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Quasi-uniform Space

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Summary. In this article, using mostly Pervin [9], Kunzi [6], [8], [7], Williams [11] and Bourbaki [3] works, we formalize in Mizar [2] the notions of quasi-uniform space, semi-uniform space and locally uniform space.

We define the topology induced by a quasi-uniform space. Finally we formalize from the sets of the form $((X \setminus \Omega) \times X) \cup (X \times \Omega)$, the Csaszar-Pervin quasi-uniform space induced by a topological space.

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1. Preliminaries

From now on X denotes a set, A denotes a subset of X, and R, S denote binary relations on X.

Now we state the propositions:

(1) $(X \setminus A) \times X \cup X \times A \subseteq X \times X$.

(2) $(X \setminus A) \times X \cup X \times A = A \times A \cup (X \setminus A) \times X$.

PROOF: $(X \setminus A) \times X \cup X \times A \subseteq A \times A \cup (X \setminus A) \times X$ by (1), [4, (87)]. $\square$

(3) $R \cdot S = \{\langle x, y \rangle$, where x, y are elements of X : there exists an element z of X such that $\langle x, z \rangle \in R$ and $\langle z, y \rangle \in S\}$.

PROOF: $R \cdot S \subseteq \{\langle x, y \rangle$, where x, y are elements of X : there exists an element z of X such that $\langle x, z \rangle \in R$ and $\langle z, y \rangle \in S\}$ by [4, (87)]. $\{\langle x, y \rangle$, where x, y are elements of X : there exists an element z of X such that $\langle x, z \rangle \in R$ and $\langle z, y \rangle \in S\} \subseteq R \cdot S$. $\square$

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Let X be a set and $\mathcal{B}$ be a family of subsets of X. One can check that $[\mathcal{B}]$ is non empty.

Let $\mathcal{B}$ be a family of subsets of $X \times X$. Note that every element of $\mathcal{B}$ is relation-like.

Let B be an element of $\mathcal{B}$. We introduce the notation $B\,[\sim]$ as a synonym of $B^{\smile}$.

Let us observe that the functor $B\,[\sim]$ yields a subset of $X \times X$. Let B_1, B_2 be elements of $\mathcal{B}$. We introduce the notation $B_1 \otimes B_2$ as a synonym of $B_1 \cdot B_2$.

One can verify that the functor $B_1 \otimes B_2$ yields a subset of $X \times X$. Now we state the propositions:

(4) Let us consider a set X, and a family G of subsets of X. If G is upper, then $\mathrm{FinMeetCl}(G)$ is upper.

(5) If R is symmetric in X, then $R^{\smile}$ is symmetric in X.

2. Uniform Space Structure

We consider uniform space structures which extend 1-sorted structures and are systems

$$\langle \text{a carrier}, \text{entourages} \rangle$$

where the carrier is a set, the entourages constitute a family of subsets of (the carrier) $\times$ (the carrier).

Let U be a uniform space structure. We say that U is void if and only if

(Def. 1) the entourages of U is empty.

Let X be a set. The functor $\mathrm{UniformSpace}(X)$ yielding a strict uniform space structure is defined by the term

(Def. 2) $\langle X, \emptyset_{2^{X \times X}} \rangle$.

The functors: the trivial uniform space and the non empty trivial uniform space yielding strict uniform space structures are defined by terms

(Def. 3) $\langle \emptyset, 2_*^{\emptyset \times \emptyset} \rangle$,

(Def. 4) there exists a family S_1 of subsets of $\{\emptyset\} \times \{\emptyset\}$ such that $S_1 = \{\{\emptyset\} \times \{\emptyset\}\}$ and the non empty trivial uniform space $= \langle \{\emptyset\}, S_1 \rangle$,

respectively. Let X be an empty set. One can verify that $\mathrm{UniformSpace}(X)$ is empty.

Let X be a non empty set. One can check that $\mathrm{UniformSpace}(X)$ is non empty.

Let X be a set. Note that $\mathrm{UniformSpace}(X)$ is void and the trivial uniform space is empty and non void and the non empty trivial uniform space is non empty and non void and there exists a uniform space structure which is empty, strict, and void and there exists a uniform space structure which is empty, strict, and non void and there exists a uniform space structure which is non empty, strict, and void and there exists a uniform space structure which is non empty, strict, and non void.

Let X be a set and S_1 be a family of subsets of $X \times X$. The functor $S_1[\sim]$ yielding a family of subsets of $X \times X$ is defined by the term

(Def. 5) the set of all $S[\sim]$ where S is an element of S_1.

Let U be a uniform space structure. The functor $U[\sim]$ yielding a uniform space structure is defined by the term

(Def. 6) $\langle$the carrier of U, (the entourages of U) $[\sim]\rangle$.

Let U be a non empty uniform space structure. One can verify that $U[\sim]$ is non empty.

3. Axioms

Let U be a uniform space structure. We say that U is upper if and only if

(Def. 7) the entourages of U is upper.

We say that U is $\cap$-closed if and only if

(Def. 8) the entourages of U is $\cap$-closed.

We say that U satisfies axiom U1 if and only if

(Def. 9) for every element S of the entourages of U, $\mathrm{id}_\alpha \subseteq S$, where α is the carrier of U.

We say that U satisfies axiom U2 if and only if

(Def. 10) for every element S of the entourages of U, $S[\sim] \in$ the entourages of U.

We say that U satisfies axiom U3 if and only if

(Def. 11) for every element S of the entourages of U, there exists an element W of the entourages of U such that $W \otimes W \subseteq S$.

Let us consider a non void uniform space structure U. Now we state the propositions:

(6) U satisfies axiom U1 if and only if for every element S of the entourages of U, there exists a binary relation R on the carrier of U such that $R = S$ and R is reflexive in the carrier of U.

(7) U satisfies axiom U1 if and only if for every element S of the entourages of U, there exists a total, reflexive binary relation R on the carrier of U such that $R = S$. The theorem is a consequence of (6).

Note that every uniform space structure which is void does not satisfy also axiom U2.

Now we state the proposition:

(8) Let us consider a uniform space structure U. Suppose U satisfies axiom U2. Let us consider an element S of the entourages of U, and elements x, y of U. Suppose $\langle x, y\rangle \in S$. Then $\langle y, x\rangle \in \bigcup$(the entourages of U).

Let us consider a non void uniform space structure U. Now we state the propositions:

(9) Suppose for every element S of the entourages of U, there exists a binary relation R on the carrier of U such that $S = R$ and R is symmetric in the carrier of U. Then U satisfies axiom U2. The theorem is a consequence of (5).

(10) Suppose for every element S of the entourages of U, there exists a binary relation R on the carrier of U such that $S = R$ and R is symmetric. Then U satisfies axiom U2. The theorem is a consequence of (9).

(11) If for every element S of the entourages of U, there exists a tolerance R of the carrier of U such that $S = R$, then U satisfies axiom U1 and axiom U2. The theorem is a consequence of (7) and (10).

Let X be an empty set. Observe that UniformSpace(X) is upper and $\cap$-closed and satisfies axiom U1 and axiom U3 and does not satisfy axiom U2 and UniformSpace($\{\emptyset\}$) is upper and $\cap$-closed and does not satisfy axiom U2 and the trivial uniform space is upper and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3 and the non empty trivial uniform space is upper and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3.

There exists a uniform space structure which is strict, empty, non void, upper, and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3 and every strict uniform space structure which is empty satisfies also axiom U1 and there exists a uniform space structure which is strict, non empty, non void, upper, and $\cap$-closed and satisfies axiom U1, axiom U2, and axiom U3.

Let S_4 be a non empty uniform space structure satisfying axiom U1, x be an element of S_4, and V be an element of the entourages of S_4. The functor $\mathrm{Nbh}(V, x)$ yielding a non empty subset of S_4 is defined by the term

(Def. 12) $\{y$, where y is an element of $S_4 : \langle x, y\rangle \in V\}$.

Now we state the proposition:

(12) Let us consider a non empty uniform space structure U satisfying axiom U1, an element x of the carrier of U, and an element V of the entourages of U. Then $x \in \mathrm{Nbh}(V, x)$.

Let U be a $\cap$-closed uniform space structure and V_1, V_2 be elements of the entourages of U. One can check that the functor $V_1 \cap V_2$ yields an element of the entourages of U. Now we state the proposition:

(13) Let us consider a non empty, $\cap$-closed uniform space structure U satisfying axiom U1, an element x of U, and elements V, W of the entourages

of U. Then $\mathrm{Nbh}(V,x) \cap \mathrm{Nbh}(W,x) = \mathrm{Nbh}(V \cap W,x)$.

Let U be a non empty uniform space structure satisfying axiom U1. Let us observe that the entourages of U has non empty elements and the entourages of U is non empty.

Let x be a point of U. The functor Neighborhood x yielding a family of subsets of U is defined by the term

(Def. 13) the set of all $\mathrm{Nbh}(V,x)$ where V is an element of the entourages of U.

Let us note that Neighborhood x is non empty.

Now we state the proposition:

(14) Let us consider a non empty uniform space structure S_4 satisfying axiom U1, an element x of the carrier of S_4, and an element V of the entourages of S_4. Then

(i) $\mathrm{Nbh}(V,x) = V^{\circ}\{x\}$, and

(ii) $\mathrm{Nbh}(V,x) = \mathrm{rng}(V \restriction \{x\})$, and

(iii) $\mathrm{Nbh}(V,x) = V^{\circ}x$, and

(iv) $\mathrm{Nbh}(V,x) = [x]_V$, and

(v) $\mathrm{Nbh}(V,x) = \mathrm{neighbourhood}(x,V)$.

Proof: $\mathrm{Nbh}(V,x) = V^{\circ}\{x\}$ by [4, (87)]. □

Let U be a non empty uniform space structure satisfying axiom U1. The functor Neighborhood U yielding a function from the carrier of U into $2^{2^{(\text{the carrier of } U)}}$ is defined by

(Def. 14) for every element x of U, $it(x) =$ Neighborhood x.

We say that U is topological if and only if

(Def. 15) $\langle$the carrier of U, Neighborhood $U\rangle$ is a topology from neighbourhoods.

4. Quasi-Uniform Space

A quasi-uniform space is an upper, $\cap$-closed uniform space structure satisfying axiom U1 and axiom U3. In the sequel Q denotes a quasi-uniform space.

Now we state the propositions:

(15) If the entourages of Q is empty, then the entourages of $Q\,[\sim] = \{\emptyset\}$.

(16) Suppose the entourages of $Q\,[\sim] = \{\emptyset\}$ and the entourages of $Q\,[\sim]$ is upper. Then the carrier of Q is empty.

Let Q be a non void quasi-uniform space. One can check that $Q\,[\sim]$ is upper and $\cap$-closed and satisfies axiom U1 and axiom U3.

Let X be a set and $\mathcal{B}$ be a family of subsets of $X \times X$. We say that $\mathcal{B}$ satisfies axiom UP1 if and only if

(Def. 16) for every element B of $\mathcal{B}$, $\mathrm{id}_X \subseteq B$.

We say that $\mathcal{B}$ satisfies axiom UP3 if and only if

(Def. 17) for every element B_1 of $\mathcal{B}$, there exists an element B_2 of $\mathcal{B}$ such that $B_2 \otimes B_2 \subseteq B_1$.

Now we state the propositions:

(17) Let us consider a non empty set X, and an empty family $\mathcal{B}$ of subsets of $X \times X$. Then $\mathcal{B}$ does not satisfy axiom UP1.

(18) Let us consider a set X, and a family $\mathcal{B}$ of subsets of $X \times X$. Suppose $\mathcal{B}$ is quasi-basis and satisfies axiom UP1 and axiom UP3. Then $\langle X, [\mathcal{B}] \rangle$ is a quasi-uniform space.

5. Semi-Uniform Space

A semi-uniform space is an upper, $\cap$-closed uniform space structure satisfying axiom U1 and axiom U2. From now on S_4 denotes a semi-uniform space.

Let us observe that every semi-uniform space is non void.

Now we state the proposition:

(19) If S_4 is empty, then $\emptyset \in$ the entourages of S_4.

Let S_4 be an empty semi-uniform space. One can verify that the entourages of S_4 has the empty element.

6. Locally Uniform Space

Let S_4 be a non empty semi-uniform space. We say that S_4 satisfies axiom UL if and only if

(Def. 18) for every element S of the entourages of S_4 and for every element x of S_4, there exists an element W of the entourages of S_4 such that $\{y$, where y is an element of $S_4 : \langle x, y \rangle \in W \otimes W\} \subseteq \mathrm{Nbh}(S, x)$.

One can verify that every non empty semi-uniform space which satisfies axiom U3 satisfies also axiom UL and there exists a non empty semi-uniform space which satisfies axiom UL.

A locally uniform space is a non empty semi-uniform space satisfying axiom UL. Now we state the propositions:

(20) Let us consider a non empty, upper uniform space structure U satisfying axiom U1, and an element x of the carrier of U. Then Neighborhood x is upper.

(21) Let us consider a non empty uniform space structure U satisfying axiom U1, an element x of U, and an element V of the entourages of U. Then $x \in \mathrm{Nbh}(V, x)$.

(22) Let us consider a non empty, $\cap$-closed uniform space structure U satisfying axiom U1, and an element x of U. Then Neighborhood x is $\cap$-closed. The theorem is a consequence of (13).

(23) Let us consider a non empty, upper, $\cap$-closed uniform space structure U satisfying axiom U1, and an element x of U. Then Neighborhood x is a filter of the carrier of U. The theorem is a consequence of (22) and (20).

Let us observe that every locally uniform space is topological.

7. Topological Space Induced by a Uniform Space Structure

Let U be a topological, non empty uniform space structure satisfying axiom U1. The FMT induced by U yielding a non empty, strict topology from neighbourhoods is defined by the term

(Def. 19) $\langle$the carrier of U, Neighborhood $U\rangle$.

The topological space induced by U yielding a topological space is defined by the term

(Def. 20) FMT2TopSpace(the FMT induced by U).

8. The Quasi-Uniform Pervin Space Induced by a Topological Space

Let X be a set and A be a subset of X. The functor qBlock(A) yielding a subset of $X \times X$ is defined by the term

(Def. 21) $(X \setminus A) \times X \cup X \times A$.

Now we state the proposition:

(24) (i) $\mathrm{id}_X \subseteq \mathrm{qBlock}(A)$, and

(ii) $\mathrm{qBlock}(A) \cdot \mathrm{qBlock}(A) \subseteq \mathrm{qBlock}(A)$.

Proof: $\mathrm{id}_X \subseteq \mathrm{qBlock}(A)$ by [4, (96)]. □

Let T be a topological space. The functor qBlocks(T) yielding a family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 22) the set of all qBlock(O) where O is an element of the topology of T.

Let T be a non empty topological space. One can check that qBlocks(T) is non empty.

Let T be a topological space. The functor FMCqBlocks(T) yielding a family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 23) FinMeetCl(qBlocks(T)).

Let X be a set. One can check that every non empty family of subsets of $X \times X$ which is $\cap$-closed is also quasi-basis.

In the sequel T denotes a topological space.

Let us consider T. One can check that FMCqBlocks(T) is $\cap$-closed and FMCqBlocks(T) is quasi-basis and FMCqBlocks(T) satisfies axiom UP1 and FMCqBlocks(T) satisfies axiom UP3.

Let T be a topological space. The Pervin quasi-uniformity of T yielding a strict quasi-uniform space is defined by the term

(Def. 24) $\langle$the carrier of T, [FMCqBlocks(T)]$\rangle$.

Now we state the propositions:

(25) Let us consider a non empty topological space T, and an element V of the entourages of the Pervin quasi-uniformity of T. Then there exists an element b of FinMeetCl(qBlocks(T)) such that $b \subseteq V$.

(26) Let us consider a non empty topological space T, and a subset V of (the carrier of T) $\times$ (the carrier of T). Suppose there exists an element b of FinMeetCl(qBlocks(T)) such that $b \subseteq V$. Then V is an element of the entourages of the Pervin quasi-uniformity of T.

(27) qBlocks(T) $\subseteq$ the entourages of the Pervin quasi-uniformity of T.

Let us consider a non void quasi-uniform space Q. Now we state the propositions:

(28) (The carrier of Q) $\times$ (the carrier of Q) $\in$ the entourages of Q.

(29) Suppose the carrier of T = the carrier of Q and qBlocks(T) $\subseteq$ the entourages of Q. Then the entourages of the Pervin quasi-uniformity of T $\subseteq$ the entourages of Q.

PROOF: The entourages of the Pervin quasi-uniformity of $T \subseteq$ the entourages of Q by (28), [1, (1)]. $\square$

Let T be a non empty topological space. One can check that the Pervin quasi-uniformity of T is non empty and the Pervin quasi-uniformity of T is topological.

Now we state the propositions:

(30) Let us consider a non empty topological space T, an element x of qBlocks (T), and an element y of the Pervin quasi-uniformity of T. Then $\{z$, where z is an element of the Pervin quasi-uniformity of $T : \langle y, z\rangle \in x\} \in$ the topology of T.

(31) Let us consider a non empty topological space T, an element x of the carrier of the Pervin quasi-uniformity of T, and an element b of FinMeetCl (qBlocks(T)). Then $\{y$, where y is an element of $T : \langle x, y\rangle \in b\} \in$ the to-

pology of T. The theorem is a consequence of (30).

(32) Let us consider a non empty, strict quasi-uniform space U, a non empty, strict formal topological space F, and an element x of F. Suppose $F = \langle$the carrier of $U, \text{Neighborhood}\, U\rangle$. Then there exists an element y of U such that

(i) $x = y$, and

(ii) $U_F(x) = \text{Neighborhood}\, y$.

(33) Let us consider a non empty topological space T. Then the open set family of the FMT induced by the Pervin quasi-uniformity of $T =$ the topology of T.

Proof: The open set family of the FMT induced by the Pervin quasi-uniformity of $T \subseteq$ the topology of T by (32), [5, (18)], (31), [12, (25)]. The topology of $T \subseteq$ the open set family of the FMT induced by the Pervin quasi-uniformity of T by (32), [10, (4)], [5, (18)], [4, (87)]. □

(34) Let us consider a non empty, strict topological space T. Then the topological space induced by the Pervin quasi-uniformity of $T = T$. The theorem is a consequence of (33).

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