Prime Factorization of Sums and Differences of Two Like Powers

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Summary. Representation of a non zero integer as a signed product of primes is unique similarly to its representations in various types of positional notations [1], [3]. The study focuses on counting the prime factors of integers in the form of sums or differences of two equal powers (thus being represented by 1 and a series of zeroes in respective digital bases).

Although the introduced theorems are not particularly important, they provide a couple of shortcuts useful for integer factorization, which could serve in further development of Mizar projects [2]. This could be regarded as one of the important benefits of proof formalization [9].

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From now on $a, b, c, d, x, j, k, l, m, n, o$ denote natural numbers, $p, q, t, z, u, v$ denote integers, and $a_1, b_1, c_1, d_1$ denote complexes.

Let $a$ be a natural number. Let us note that $a$ is trivial if and only if the condition (Def. 1) is satisfied.

(Def. 1) $a \leq 1$.

Let $a$ be a complex. Let us note that the functor $a^2$ yields a set and is defined by the term
Let $a, b$ be integers. The functors: $\gcd(a, b)$ and $\text{lcm}(a, b)$ yielding natural numbers are defined by terms

$\gcd(|a|, |b|),$ $\text{lcm}(|a|, |b|),$ respectively. Let $a, b$ be positive real numbers. Note that $\max(a, b)$ is positive and $\min(a, b)$ is positive.

Let $a$ be a non zero integer and $b$ be an integer. One can check that $\gcd(a, b)$ is non zero.

Let $a$ be a non zero complex and $n$ be a natural number. Let us observe that $a^n$ is non zero.

Let $a$ be a non trivial natural number and $n$ be a non zero natural number. Note that $a^n$ is non trivial.

Let $a$ be an integer. One can check that $|a|$ is natural.

Let $a$ be an even integer. Note that $|a|$ is even.

Let $a$ be a natural number. Let us note that $\text{lcm}(a, a)$ reduces to $a$ and $\gcd(a, a)$ reduces to $a$.

Let $a$ be a non zero integer and $b$ be an integer. Note that $\gcd(a, b)$ is positive.

Let $a, b$ be integers. One can check that $\gcd(a, \gcd(a, b))$ reduces to $\gcd(a, b)$ and $\text{lcm}(a, \text{lcm}(a, b))$ reduces to $\text{lcm}(a, b)$.

Let $a$ be an integer. Observe that $\gcd(a, 1)$ reduces to 1 and $\gcd(a + 1, a)$ reduces to 1.

Now we state the proposition:

$(4)$ Let us consider integers $t, z$. Then $\gcd(t^n, z^n) = (\gcd(t, z))^n$.

Let $a$ be an integer and $n$ be a natural number.

One can verify that $\gcd((a + 1)^n, a^n)$ reduces to 1.

Let us consider $a_1$ and $b_1$. One can verify that $a_1^{-b_1}$ reduces to 0.

Let $a$ be a non negative real number and $n$ be a natural number. One can verify that $a^n$ is non negative and there exists an odd natural number which is non trivial and there exists an even natural number which is non trivial.

Let $a$ be a positive real number and $n$ be a natural number. One can verify that $a^n$ is positive.

Let $a$ be an integer. One can verify that $a \cdot a$ is square and $\frac{a}{a}$ is square and there exists an element of $\mathbb{N}$ which is non square and every element of $\mathbb{N}$ which is prime is also non square and there exists a prime natural number which is even and there exists a prime natural number which is odd and every integer which is prime is also non square.

Let $a$ be a square element of $\mathbb{N}$. Observe that $\sqrt{a}$ is natural.
Let \( a \) be an integer. Let us note that \( a^2 \) is square and \( a \cdot a \) is square and there exists an integer which is non square and every natural number which is zero is also trivial and there exists a natural number which is square and there exists an element of \( \mathbb{N} \) which is non zero and there exists a square element of \( \mathbb{N} \) which is non trivial and every natural number which is trivial is also square and every integer which is non square is also non zero.

Now we state the propositions:

5. Let us consider integers \( a, b, c, d \). If \( a \mid b \) and \( c \mid d \), then \( a \cdot c \mid b \cdot d \).

**Proof:** If \( a \mid b \), then \( \text{lcm}(a, b) = |b| \) by \( [8, (16)] \). □

Let \( a \) be an integer. Observe that \( \text{lcm}(a, 0) \) reduces to 0.

Let \( a \) be a natural number. Note that \( \text{lcm}(a, 1) \) reduces to \( a \).

Let us consider \( a \) and \( b \). Let us observe that \( \text{lcm}(a \cdot b, a) \) reduces to \( a \cdot b \) and \( \text{lcm}(\gcd(a, b), b) \) reduces to \( b \) and \( \gcd(a, \text{lcm}(a, b)) \) reduces to \( a \).

Let us consider integers \( a, b \). Now we state the propositions:

7. \(|a \cdot b| = (\gcd(a, b)) \cdot \text{lcm}(a, b)\).

8. \( \text{lcm}(a^n, b^n) = \text{lcm}(a, b)^n \). The theorem is a consequence of (4) and (7).

Let \( a \) be a square element of \( \mathbb{N} \) and \( b \) be a square element of \( \mathbb{N} \). One can check that \( \gcd(a, b) \) is square and \( \text{lcm}(a, b) \) is square.

Let \( a, b \) be square integers. One can verify that \( \gcd(a, b) \) is square and \( \text{lcm}(a, b) \) is square.

Now we state the proposition:

9. Let us consider an integer \( t \). Then \( t \) is odd if and only if \( \gcd(t, 2) = 1 \).

**Proof:** If \( t \) is odd, then \( \gcd(t, 2) = 1 \) by \( [13, (1)] \). □

Let \( t \) be an integer. One can check that \( t \) is odd if and only if the condition (Def. 5) is satisfied.

(Def. 5) \( \gcd(t, 2) = 1 \).

Let \( a \) be an odd integer. Let us observe that \(|a| \) is odd and \(-a \) is odd.

Let \( a, b \) be even integers. Note that \( \gcd(a, b) \) is even.

Let \( a \) be an integer and \( b \) be an odd integer. Note that \( \gcd(a, b) \) is odd.

Let \( a \) be a natural number. One can check that \(|-a| \) reduces to \( a \).

Let \( t, z \) be even integers. One can check that \( t + z \) is even and \( t - z \) is even and \( t \cdot z \) is even.

Let \( t, z \) be odd integers. Note that \( t + z \) is even and \( t - z \) is even and \( t \cdot z \) is odd.

Let \( t \) be an odd integer and \( z \) be an even integer. Let us observe that \( t + z \) is odd and \( t - z \) is odd and \( t \cdot z \) is even.

Now we state the proposition:
(10) Let us consider a non zero, square integer \(a\), and an integer \(b\). If \(a \cdot b\) is square, then \(b\) is square.

Let \(a\) be a square element of \(\mathbb{N}\) and \(n\) be a natural number. Let us observe that \(a^n\) is square.

Let \(a\) be a square integer. Note that \(a^n\) is square.

Let \(a\) be a non zero, square integer and \(b\) be a non square integer. Let us note that \(a \cdot b\) is non square.

Let \(a\) be an element of \(\mathbb{N}\) and \(b\) be an even natural number. Note that \(a^b\) is square.

Let \(a\) be a non square element of \(\mathbb{N}\) and \(b\) be a non square integer. Let us observe that \(a \cdot b\) is non square.

Let \(a\) be a non zero, square integer and \(n\), \(m\) be natural numbers. Let us observe that \(a^n + a^m\) is non square.

Let \(a\) be a non trivial element of \(\mathbb{N}\). One can verify that \(a - 1\) is non zero.

Let \(a\) be a non zero, square integer and \(p\) be a prime natural number. Note that \(p \cdot a\) is non square.

Let \(a\) be a non trivial element of \(\mathbb{N}\). Let us observe that \(a \cdot b\) is non square.

Let \(a\), \(b\) be non zero integers. Let us note that \(\frac{\text{lcm}(a,b)}{\text{gcd}(a,b)}\) is integer and \(\frac{\text{lcm}(a,b)}{b}\) is integer and \(\frac{\text{gcd}(a,b)}{\text{lcm}(a,b)}\) is integer.

Let \(a\) be an even integer. One can verify that \(\text{gcd}(a,2)\) reduces to 2.

Let us observe that there exists an even natural number which is non zero.
Let $a$ be an even integer and $n$ be a non zero natural number. Let us observe that $a \cdot n$ is even and $a^n$ is even.

Let $a$ be an integer and $n$ be a zero natural number. One can check that $a \cdot n$ is even and $a^n$ is odd.

Let $a$ be an element of $\mathbb{N}$. Note that $|a|$ reduces to $a$.

One can check that every integer which is non negative is also natural.

Let $a$ be a non negative real number and $n$ be a non zero natural number. Let us note that \( n \sqrt[ n]{a^n} \) reduces to $a$ and \( (n \sqrt[ n]{a})^n \) reduces to $a$.

Now we state the propositions:

(11) If $a \not| b$, then $a \cdot c \not| b$.

(12) Let us consider non negative real numbers $a$, $b$, and a positive natural number $n$. Then $a^n = b^n$ if and only if $a = b$.

Let $a$ be a real number and $n$ be an even natural number. One can verify that $a^n$ is non negative.

Let $a$ be a negative real number and $n$ be an odd natural number. One can verify that $a^n$ is negative.

Now we state the propositions:

(13) Let us consider real numbers $a$, $b$, and an odd natural number $n$. Then $a^n = b^n$ if and only if $a = b$. The theorem is a consequence of (12).

(14) If $a$ and $b$ are relatively prime, then for every non zero natural number $n$, $a \cdot b = c^n$ iff $\sqrt[ n]{a}$, $\sqrt[ n]{b} \in \mathbb{N}$ and $c = \sqrt[ n]{a} \cdot \sqrt[ n]{b}$.

Proof: If $a \cdot b = c^n$, then $\sqrt[ n]{a}$, $\sqrt[ n]{b} \in \mathbb{N}$ and $c = \sqrt[ n]{a} \cdot \sqrt[ n]{b}$ by \([14], (30)], \([11], (14)]\). □

(15) Let us consider a non zero natural number $n$, an integer $a$, and an integer $b$. Then $b^n | a^n$ if and only if $b | a$.

Proof: If $b^n | a^n$, then $b | a$ by \([10], (1)], \([14], (3)], (4), \([5], (3)]\). □

(16) Let us consider an integer $a$, and natural numbers $m, n$. If $m \geq n$, then $a^n | a^m$.

(17) Let us consider integers $a$, $b$. If $a | b$ and $b^n | c$, then $a^m | c$. The theorem is a consequence of (4).

(18) Let us consider integers $a$, $p$. If $p^{2 \cdot n + k} | a^2$, then $p^n | a$. The theorem is a consequence of (16), (4), and (12).

(19) Let us consider odd, square elements $a$, $b$ of $\mathbb{N}$. Then $8 | a - b$.

Let us consider odd natural numbers $a$, $b$. Now we state the propositions:

(20) If $4 | a - b$, then $4 \not| a^n + b^n$.

(21) If $4 | a^n + b^n$, then $4 \not| a^{2 \cdot n} + b^{2 \cdot n}$.

(22) If $4 | a^n - b^n$, then $4 \not| a^{2 \cdot n} + b^{2 \cdot n}$.
Let us consider odd natural numbers $a, b$. If $2^m \mid a^n - b^n$, then $2^{m+1} \mid a^{2n} - b^{2n}$.

$a_1^3 - b_1^3 = (a_1 - b_1) \cdot (a_1^2 + b_1^2 + a_1 \cdot b_1)$. The theorem is a consequence of (2).

Let us consider an odd natural number $n$. Then $3 \mid a^n + b^n$ if and only if $3 \mid a + b$.

**Proof:** Consider $k$ such that $n = 2 \cdot k + 1$. If $3 \mid a^n + b^n$, then $3 \mid a + b$ by [14, (173)], [5, (4)], [8, (1), (10)]. □

Let us consider an integer $c$. If $c \mid a - b$, then $c \mid a^n - b^n$.

Let us consider an odd natural number $n$. Then $3 \mid a^n - b^n$ if and only if $3 \mid a - b$.

**Proof:** Consider $k$ such that $n = 2 \cdot k + 1$. If $3 \mid a^n - b^n$, then $3 \mid a - b$ by [14, (173)], [8, (10)], [5, (4)], [8, (1)]. □

Let us consider a natural number $n$. Then $a^n \equiv (a - b)^n \pmod{b}$.

Let us consider a non trivial natural number $a$. Then there exists a prime natural number $n$ such that $n \mid a$.

Let us consider a prime natural number $p$. If $p \mid (p+(k+1)) \cdot (p-(k+1))$, then $k + 1 \geq p$.

Let us consider a prime natural number $p$, and a non zero natural number $k$. If $k < p$, then $p \nmid p^2 - k^2$. The theorem is a consequence of (30).

Let us consider integers $a, b$, and an odd, prime natural number $p$. If $p \nmid b$, then if $p \mid a - b$, then $p \nmid a + b$.

Let us consider a non zero, square element $a$ of $\mathbb{N}$, and a prime natural number $p$. If $p \mid a$, then $a + p$ is not square.

Let us consider a non zero, square element $a$ of $\mathbb{N}$, and a prime natural number $p$. If $a + p$ is square, then $p = 2 \cdot \sqrt{a} + 1$.

Let us consider integers $a, b, c$. Suppose $a$ and $b$ are relatively prime. Then $\gcd(c, a \cdot b) = (\gcd(c, a)) \cdot (\gcd(c, b))$.

Let us consider a prime natural number $p$. If $a \mid p^n$, then there exists $k$ such that $a = p^k$.

Let us consider non zero natural numbers $a, b$ and a prime natural number $p$. Now we state the propositions:

If $a + b = p$, then $a$ and $b$ are relatively prime.

If $a^n + b^n = p^n$, then $a$ and $b$ are relatively prime.

Let us consider non zero natural numbers $a, b$. If $c \geq a + b$, then $c^{k+1} \cdot (a + b) > a^{k+2} + b^{k+2}$. 


Let us consider natural numbers \(a, c\), and a non zero natural number \(b\). If \(a \cdot b < c < a \cdot (b + 1)\), then \(a \nmid c\) and \(c \nmid a\).

Let us consider real numbers \(a, b\). Then \(a + b = \min(a, b) + \max(a, b)\).

Let us consider non negative real numbers \(a, b\). Then

(i) \(\max(a^n, b^n) = (\max(a, b))^n\), and
(ii) \(\min(a^n, b^n) = (\min(a, b))^n\).

Let us consider a prime natural number \(p\). Suppose \(a \cdot b = p^n\). Then there exist natural numbers \(k, l\) such that

(i) \(a = p^k\), and
(ii) \(b = p^l\), and
(iii) \(k + l = n\).

Let us consider non trivial natural numbers \(a, b\). If \(a\) and \(b\) are relatively prime, then \(a \nmid b\) and \(b \nmid a\).

Let us consider a non trivial natural number \(a\), and a prime natural number \(p\). If \(p > a\), then \(p \nmid a\) and \(a \nmid p\). The theorem is a consequence of (44).

Let us consider a prime natural number \(p\). Then

(i) \(\gcd(a, p) = 1\), or
(ii) \(\gcd(a, p) = p\).

Let us consider a non trivial natural number \(a\), and a prime natural number \(p\). If \(a \mid p^n\), then \(p \mid a\). The theorem is a consequence of (46).

Let us consider odd natural numbers \(a, b\), and an even natural number \(m\). Then \(2\text{-count}(a^m + b^m) = 1\).

Let us consider a non zero natural number \(a\). Then there exists an odd natural number \(k\) such that \(a = 2^{\text{2-count}(a)} \cdot k\).

Let us consider a non zero natural number \(b\). Suppose \(a > b\). Then there exists a prime natural number \(p\) such that \(p\text{-count}(a) > p\text{-count}(b)\).

**Proof:** If for every prime natural number \(p\), \(p\text{-count}(a) \leq p\text{-count}(b)\), then \(a \leq b\) by [12] (20), [11] (14). □

Let us consider natural numbers \(a, b, c\). Suppose \(a \neq 1\) and \(b \neq 0\) and \(c \neq 0\) and \(b > a\text{-count}(c)\). Then \(a^b \nmid c\). The theorem is a consequence of (11).

Let us consider a non zero integer \(b\) and an integer \(a\). Now we state the propositions:

(52) If \(|a| \neq 1\), then \(a^{\lfloor a^{-\text{count}(|b|)} \rfloor} \mid b\) and \(a^{\lfloor a^{-\text{count}(|b|)} \rfloor} + 1 \nmid b\).

(53) If \(|a| \neq 1\), then if \(a^n \mid b\) and \(a^{n+1} \nmid b\), then \(n = |a|^{-\text{count}(|b|)}\).
(54) Let us consider a non zero natural number \( b \), and a non trivial natural number \( a \). Then \( a \mid b \) if and only if \( a\)-count(\( \gcd(a, b) \)) = 1.  

**Proof:** If \( a \mid b \), then \( a\)-count(\( \gcd(a, b) \)) = 1 by \([14] (3), [6] (22)\]. □  

(55) Let us consider non zero natural numbers \( b, n \), and a non trivial natural number \( a \). Then \( a\)-count(\( \gcd(a, b) \)) = 1 if and only if \( a^n\)-count((\( \gcd(a, b) \))^n) = 1. The theorem is a consequence of (15), (54), and (4).  

(56) Let us consider a non zero natural number \( b \), and a non trivial natural number \( a \). Then \( a\)-count(\( \gcd(a, b) \)) = 0 if and only if \( a\)-count(\( \gcd(a, b) \)) ≠ 1. The theorem is a consequence of (54).  

Let \( a, b \) be integers. The functor \( a\)-count(\( b \)) yielding a natural number is defined by the term  
(Def. 6) \(|a|\)-count(\( |b| \)).  

Let \( a \) be an integer. Assume \(|a| \neq 1\). Let \( b \) be a non zero integer. One can check that the functor \( a\)-count(\( b \)) is defined by  
(Def. 7) \( a^i \mid b \) and \( a^{i+1} \nmid b \).  

Now we state the propositions:  

(57) Let us consider a prime natural number \( p \), and non zero integers \( a, b \). Then \( p\)-count(\( a \cdot b \)) = (\( p\)-count(\( a \))) + (\( p\)-count(\( b \))).  

(58) Let us consider a non trivial natural number \( a \), and a non zero natural number \( b \). Then \( a\)-count(\( b \)) ≤ \( b \).  

(59) Let us consider a non trivial natural number \( a \), and a non zero integer \( b \). Then \( a^n \mid b \) if and only if \( n \leq a\)-count(\( b \)).  

**Proof:** If \( a^n \mid b \), then \( n \leq a\)-count(\( b \)) by \([8] (9), [7] (89), [1] (13)\). If \( a^n \nmid b \), then \( a\)-count(\( b \)) < \( n \) by \([8] (9), [7] (89)\). □  

(60) Let us consider a non trivial natural number \( a \), a non zero integer \( b \), and a non zero natural number \( n \). Then \( n \cdot (a\)-count(\( b \)) < \( n \cdot ((a\)-count(\( b \)) + 1) \). The theorem is a consequence of (4) and (59).  

(61) Let us consider a non trivial natural number \( a \), and non zero natural numbers \( b, n \). If \( b < a \), then \( a\)-count(\( b^n \)) < \( n \). The theorem is a consequence of (60).  

(62) Let us consider a non trivial natural number \( a \), and a non zero natural number \( b \). If \( b < a^n \), then \( a\)-count(\( b \)) < \( n \). The theorem is a consequence of (59).  

(63) Let us consider non zero natural numbers \( a, b \), and a non trivial natural number \( n \). Then \( a + b\)-count(\( a^n + b^n \)) < \( n \). The theorem is a consequence of (62).  

(64) Let us consider non zero natural numbers \( a, b \). Then \( \gcd(a, b) = 1 \) if and only if for every non trivial natural number \( c \), (\( c\)-count(\( a \))) · (\( c\)-count(\( b \))) = 0.
Proof: If $\gcd(a, b) = 1$, then for every non trivial natural number $c$, $(c\text{-count}(a)) \cdot (c\text{-count}(b)) = 0$ by \[6 \ (27)\]. If for every prime natural number $c$, $(c\text{-count}(a)) \cdot (c\text{-count}(b)) = 0$, then $\gcd(a, b) = 1$ by \[6 \ (27)\]. □

Let us consider a non zero, even natural number $m$ and odd natural numbers $a, b$. Now we state the propositions:

(65) If $a \neq b$, then $2\text{-count}(a^2 \cdot m - b^2 \cdot m) \geq (2\text{-count}(a^m - b^m)) + 1$. The theorem is a consequence of (12), (23), and (59).

(66) If $a \neq b$, then $2\text{-count}(a^2 \cdot m - b^2 \cdot m) = (2\text{-count}(a^m - b^m)) + 1$. The theorem is a consequence of (12), (57), and (48).

Let us consider a prime natural number $p$ and integers $a, b$. Now we state the propositions:

(67) If $|a| \neq |b|$, then $p\text{-count}(a^2 - b^2) = (p\text{-count}(a - b)) + (p\text{-count}(a + b))$.

(68) If $|a| \neq |b|$, then $p\text{-count}(a^3 - b^3) = (p\text{-count}(a - b)) + (p\text{-count}(a^2 + a \cdot b + b^2))$. The theorem is a consequence of (24).

(69) Let us consider non zero natural numbers $a, b$. Then $\frac{a}{\gcd(a, b)} = \frac{\text{lcm}(a, b)}{b}$.

Let us consider a non zero natural number $b$. Now we state the propositions:

(70) $\text{lcm}(a, a \cdot n + b) = ((\frac{a}{b}) + 1) \cdot \text{lcm}(a, b)$. The theorem is a consequence of (69).

(71) $\text{lcm}(a, (n \cdot a + 1) \cdot b) = (n \cdot a + 1) \cdot \text{lcm}(a, b)$. The theorem is a consequence of (70).

(72) Let us consider a non trivial natural number $a$, and non zero natural numbers $n, b$. Then $a\text{-count}(b) \geq n \cdot (a^n\text{-count}(b))$. The theorem is a consequence of (51).

Let us consider odd integers $a, b$. Now we state the propositions:

(73) $4 \mid a - b$ if and only if $4 \nmid a + b$.

(74) $2\text{-count}(a^2 + b^2) = 1$. The theorem is a consequence of (5) and (73).

(75) Let us consider a prime natural number $p$, and natural numbers $a, b$. Suppose $a \neq b$. Then $p\text{-count}(a + b) \geq p\text{-count}(\gcd(a, b))$.

(76) Let us consider a non zero integer $a$, a non trivial natural number $b$, and an integer $c$. If $a = b^{p\text{-count}(a)} \cdot c$, then $b \nmid c$.

Let $a$ be a non zero integer and $b$ be a non trivial natural number. Let us note that $\frac{a}{p\text{-count}(a)}$ is integer and $\frac{a}{2^{p\text{-count}(a)}}$ is integer and $\frac{a}{2^{p\text{-count}(a)}}$ is odd.

Now we state the proposition:

(77) Let us consider a non zero integer $a$, and a non trivial natural number $b$. Then $b\text{-count}(a) = 0$ if and only if $b \nmid a$.

Let $a$ be an odd integer. Observe that $2\text{-count}(a)$ is zero.
Observe that \( \frac{a}{2^{\text{count}(a)}} \) reduces to \( a \).

Now we state the propositions:

(78) Let us consider a prime natural number \( a \), a non zero integer \( b \), and a natural number \( c \). Then \( a\text{-count}(b^c) = c \cdot (a\text{-count}(b)) \).

(79) Let us consider non zero natural numbers \( a \), \( b \), and an odd natural number \( n \). Then \( \frac{a^{n+2}+b^{n+2}}{a+b} = a^{n+1} + b^{n+1} - a \cdot b \cdot (\frac{a^n+b^n}{a+b}) \). The theorem is a consequence of (3).

(80) Let us consider odd integers \( a \), \( b \), and a natural number \( n \). Then \( 2\text{-count}(a^2n+1 - b^2n+1) = 2\text{-count}(a - b) \). The theorem is a consequence of (13), (2), and (57).

(81) Let us consider odd integers \( a \), \( b \), and an odd natural number \( m \). Then \( 2\text{-count}(a^m+b^m) = 2\text{-count}(a+b) \). The theorem is a consequence of (80).

(82) Let us consider odd natural numbers \( a \), \( b \). Suppose \( a \neq b \). Then \( 1 = \min(2\text{-count}(a-b), 2\text{-count}(a+b)) \).

Let us consider a non trivial natural number \( a \) and non zero integers \( b \), \( c \).

Now we state the propositions:

(83) If \( a\text{-count}(b) > a\text{-count}(c) \), then \( a^{a\text{-count}(c)} \mid b \) and \( a^{a\text{-count}(b)} \nmid c \).

(84) If \( a^{a\text{-count}(b)} \mid c \) and \( a^{a\text{-count}(c)} \mid b \), then \( a\text{-count}(b) = a\text{-count}(c) \). The theorem is a consequence of (83).

(85) Let us consider integers \( a \), \( b \), and natural numbers \( m \), \( n \). If \( a^n \mid b \) and \( a^m \nmid b \), then \( m > n \). The theorem is a consequence of (16).

Let us consider a non trivial natural number \( a \) and non zero integers \( b \), \( c \).

Now we state the propositions:

(86) If \( a\text{-count}(b) = a\text{-count}(c) \) and \( a^n \mid b \), then \( a^n \mid c \). The theorem is a consequence of (85).

(87) \( a\text{-count}(b) = a\text{-count}(c) \) if and only if for every natural number \( n \), \( a^n \mid b \) iff \( a^n \mid c \).

Proof: If \( a\text{-count}(b) \neq a\text{-count}(c) \), then there exists a natural number \( n \) such that \( a^n \mid b \) and \( a^n \nmid c \) or \( a^n \mid c \) and \( a^n \nmid b \) by (83), ([1] (13)], [7] (89)], [9] (9)]. \( \square \)

(88) Let us consider odd integers \( a \), \( b \). Suppose \( |a| \neq |b| \). Then

(i) \( 2\text{-count}((a - b)^2) \neq 2\text{-count}((a + b)^2) \), and

(ii) \( 2\text{-count}((a - b)^2) \neq (2\text{-count}(a^2)) - b^2 \).

The theorem is a consequence of (78), (73), and (87).

(89) Let us consider a non trivial natural number \( b \), and a non zero integer \( a \). Then \( b\text{-count}(a) \neq 0 \) if and only if \( b \mid a \).

Proof: \( b\text{-count}(|a|) \neq 0 \) iff \( b \mid |a| \) by [6] (27)]. \( \square \)
(90) Let us consider a non trivial natural number $b$, and a non zero natural number $a$. Then $b\text{-count}(a) = 0$ if and only if $a \mod b \neq 0$. The theorem is a consequence of (89).

(91) Let us consider a prime natural number $p$, and a non trivial natural number $a$. Then $a\text{-count}(p) \leq 1$.

(92) Let us consider non trivial natural numbers $a$, $b$, and a non zero natural number $c$. Then $a^{(a\text{-count}(b)) \cdot (b\text{-count}(c))} \leq c$. The theorem is a consequence of (58).

(93) Let us consider a prime natural number $p$, a non trivial natural number $a$, and a non zero natural number $b$. Then $a\text{-count}(p^b) \leq b$. The theorem is a consequence of (89) and (59).

(94) Let us consider non trivial natural numbers $a$, $b$, and a non zero natural number $c$. Then $a\text{-count}(b \cdot c) \leq a\text{-count}(c)$. The theorem is a consequence of (17).

(95) Let us consider non trivial natural numbers $a$, $b$, and a non zero natural number $c$. Then $(a\text{-count}(b) \cdot (b\text{-count}(c))) \leq a\text{-count}(c)$. The theorem is a consequence of (92).

(96) Let us consider a non trivial natural number $a$, and an odd natural number $b$. Then $2\text{-count}(a \cdot b) = 2\text{-count}(a)$.

Let us consider a non trivial natural number $a$. Now we state the propositions:

(97) $a^{n+1} + a^n < a^{n+2}$.

(98) $(a + 1)^n + (a + 1)^n < (a + 1)^{n+1}$.

(99) Let us consider a non trivial, odd natural number $a$. Then $a^n + a^n < a^{n+1}$. The theorem is a consequence of (98).

(100) Let us consider a non trivial natural number $p$. If $a \nmid b$, then $(p^a)^c \neq p^b$.

(101) Let us consider non zero integers $a$, $b$, and a non zero natural number $n$. Suppose there exists a prime natural number $p$ such that $n \nmid p\text{-count}(a)$. Then $a \neq b^n$.

(102) Let us consider non zero integers $a$, $b$, and a non zero natural number $n$. Suppose $a = b^n$. Let us consider a prime natural number $p$. Then $n \mid p\text{-count}(a)$.

(103) Let us consider positive real numbers $a$, $b$, and a non trivial natural number $n$. Then $(a + b)^n > a^n + b^n$. The theorem is a consequence of (42) and (41).

(104) Let us consider non zero integers $a$, $b$, and an odd, prime natural number $p$. Suppose $|a| \neq |b|$ and $p \nmid b$. Then $p\text{-count}(a^2 - b^2) = \max(p\text{-count}(a - b), p\text{-count}(a + b))$. The theorem is a consequence of (32), (77), and (57).
Let us consider a non trivial natural number \( a \), and a non zero integer \( b \). Then \( a\text{-count}(a^n \cdot b) = n + (a\text{-count}(b)) \).

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