

Leibniz Series for π^1

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Summary. In this article we prove the Leibniz series for π which states that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2 \cdot n + 1}.$$

The formalization follows K. Knopp [8], [1] and [6]. *Leibniz's Series for Pi* is item #26 from the “Formalizing 100 Theorems” list maintained by Freek Wiedijk at <http://www.cs.ru.nl/F.Wiedijk/100/>.

MSC: 40G99 03B35

Keywords: π approximation; Leibniz theorem; Leibniz series

MML identifier: LEIBNIZ1, version: 8.1.05 5.39.1282

1. PRELIMINARIES

From now on i, n, m denote natural numbers, r, s denote real numbers, and A denotes a non empty, closed interval subset of \mathbb{R} .

Now we state the proposition:

(1) $\text{rng}(\text{the function } \tan \upharpoonright]-\frac{\pi}{2}, \frac{\pi}{2}[) = \mathbb{R}$.

PROOF: Set $P = \frac{\pi}{2}$. Set $I =]-P, P[$. $\mathbb{R} \subseteq \text{rng}(\text{the function } \tan \upharpoonright I)$ by [4, (50)], [20, (30)], [14, (15)], [16, (1)]. \square

¹This work has been financed by the resources of the Polish National Science Centre granted by decision no. DEC-2012/07/N/ST6/02147.

One can verify that the function arctan is total and the function arctan is differentiable.

Now we state the propositions:

- (2) (The function arctan)'(r) = $\frac{1}{1+r^2}$.
- (3) Let us consider an open subset Z of \mathbb{R} . Then
 - (i) the function arctan is differentiable on Z , and
 - (ii) for every r such that $r \in Z$ holds (the function arctan)' $_{|Z}(r) = \frac{1}{1+r^2}$.

The theorem is a consequence of (2).

Let us consider n . One can verify that \square^n is continuous.

Now we state the propositions:

- (4) (i) $\text{dom}(\frac{\square^n}{\square^0 + \square^2}) = \mathbb{R}$, and
 - (ii) $\frac{\square^n}{\square^0 + \square^2}$ is continuous, and
 - (iii) $(\frac{\square^n}{\square^0 + \square^2})(r) = \frac{r^n}{1+r^2}$.

(5) $\int_A (\frac{\square^0}{\square^0 + \square^2})(x) dx =$

(the function arctan)(sup A) – (the function arctan)(inf A).

PROOF: Set $Z_0 = \square^0$. Set $Z_2 = \square^2$. Set $f = \frac{Z_0}{Z_0 + Z_2}$. $\text{dom } f = \mathbb{R}$. f is continuous. If $r \in \mathbb{R}$, then $f(r) = \frac{1}{1+r^2}$ by [13, (4)], (4). For every element x of \mathbb{R} such that $x \in \text{dom}(\text{the function arctan})'_{|\mathbb{R}}$ holds (the function arctan)' $_{|\mathbb{R}}(x) = f(x)$. \square

(6) $\int_A ((-1)^i \cdot (\frac{\square^{2 \cdot n}}{\square^0 + \square^2}))(x) dx = (-1)^i \cdot (\frac{1}{2 \cdot n + 1}) \cdot (\text{sup } A)^{2 \cdot n + 1} - (\frac{1}{2 \cdot n + 1}) \cdot (\text{inf } A)^{2 \cdot n + 1} + \int_A ((-1)^{i+1} \cdot (\frac{\square^{2 \cdot (n+1)}}{\square^0 + \square^2}))(x) dx.$

PROOF: Set $I_1 = (-1)^i$. Set $i_1 = i + 1$. Set $n_1 = n + 1$. Set $I_2 = (-1)^{i_1}$. Set $Z_0 = \square^0$. Set $Z_2 = \square^2$. Set $Z_{2n} = \square^{2 \cdot n}$. Set $f = I_1 \cdot Z_{2n}$. Set $g = I_2 \cdot (\frac{\square^{2 \cdot n_1}}{Z_0 + Z_2})$. $\text{dom } g = \mathbb{R}$. For every element x of \mathbb{R} , $(I_1 \cdot (\frac{Z_{2n}}{Z_0 + Z_2}))(x) = (f + g)(x)$ by [13, (6)], [17, (36)], (4). $f + g = I_1 \cdot (\frac{Z_{2n}}{Z_0 + Z_2}) \cdot \frac{\square^{2 \cdot n_1}}{Z_0 + Z_2}$ is continuous. \square

(7) Suppose $A = [0, r]$ and $r \geq 0$. Then $|\int_A (\frac{\square^{2 \cdot n}}{\square^0 + \square^2})(x) dx| \leq (\frac{1}{2 \cdot n + 1}) \cdot r^{2 \cdot n + 1}.$

PROOF: Set $Z_0 = \square^0$. Set $Z_2 = \square^2$. Set $N = 2 \cdot n$. Set $Z_n = \square^N$. Set $f = \frac{Z_n}{Z_0 + Z_2}$. f is continuous and $\text{dom } f = \mathbb{R}$. Reconsider $f_1 = f \upharpoonright A$ as a function from A into \mathbb{R} . Reconsider $Z_1 = Z_n \upharpoonright A$ as a function from A into \mathbb{R} . For every r such that $r \in A$ holds $f_1(r) \leq Z_1(r)$ by [4, (49)], [17,

(36)], [18, (3)], (4). For every object x such that $x \in \mathbb{R}$ holds $f(x) = |f|(x)$ by [13, (8)], (4). \square

2. EULER TRANSFORMATION

Let a be a sequence of real numbers. The alternating series of a yielding a sequence of real numbers is defined by

(Def. 1) $it(i) = (-1)^i \cdot a(i)$.

Now we state the proposition:

- (8) Let us consider a sequence a of real numbers. Suppose a is non-negative yielding, non-increasing, and convergent and $\lim a = 0$. Then
- (i) the alternating series of a is summable, and
 - (ii) for every n , $(\sum_{\alpha=0}^{\kappa} (\text{the alternating series of } a)(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot n) \geq \sum(\text{the alternating series of } a) \geq (\sum_{\alpha=0}^{\kappa} (\text{the alternating series of } a)(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot n + 1)$.

PROOF: Set $A =$ the alternating series of a . Set $P = (\sum_{\alpha=0}^{\kappa} A(\alpha))_{\kappa \in \mathbb{N}}$. Define $\mathcal{T}[\text{natural number, object}] \equiv \$_2 = P(2 \cdot \$_1)$. Define $\mathcal{S}[\text{natural number, object}] \equiv \$_2 = P(2 \cdot \$_1 + 1)$. Consider T being a function from \mathbb{N} into \mathbb{R} such that for every element x of \mathbb{N} , $\mathcal{T}[x, T(x)]$ from [5, Sch. 3].

Consider S being a function from \mathbb{N} into \mathbb{R} such that for every element x of \mathbb{N} , $\mathcal{S}[x, S(x)]$ from [5, Sch. 3]. For every natural number n , $S(n) \leq S(n+1)$. For every natural number n , $T(n) \geq T(n+1)$. For every natural number n , $T(n) \geq S(n)$. For every natural number n , $T(n) > S(0) - 1$ by [10, (6)]. For every natural number n , $S(n) < T(0) + 1$ by [10, (8)].

Define $\mathcal{D}(\text{natural number}) = 2 \cdot \$_1 + 1$. Consider D being a function from \mathbb{N} into \mathbb{N} such that for every element x of \mathbb{N} , $\mathcal{D}(x) = D(x)$ from [5, Sch. 8]. Reconsider $D_1 = D$ as a many sorted set indexed by \mathbb{N} . For every natural number n , $D(n) < D(n+1)$ by [2, (13)]. Reconsider $a_2 = a \cdot D_1$ as a sequence of real numbers.

For every object x such that $x \in \mathbb{N}$ holds $a_2(x) = (T - S)(x)$ by [4, (12)]. For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|P(m) - \lim T| < p$ by [19, (9)]. \square

3. MAIN THEOREM

Let us consider r . The Leibniz series of r yielding a sequence of real numbers is defined by

(Def. 2) $it(n) = \frac{(-1)^n \cdot r^{2 \cdot n + 1}}{2 \cdot n + 1}$.

The Leibniz series yielding a sequence of real numbers is defined by the term

(Def. 3) the Leibniz series of 1.

Now we state the propositions:

(9) Suppose $r \in [-1, 1]$. Then

(i) |the Leibniz series of r | is non-negative yielding, non-increasing, and convergent, and

(ii) \lim |the Leibniz series of r | = 0.

PROOF: Set $r_1 =$ the Leibniz series of r . Set $A = |r_1|$. $A(n) = \frac{|r|^{2 \cdot n + 1}}{2 \cdot n + 1}$ by [15, (1)], [3, (67), (65)]. $A(n) \geq A(n + 1)$ by [3, (46)], [15, (1)], [13, (6)], [2, (13)]. Set $C = \{0\}_{n \in \mathbb{N}}$. Define \mathcal{F} (natural number) = $\frac{1}{\mathbb{S}_1 + \frac{1}{2}}$. Consider f being a sequence of real numbers such that $f(n) = \mathcal{F}(n)$ from [11, Sch. 1]. $C(n) \leq A(n) \leq f(n)$ by [11, (57)], [3, (46)], [13, (11)], [2, (11)]. \square

(10) (i) if $r \geq 0$, then the alternating series of |the Leibniz series of r | = the Leibniz series of r , and

(ii) if $r < 0$, then $(-1) \cdot$ (the alternating series of |the Leibniz series of r |) = the Leibniz series of r .

PROOF: Set $r_1 =$ the Leibniz series of r . Set $A = |r_1|$. Set $a_1 =$ the alternating series of A . $a_1(n) = (-1)^n \cdot (\frac{|r|^{2 \cdot n + 1}}{2 \cdot n + 1})$ by [15, (1)], [3, (67), (65)]. If $r \geq 0$, then $a_1 = r_1$. \square

(11) If $r \in [-1, 1]$, then the Leibniz series of r is summable. The theorem is a consequence of (9), (8), and (10).

(12) Suppose $A = [0, r]$ and $r \geq 0$. Then (the function arctan)(r) = $(\sum_{\alpha=0}^{\kappa}$ (the Leibniz series of r)(α)) $_{\kappa \in \mathbb{N}}$ (n) + $\int_{\square^0}^A ((-1)^{n+1} \cdot (\frac{\square^{2 \cdot (n+1)}}{\square^0 + \square^2}))(x)dx$.

PROOF: Set $Z_0 = \square^0$. Set $Z_2 = \square^2$. Set $r_1 =$ the Leibniz series of r . Define \mathcal{P} [natural number] \equiv (the function arctan)(r) = $(\sum_{\alpha=0}^{\kappa} r_1(\alpha))_{\kappa \in \mathbb{N}}$ (\mathbb{S}_1) + $\int_{\square^0}^A ((-1)^{\mathbb{S}_1 + 1} \cdot (\frac{\square^{2 \cdot (\mathbb{S}_1 + 1)}}{Z_0 + Z_2}))(x)dx$. $\mathcal{P}[0]$ by (5), [14, (43)], [13, (4)], [9, (21)].

If $\mathcal{P}[i]$, then $\mathcal{P}[i + 1]$ by [13, (11)], [2, (11)], (6). $\mathcal{P}[i]$ from [2, Sch. 2]. \square

(13) If $0 \leq r \leq 1$, then (the function arctan)(r) = \sum (the Leibniz series of r).

PROOF: Set $r_1 =$ the Leibniz series of r . Set $P = (\sum_{\alpha=0}^{\kappa} r_1(\alpha))_{\kappa \in \mathbb{N}}$. Set $A =$ (the function \arctan)(r). Define \mathcal{I} (natural number) $= \frac{\square^{2 \cdot s_1}}{\square^0 + \square^2}$. P is convergent. For every s such that $0 < s$ there exists n such that for every m such that $n \leq m$ holds $|P(m) - A| < s$ by [12, (3)], (4), [7, (11), (10)].
□

(14) LEIBNIZ SERIES FOR π :

$$\frac{\pi}{4} = \sum(\text{the Leibniz series}).$$

(15) $(\sum_{\alpha=0}^{\kappa}(\text{the Leibniz series})(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot n + 1) \leq \sum(\text{the Leibniz series}) \leq (\sum_{\alpha=0}^{\kappa}(\text{the Leibniz series})(\alpha))_{\kappa \in \mathbb{N}}(2 \cdot n)$. The theorem is a consequence of (9), (10), and (8).

(16) (i) $(\sum_{\alpha=0}^{\kappa}(\text{the Leibniz series})(\alpha))_{\kappa \in \mathbb{N}}(1) = \frac{2}{3}$, and

(ii) if n is odd, then $(\sum_{\alpha=0}^{\kappa}(\text{the Leibniz series})(\alpha))_{\kappa \in \mathbb{N}}(n+2) = (\sum_{\alpha=0}^{\kappa}(\text{the Leibniz series})(\alpha))_{\kappa \in \mathbb{N}}(n) + \frac{2}{4 \cdot n^2 + 16 \cdot n + 15}$.

(17) π APPROXIMATION:

$$\frac{313}{100} < \pi < \frac{315}{100}. \text{ The theorem is a consequence of (16), (14), and (15).}$$

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Received October 18, 2016
