Quasi-uniform Space

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Summary. In this article, using mostly Pervin [9], Kunzi [6], [8], [7], Williams [11] and Bourbaki [3] works, we formalize in Mizar [2] the notions of quasi-uniform space, semi-uniform space and locally uniform space.

We define the topology induced by a quasi-uniform space. Finally we formalize from the sets of the form \(((X \setminus \Omega) \times X) \cup (X \times \Omega)\), the Csaszar-Pervin quasi-uniform space induced by a topological space.

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1. Preliminaries

From now on \(X\) denotes a set, \(A\) denotes a subset of \(X\), and \(R, S\) denote binary relations on \(X\).

Now we state the propositions:

(1) \(((X \setminus A) \times X) \cup X \times A \subseteq X \times X\).

(2) \(((X \setminus A) \times X) \cup X \times A = A \times A \cup (X \setminus A) \times X\).

Proof: \(((X \setminus A) \times X) \cup X \times A \subseteq A \times A \cup (X \setminus A) \times X\) by (1), [4] (87). \(\square\)

(3) \(R \cdot S = \{\langle x, y \rangle\}\), where \(x, y\) are elements of \(X\): there exists an element \(z\) of \(X\) such that \(\langle x, z \rangle \in R\) and \(\langle z, y \rangle \in S\).

Proof: \(R \cdot S \subseteq \{\langle x, y \rangle\}\), where \(x, y\) are elements of \(X\): there exists an element \(z\) of \(X\) such that \(\langle x, z \rangle \in R\) and \(\langle z, y \rangle \in S\) by [4] (87). \(\langle x, y \rangle\), where \(x, y\) are elements of \(X\): there exists an element \(z\) of \(X\) such that \(\langle x, z \rangle \in R\) and \(\langle z, y \rangle \in S\) \(\subseteq R \cdot S\). \(\square\)
Let $X$ be a set and $\mathcal{B}$ be a family of subsets of $X$. One can check that $[\mathcal{B}]$ is non empty.

Let $\mathcal{B}$ be a family of subsets of $X \times X$. Note that every element of $\mathcal{B}$ is relation-like.

Let $B$ be an element of $\mathcal{B}$. We introduce the notation $B[\sim]$ as a synonym of $B^\sim$.

Let us observe that the functor $B[\sim]$ yields a subset of $X \times X$. Let $B_1, B_2$ be elements of $\mathcal{B}$. We introduce the notation $B_1 \otimes B_2$ as a synonym of $B_1 \cdot B_2$.

One can verify that the functor $B_1 \otimes B_2$ yields a subset of $X \times X$. Now we state the propositions:

(4) Let us consider a set $X$, and a family $G$ of subsets of $X$. If $G$ is upper, then FinMeetCl($G$) is upper.

(5) If $R$ is symmetric in $X$, then $R^\sim$ is symmetric in $X$.

2. Uniform Space Structure

We consider uniform space structures which extend 1-sorted structures and are systems

$$\langle \text{a carrier, entourages}\rangle$$

where the carrier is a set, the entourages constitute a family of subsets of $(\text{the carrier}) \times (\text{the carrier})$.

Let $U$ be a uniform space structure. We say that $U$ is void if and only if

(Def. 1) the entourages of $U$ is empty.

Let $X$ be a set. The functor UniformSpace($X$) yielding a strict uniform space structure is defined by the term

(Def. 2) $\langle X, \emptyset_2 \times x \rangle$.

The functors: the trivial uniform space and the non empty trivial uniform space yielding strict uniform space structures are defined by terms

(Def. 3) $\langle \emptyset, 2^{\emptyset \times \emptyset} \rangle$,

(Def. 4) there exists a family $S_1$ of subsets of $\{\emptyset\} \times \{\emptyset\}$ such that $S_1 = \{\emptyset\} \times \{\emptyset\}$ and the non empty trivial uniform space = $\langle \{\emptyset\}, S_1 \rangle$, respectively. Let $X$ be an empty set. One can verify that UniformSpace($X$) is empty.

Let $X$ be a non empty set. One can check that UniformSpace($X$) is non empty.

Let $X$ be a set. Note that UniformSpace($X$) is void and the trivial uniform space is empty and non void and the non empty trivial uniform space is non empty and non void and there exists a uniform space structure which is empty,
strict, and void and there exists a uniform space structure which is empty, strict, and non void and there exists a uniform space structure which is non empty, strict, and void and there exists a uniform space structure which is non empty, strict, and non void.

Let $X$ be a set and $S_1$ be a family of subsets of $X \times X$. The functor $S_1[\sim]$ yielding a family of subsets of $X \times X$ is defined by the term

(Def. 5) the set of all $S[\sim]$ where $S$ is an element of $S_1$.

Let $U$ be a uniform space structure. The functor $U[\sim]$ yielding a uniform space structure is defined by the term

(Def. 6) $\langle \text{the carrier of } U, (\text{the entourages of } U)[\sim] \rangle$.

Let $U$ be a non empty uniform space structure. One can verify that $U[\sim]$ is non empty.

3. Axioms

Let $U$ be a uniform space structure. We say that $U$ is upper if and only if

(Def. 7) the entourages of $U$ is upper.

We say that $U$ is $\cap$-closed if and only if

(Def. 8) the entourages of $U$ is $\cap$-closed.

We say that $U$ satisfies axiom U1 if and only if

(Def. 9) for every element $S$ of the entourages of $U$, $\text{id}_\alpha \subseteq S$, where $\alpha$ is the carrier of $U$.

We say that $U$ satisfies axiom U2 if and only if

(Def. 10) for every element $S$ of the entourages of $U$, $S[\sim] \in$ the entourages of $U$.

We say that $U$ satisfies axiom U3 if and only if

(Def. 11) for every element $S$ of the entourages of $U$, there exists an element $W$ of the entourages of $U$ such that $W \otimes W \subseteq S$.

Let us consider a non void uniform space structure $U$. Now we state the propositions:

(6) $U$ satisfies axiom U1 if and only if for every element $S$ of the entourages of $U$, there exists a binary relation $R$ on the carrier of $U$ such that $R = S$ and $R$ is reflexive in the carrier of $U$.

(7) $U$ satisfies axiom U1 if and only if for every element $S$ of the entourages of $U$, there exists a total, reflexive binary relation $R$ on the carrier of $U$ such that $R = S$. The theorem is a consequence of (6).

Note that every uniform space structure which is void does not satisfy also axiom U2.

Now we state the proposition:
(8) Let us consider a uniform space structure $U$. Suppose $U$ satisfies axiom U2. Let us consider an element $S$ of the entourages of $U$, and elements $x$, $y$ of $U$. Suppose $\langle x, y \rangle \in S$. Then $\langle y, x \rangle \in \bigcup$ (the entourages of $U$).

Let us consider a non void uniform space structure $U$. Now we state the propositions:

(9) Suppose for every element $S$ of the entourages of $U$, there exists a binary relation $R$ on the carrier of $U$ such that $S = R$ and $R$ is symmetric in the carrier of $U$. Then $U$ satisfies axiom U2. The theorem is a consequence of (5).

(10) Suppose for every element $S$ of the entourages of $U$, there exists a binary relation $R$ on the carrier of $U$ such that $S = R$ and $R$ is symmetric. Then $U$ satisfies axiom U2. The theorem is a consequence of (9).

(11) If for every element $S$ of the entourages of $U$, there exists a tolerance $R$ of the carrier of $U$ such that $S = R$, then $U$ satisfies axiom U1 and axiom U2. The theorem is a consequence of (7) and (10).

Let $X$ be an empty set. Observe that UniformSpace($X$) is upper and $\cap$-closed and satisfies axiom U1 and axiom U3 and does not satisfy axiom U2 and UniformSpace($\emptyset$) is upper and $\cap$-closed and does not satisfy axiom U2 and the trivial uniform space is upper and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3 and the non empty trivial uniform space is upper and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3.

There exists a uniform space structure which is strict, empty, non void, upper, and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3 and every strict uniform space structure which is empty satisfies also axiom U1 and there exists a uniform space structure which is strict, non empty, non void, upper, and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3.

Let $S_4$ be a non empty uniform space structure satisfying axiom U1, $x$ be an element of $S_4$, and $V$ be an element of the entourages of $S_4$. The functor $\text{Nbh}(V, x)$ yielding a non empty subset of $S_4$ is defined by the term

(Def. 12) $\{y, \text{where } y \text{ is an element of } S_4 : \langle x, y \rangle \in V\}$.

Now we state the proposition:

(12) Let us consider a non empty uniform space structure $U$ satisfying axiom U1, an element $x$ of the carrier of $U$, and an element $V$ of the entourages of $U$. Then $x \in \text{Nbh}(V, x)$.

Let $U$ be a $\cap$-closed uniform space structure and $V_1$, $V_2$ be elements of the entourages of $U$. One can check that the functor $V_1 \cap V_2$ yields an element of the entourages of $U$. Now we state the proposition:

(13) Let us consider a non empty, $\cap$-closed uniform space structure $U$ satisfying axiom U1, an element $x$ of $U$, and elements $V$, $W$ of the entourages
of $U$. Then $\text{Nbh}(V, x) \cap \text{Nbh}(W, x) = \text{Nbh}(V \cap W, x)$.

Let $U$ be a non empty uniform space structure satisfying axiom U1. Let us observe that the entourages of $U$ has non empty elements and the entourages of $U$ is non empty.

Let $x$ be a point of $U$. The functor Neighborhood $x$ yielding a family of subsets of $U$ is defined by the term

(Def. 13) the set of all $\text{Nbh}(V, x)$ where $V$ is an element of the entourages of $U$.

Let us note that Neighborhood $x$ is non empty.

Now we state the proposition:

(14) Let us consider a non empty uniform space structure $S_4$ satisfying axiom U1, an element $x$ of the carrier of $S_4$, and an element $V$ of the entourages of $S_4$. Then

(i) $\text{Nbh}(V, x) = V^\circ \{x\}$, and
(ii) $\text{Nbh}(V, x) = \text{rng}(V \upharpoonright \{x\})$, and
(iii) $\text{Nbh}(V, x) = V^\circ x$, and
(iv) $\text{Nbh}(V, x) = [x]_V$, and
(v) $\text{Nbh}(V, x) = \text{neighbourhood}(x, V)$.

Proof: $\text{Nbh}(V, x) = V^\circ \{x\}$ by [4, (87)]. $\square$

Let $U$ be a non empty uniform space structure satisfying axiom U1. The functor Neighborhood $U$ yielding a function from the carrier of $U$ into $2^{(\text{the carrier of } U)}$ is defined by

(Def. 14) for every element $x$ of $U$, $it(x) = \text{Neighborhood } x$.

We say that $U$ is topological if and only if

(Def. 15) $\langle \text{the carrier of } U, \text{Neighborhood } U \rangle$ is a topology from neighbourhoods.

4. Quasi-Uniform Space

A quasi-uniform space is an upper, $\cap$-closed uniform space structure satisfying axiom U1 and axiom U3. In the sequel $Q$ denotes a quasi-uniform space.

Now we state the propositions:

(15) If the entourages of $Q$ is empty, then the entourages of $Q \sim = \{\emptyset\}$.
(16) Suppose the entourages of $Q \sim = \{\emptyset\}$ and the entourages of $Q \sim$ is upper. Then the carrier of $Q$ is empty.

Let $Q$ be a non void quasi-uniform space. One can check that $Q \sim$ is upper and $\cap$-closed and satisfies axiom U1 and axiom U3.

Let $X$ be a set and $\mathcal{B}$ be a family of subsets of $X \times X$. We say that $\mathcal{B}$ satisfies axiom UP1 if and only if
(Def. 16) for every element $B$ of $\mathcal{B}$, $\text{id}_X \subseteq B$.

We say that $\mathcal{B}$ satisfies axiom UP3 if and only if

(Def. 17) for every element $B_1$ of $\mathcal{B}$, there exists an element $B_2$ of $\mathcal{B}$ such that $B_2 \otimes B_2 \subseteq B_1$.

Now we state the propositions:

(17) Let us consider a non empty set $X$, and an empty family $\mathcal{B}$ of subsets of $X \times X$. Then $\mathcal{B}$ does not satisfy axiom UP1.

(18) Let us consider a set $X$, and a family $\mathcal{B}$ of subsets of $X \times X$. Suppose $\mathcal{B}$ is quasi-basis and satisfies axiom UP1 and axiom UP3. Then $\langle X, [\mathcal{B}] \rangle$ is a quasi-uniform space.

5. Semi-Uniform Space

A semi-uniform space is an upper, $\cap$-closed uniform space structure satisfying axiom U1 and axiom U2. From now on $S_4$ denotes a semi-uniform space.

Let us observe that every semi-uniform space is non void.

Now we state the proposition:

(19) If $S_4$ is empty, then $\emptyset \in$ the entourages of $S_4$.

Let $S_4$ be an empty semi-uniform space. One can verify that the entourages of $S_4$ has the empty element.

6. Locally Uniform Space

Let $S_4$ be a non empty semi-uniform space. We say that $S_4$ satisfies axiom UL if and only if

(Def. 18) for every element $S$ of the entourages of $S_4$ and for every element $x$ of $S_4$, there exists an element $W$ of the entourages of $S_4$ such that $\{y, \text{ where } y \text{ is an element of } S_4 : \langle x, y \rangle \in W \otimes W \} \subseteq \text{Nbh}(S, x)$.

One can verify that every non empty semi-uniform space which satisfies axiom U3 satisfies also axiom UL and there exists a non empty semi-uniform space which satisfies axiom UL.

A locally uniform space is a non empty semi-uniform space satisfying axiom UL. Now we state the propositions:

(20) Let us consider a non empty, upper uniform space structure $U$ satisfying axiom U1, and an element $x$ of the carrier of $U$. Then Neighborhood $x$ is upper.
(21) Let us consider a non empty uniform space structure $U$ satisfying axiom $U1$, an element $x$ of $U$, and an element $V$ of the entourages of $U$. Then $x \in \text{Nbh}(V, x)$.

(22) Let us consider a non empty, $\cap$-closed uniform space structure $U$ satisfying axiom $U1$, and an element $x$ of $U$. Then Neighborhood $x$ is $\cap$-closed. The theorem is a consequence of (13).

(23) Let us consider a non empty, upper, $\cap$-closed uniform space structure $U$ satisfying axiom $U1$, and an element $x$ of $U$. Then Neighborhood $x$ is a filter of the carrier of $U$. The theorem is a consequence of (22) and (20).

Let us observe that every locally uniform space is topological.

7. Topological Space Induced by a Uniform Space Structure

Let $U$ be a topological, non empty uniform space structure satisfying axiom $U1$. The FMT induced by $U$ yielding a non empty, strict topology from neighbourhoods is defined by the term

\[(\text{Def. 19}) \langle \text{the carrier of } U, \text{Neighborhood } U \rangle.\]

The topological space induced by $U$ yielding a topological space is defined by the term

\[(\text{Def. 20}) \text{FMT2TopSpace}(\text{the FMT induced by } U).\]

8. The Quasi-Uniform Pervin Space Induced by a Topological Space

Let $X$ be a set and $A$ be a subset of $X$. The functor $q\text{Block}(A)$ yielding a subset of $X \times X$ is defined by the term

\[(\text{Def. 21}) (X \setminus A) \times X \cup X \times A.\]

Now we state the proposition:

(24) (i) $id_X \subseteq q\text{Block}(A)$, and
(ii) $q\text{Block}(A) \cdot q\text{Block}(A) \subseteq q\text{Block}(A)$.

\[\text{Proof: } id_X \subseteq q\text{Block}(A) \text{ by } [11] (96). \square\]

Let $T$ be a topological space. The functor $q\text{Blocks}(T)$ yielding a family of subsets of (the carrier of $T) \times (the carrier of T)$ is defined by the term

\[(\text{Def. 22}) \text{the set of all } q\text{Block}(O) \text{ where } O \text{ is an element of the topology of } T.\]

Let $T$ be a non empty topological space. One can check that $q\text{Blocks}(T)$ is non empty.

Let $T$ be a topological space. The functor $FMCq\text{Blocks}(T)$ yielding a family of subsets of (the carrier of $T) \times (the carrier of T)$ is defined by the term
Let $X$ be a set. One can check that every non empty family of subsets of $X \times X$ which is $\cap$-closed is also quasi-basis.

In the sequel $T$ denotes a topological space.

Let us consider $T$. One can check that $FMCqBlocks(T)$ is $\cap$-closed and $FMCqBlocks(T)$ is quasi-basis and $FMCqBlocks(T)$ satisfies axiom UP1 and $FMCqBlocks(T)$ satisfies axiom UP3.

Let $T$ be a topological space. The Pervin quasi-uniformity of $T$ yielding a strict quasi-uniform space is defined by the term

(Def. 24) \( \langle \text{the carrier of } T, [FMCqBlocks(T)] \rangle \).

Now we state the propositions:

(25) Let us consider a non empty topological space $T$, and an element $V$ of the entourages of the Pervin quasi-uniformity of $T$. Then there exists an element $b$ of $\text{FinMeetCl}(qBlocks(T))$ such that $b \subseteq V$.

(26) Let us consider a non empty topological space $T$, and a subset $V$ of $(\text{the carrier of } T) \times (\text{the carrier of } T)$. Suppose there exists an element $b$ of $\text{FinMeetCl}(qBlocks(T))$ such that $b \subseteq V$. Then $V$ is an element of the entourages of the Pervin quasi-uniformity of $T$.

(27) $qBlocks(T) \subseteq \text{the entourages of the Pervin quasi-uniformity of } T$.

Let us consider a non void quasi-uniform space $Q$. Now we state the propositions:

(28) $(\text{the carrier of } Q) \times (\text{the carrier of } Q) \in \text{the entourages of } Q$.

(29) Suppose the carrier of $T =$ the carrier of $Q$ and $qBlocks(T) \subseteq \text{the entourages of } Q$. Then the entourages of the Pervin quasi-uniformity of $T \subseteq \text{the entourages of } Q$.

Proof: The entourages of the Pervin quasi-uniformity of $T \subseteq \text{the entourages of } Q$ by (28), \( \prod (1) \)\]. □

Let $T$ be a non empty topological space. One can check that the Pervin quasi-uniformity of $T$ is non empty and the Pervin quasi-uniformity of $T$ is topological.

Now we state the propositions:

(30) Let us consider a non empty topological space $T$, an element $x$ of $qBlocks(T)$, and an element $y$ of the Pervin quasi-uniformity of $T$. Then \( \{ z, \text{where } z \text{ is an element of the Pervin quasi-uniformity of } T : \langle y, z \rangle \in x \} \in \text{the topology of } T \).

(31) Let us consider a non empty topological space $T$, an element $x$ of the carrier of the Pervin quasi-uniformity of $T$, and an element $b$ of $\text{FinMeetCl}(qBlocks(T))$. Then \( \{ y, \text{where } y \text{ is an element of } T : \langle x, y \rangle \in b \} \in \text{the to-} \)}
polity of $T$. The theorem is a consequence of (30).

(32) Let us consider a non empty, strict quasi-uniform space $U$, a non empty, strict formal topological space $F$, and an element $x$ of $F$. Suppose $F = \langle \text{the carrier of } U, \text{Neighborhood } U \rangle$. Then there exists an element $y$ of $U$ such that

(i) $x = y$, and

(ii) $U_F(x) = \text{Neighborhood } y$.

(33) Let us consider a non empty topological space $T$. Then the open set family of the FMT induced by the Pervin quasi-uniformity of $T = \text{the topology of } T$.

**Proof:** The open set family of the FMT induced by the Pervin quasi-uniformity of $T \subseteq \text{the topology of } T$ by (32), [5, (18)], (31), [12, (25)]. The topology of $T \subseteq \text{the open set family of the FMT induced by the Pervin quasi-uniformity of } T$ by (32), [10, (4)], [5, (18)], [4, (87)]. □

(34) Let us consider a non empty, strict topological space $T$. Then the topological space induced by the Pervin quasi-uniformity of $T = T$. The theorem is a consequence of (33).

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**References**


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