3 = \bigcup \operatorname{rng} f \cup F$. For every objects $x$, $y$ such that $x \in c$ and $y \in c \dot{\rightarrow} 2^L$ there exists an object $z$ such that $z \in 2^L$ and $\mathcal{H}[x, y, z]$ by [8, (46)]. Consider $h$ being a function from $c \times (c \dot{\rightarrow} 2^L)$ into $2^L$ such that for every objects $x$, $y$ such that $x \in c$ and $y \in c \dot{\rightarrow} 2^L$ holds $\mathcal{H}[x, y, h(x, y)]$ from [5, Sch. 1]. Consider $f$ being a function from $c$ into $2^L$ such that $f$ is recursively expressed by $h$. Reconsider $G = \bigcup \operatorname{rng}(f$ **qua** $(2^L)$-valued binary relation$)$ as a subset of PL-WFF. Set $i_1 =$ the internal relation of $R_1$. For every $A$ and $B$ such that $\langle A, B \rangle \in R_2$ holds $f(A) \subseteq f(B)$ by [4, (1)], [2, (4), (29), (9)]. $\operatorname{rng} f$ is $\subseteq$-linear. Define $\mathcal{S}[\text{element of } R_1] \equiv f(\$_1)$ is consistent. For every element $x$ of $R_1$ such that for every element $y$ of $R_1$ such that $y \neq x$ and $\langle y, x \rangle \in i_1$ holds $\mathcal{S}[y]$ holds $\mathcal{S}[x]$ by [2, (9)], [7, (32)], [2, (1)], [15, (42)]. For every element $A$ of $R_1$, $\mathcal{S}[A]$ from [12, Sch. 3]. $F \subseteq G$ by [6, (3)]. $G$ is consistent by (65), (16), [15, (42)], (66). $G$ is maximal by [6, (3)], (17), [13, (16)], (66). $\square$

(69) If $F$ is maximal and consistent, then for every $p$, $F \vdash p$ iff $p \in F$. The theorem is a consequence of (53).

(70) If $F \models A$, then $F \vdash A$.

Proof: Consider $G$ such that $F \cup \{\neg A\} \subseteq G$ and $G$ is consistent and $G$ is maximal. Set $M = \{\operatorname{Prop} n$, where $n$ is an element of $\mathbb{N} : \operatorname{Prop} n \in G\}$.

$M \subseteq AP$. Define $\mathcal{P}[\text{element of PL-WFF}] \equiv \$_1 \in G$ iff $M \models \$_1$. $\mathcal{P}[\bot_{\mathrm{PL}}]$. For every $n$, $\mathcal{P}[\mathrm{Prop}\, n]$. For every $r$ and $s$ such that $\mathcal{P}[r]$ and $\mathcal{P}[s]$ holds $\mathcal{P}[r \Rightarrow s]$. For every $B$, $\mathcal{P}[B]$ from *PLInd*. $M \not\models A$. $\square$

(71) $A$ is a tautology if and only if $\emptyset_{\text{PL-WFF}} \vdash A$.

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