

Chebyshev Distance

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Summary. In [21], Marco Riccardi formalized that \mathbb{R}^n -basis n is a basis (in the algebraic sense defined in [26]) of \mathcal{E}_T^n and in [20] he has formalized that \mathcal{E}_T^n is second-countable, we build (in the topological sense defined in [23]) a denumerable base of \mathcal{E}_T^n .

Then we introduce the n -dimensional intervals (interval in n -dimensional Euclidean space, *pavé (borné) de \mathbb{R}^n* [16], *semi-intervalle (borné) de \mathbb{R}^n* [22]).

We conclude with the definition of Chebyshev distance [11].

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1. PRELIMINARIES

From now on n denotes a natural number, r, s denote real numbers, x, y denote elements of \mathcal{R}^n , p, q denote points of \mathcal{E}_T^n , and e denotes a point of \mathcal{E}^n .

Now we state the propositions:

- (1) $|x - y| = |y - x|$.
- (2) Let us consider a natural number i . If $i \in \text{Seg } n$, then $|x|(i) \in \mathbb{R}$.
- (3) Let us consider elements x, y of \mathbb{R} , and extended reals x_1, y_1 . If $x \leq x_1$ and $y \leq y_1$, then $x + y \leq x_1 + y_1$.
- (4) Let us consider real numbers a, c , and an extended real number b . Suppose $a < b$ and $[a, b] \subseteq [a, c]$. Then b is a real number.
- (5) Let us consider a non empty set D , and a non empty subset D_1 of D . Then $D_1^n \subseteq D^n$.

(6) Let us consider a non empty set X , and a function f . Suppose $f = \text{Seg } n \mapsto X$. Then f is a non-empty, n -element finite sequence.

Let n be a natural number. The functor $\mathbb{R}(n)$ yielding a non-empty, n -element finite sequence is defined by the term

(Def. 1) $\text{Seg } n \mapsto \mathbb{R}$.

Now we state the propositions:

(7) $\mathbb{R}(n) = \text{Seg } n \mapsto$ the carrier of \mathbb{R}^1 .

(8) $\prod(\text{Seg } n \mapsto \mathbb{R}) = \mathcal{R}^n$.

(9) $\prod \mathbb{R}(n) = \mathcal{R}^n$.

(10) Let us consider a set X . Then $\prod(\text{Seg } n \mapsto X) = X^n$.

(11) Let us consider a non empty set D , and an n -tuple x of D . Then $x \in D^{\text{Seg } n}$.

(12) Let us consider a subset O_1 of \mathcal{E}_T^n , and a subset O_2 of $(\mathcal{E}^n)_{\text{top}}$. If $O_1 = O_2$, then O_1 is open iff O_2 is open.

(13) Suppose $e = p$. Then the set of all $\text{OpenHypercube}(e, \frac{1}{m})$ where m is a non zero element of \mathbb{N} = the set of all $\text{OpenHypercube}(p, \frac{1}{m})$ where m is a non zero element of \mathbb{N} .

(14) If $q \in \text{OpenHypercube}(p, r)$, then $p \in \text{OpenHypercube}(q, r)$.

(15) If $q \in \text{OpenHypercube}(p, \frac{r}{2})$,
then $\text{OpenHypercube}(q, \frac{r}{2}) \subseteq \text{OpenHypercube}(p, r)$.

Let x be an element of $\mathbb{R} \times \mathbb{R}$. The functors: $(x)_1$ and $(x)_2$ yield elements of \mathbb{R} . Let n be a natural number and x be an element of $\mathcal{R}^n \times \mathcal{R}^n$. The functors: $(x)_1$ and $(x)_2$ yield elements of \mathcal{R}^n . Now we state the proposition:

(16) Let us consider an n -element finite sequence f of elements of $\mathbb{R} \times \mathbb{R}$. Then there exists an element x of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \text{Seg } n$ holds $(x)_1(i) = (f_i)_1$ and $(x)_2(i) = (f_i)_2$.

2. THE SET OF n -TUPLES OF RATIONAL NUMBERS

Let us consider n . The functor \mathcal{Q}^n yielding a set of finite sequences of \mathbb{Q} is defined by the term

(Def. 2) \mathcal{Q}^n .

Now we state the proposition:

(17) $\mathcal{Q}^0 = \{0\}$.

One can check that \mathcal{Q}^0 is trivial.

Let us consider n . One can check that \mathcal{Q}^n is non empty and every element of \mathcal{Q}^n is n -element and \mathcal{Q}^n is countable.

Let n be a positive natural number. Let us note that \mathcal{Q}^n is infinite and \mathcal{Q}^n is denumerable.

Now we state the proposition:

(18) \mathcal{Q}^n is a dense subset of \mathcal{E}_T^n .

PROOF: \mathcal{Q}^n is a subset of \mathcal{R}^n . Reconsider $R = \mathcal{Q}^n$ as a subset of \mathcal{E}_T^n . For every subset Q of \mathcal{E}_T^n such that $Q \neq \emptyset$ and Q is open holds R meets Q by [10, (67)], (12), [15, (23)], [13, (39)]. \square

Let us consider n . One can check that \mathcal{Q}^n is countable and dense as a subset of \mathcal{E}_T^n .

3. A COUNTABLE BASE OF AN n -DIMENSIONAL EUCLIDEAN SPACE

(VERSION 1: [20]):

Let n be a natural number. Let us observe that there exists a basis of \mathcal{E}_T^n which is countable.

Let us consider n and e . Note that $\text{OpenHypercubes } e$ is countable.

The functor $\text{OpenHypercubes-}\mathbb{Q}(n)$ yielding a non empty set is defined by the term

(Def. 3) $\{\text{OpenHypercubes } q, \text{ where } q \text{ is a point of } \mathcal{E}^n : q \in \mathcal{Q}^n\}$.

Let q be an element of \mathcal{Q}^n . The functor ${}^{\textcircled{q}}$ yielding a point of \mathcal{E}^n is defined by the term

(Def. 4) q .

Let q be an object. Assume $q \in \mathcal{Q}^n$. The functor $\text{object2}\mathbb{Q}(q, n)$ yielding an element of \mathcal{Q}^n is defined by the term

(Def. 5) q .

Let us note that $\text{OpenHypercubes-}\mathbb{Q}(n)$ is countable and $\bigcup \text{OpenHypercubes-}\mathbb{Q}(n)$ is countable.

Now we state the propositions:

(19) $\bigcup \text{OpenHypercubes-}\mathbb{Q}(n)$ is an open family of subsets of \mathcal{E}_T^n . The theorem is a consequence of (12).

(20) Let us consider a non empty, open subset O of \mathcal{E}_T^n . Then there exists an element p of \mathcal{Q}^n such that $p \in O$. The theorem is a consequence of (18).

(21) Let us consider a family \mathcal{B} of subsets of \mathcal{E}_T^n .

Suppose $\mathcal{B} = \bigcup \text{OpenHypercubes-}\mathbb{Q}(n)$. Then \mathcal{B} is quasi basis.

PROOF: F is quasi basis by (12), [15, (23)], [10, (67)], (20). \square

Let us consider n . Observe that $\bigcup \text{OpenHypercubes-}\mathbb{Q}(n)$ is non empty.

The functor $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$ yielding a countable, open family of subsets of \mathcal{E}_T^n is defined by the term

(Def. 6) $\bigcup \text{OpenHypercubes}\mathbb{Q}(n)$.

Now we state the proposition:

(22) $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n) = \{\text{OpenHypercube}(q, \frac{1}{m}),$
where q is a point of \mathcal{E}^n , m is a positive natural number : $q \in \mathcal{Q}^n\}$.

(VERSION 2):

Let n be a natural number. Observe that there exists a basis of \mathcal{E}_T^n which is countable.

Now we state the propositions:

(23) $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$ is a countable basis of \mathcal{E}_T^n .

(24) Let us consider an open subset O of \mathcal{E}_T^n . Then there exists a subset Y of $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$ such that

- (i) Y is countable, and
- (ii) $O = \bigcup Y$.

The theorem is a consequence of (21).

Let us consider an open, non empty subset O of \mathcal{E}_T^n . Now we state the propositions:

(25) There exists a subset Y of $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$ such that

- (i) Y is not empty, and
- (ii) $O = \bigcup Y$, and
- (iii) there exists a function g from \mathbb{N} into Y such that for every object x , $x \in O$ iff there exists an object y such that $y \in \mathbb{N}$ and $x \in g(y)$.

The theorem is a consequence of (24).

(26) There exists a sequence s of $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$ such that for every object x , $x \in O$ iff there exists an object y such that $y \in \mathbb{N}$ and $x \in s(y)$. The theorem is a consequence of (25).

(27) There exists a sequence s of $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$ such that $O = \bigcup s$. The theorem is a consequence of (26).

4. THE SET OF ALL LEFT OPEN REAL BOUNDED INTERVALS

The set of all left open real bounded intervals yielding a family of subsets of \mathbb{R} is defined by the term

(Def. 7) the set of all $]a, b]$ where a, b are real numbers.

Let us note that the set of all left open real bounded intervals is non empty.

Now we state the propositions:

- (28) The set of all left open real bounded intervals $\subseteq \{I, \text{ where } I \text{ is a subset of } \mathbb{R} : I \text{ is left open interval}\}$.
- (29) The set of all left open real bounded intervals is \cap -closed and \setminus_{fp} -closed and has the empty element and countable cover.
- (30) The set of all left open real bounded intervals is a semiring of \mathbb{R} .

5. THE SET OF ALL RIGHT OPEN REAL BOUNDED INTERVALS

The set of all right open real bounded intervals yielding a family of subsets of \mathbb{R} is defined by the term

(Def. 8) the set of all $[a, b[$ where a, b are real numbers.

Observe that the set of all right open real bounded intervals is non empty.

Now we state the propositions:

- (31) The set of all right open real bounded intervals $\subseteq \{I, \text{ where } I \text{ is a subset of } \mathbb{R} : I \text{ is right open interval}\}$.
- (32) The set of all right open real bounded intervals has the empty element.
- (33) (i) the set of all right open real bounded intervals is \cap -closed, and
(ii) the set of all right open real bounded intervals is \setminus_{fp} -closed and has the empty element.

The theorem is a consequence of (31), (32), and (4).

- (34) The set of all right open real bounded intervals has countable cover.

PROOF: Define $\mathcal{F}[\text{object}, \text{object}] \equiv \$_1$ is an element of \mathbb{N} and $\$_2 \in$ the set of all right open real bounded intervals and there exists a real number x such that $x = \$_1$ and $\$_2 = [-x, x[$. For every object x such that $x \in \mathbb{N}$ there exists an object y such that $y \in$ the set of all right open real bounded intervals and $\mathcal{F}[x, y]$. Consider f being a function such that $\text{dom } f = \mathbb{N}$ and $\text{rng } f \subseteq$ the set of all right open real bounded intervals and for every object x such that $x \in \mathbb{N}$ holds $\mathcal{F}[x, f(x)]$ from [7, Sch. 6]. $\text{rng } f$ is countable by [27, (4)], [14, (58)]. $\text{rng } f$ is a cover of \mathbb{R} by [2, (2)], [12, (8)], [3, (13)], [17, (45)]. \square

- (35) The set of all right open real bounded intervals is a semiring of \mathbb{R} .

6. FINITE PRODUCT OF LEFT OPEN INTERVALS

In the sequel n denotes a non zero natural number.

Let n be a non zero natural number. The functor $\text{LeftOpenIntervals}(n)$ yielding a classical semiring family of $\mathbb{R}(n)$ is defined by the term

(Def. 9) $\text{Seg } n \longmapsto$ (the set of all left open real bounded intervals).

Now we state the propositions:

(36) $\text{LeftOpenIntervals}(n) = \text{Seg } n \longmapsto$ the set of all $]a, b]$ where a, b are real numbers.

(37) $\text{MeasurableRectangle LeftOpenIntervals}(n)$ is a semiring of \mathcal{R}^n . The theorem is a consequence of (8).

Let us consider an object x .

Let us assume that $x \in \text{MeasurableRectangle LeftOpenIntervals}(n)$. Now we state the propositions:

(38) There exists a subset y of \mathcal{R}^n such that

(i) $x = y$, and

(ii) for every natural number i such that $i \in \text{Seg } n$ there exist real numbers a, b such that for every element t of \mathcal{R}^n such that $t \in y$ holds $t(i) \in]a, b]$.

The theorem is a consequence of (37).

(39) There exists a subset y of \mathcal{R}^n and there exists an n -element finite sequence f of elements of $\mathbb{R} \times \mathbb{R}$ such that $x = y$ and for every element t of \mathcal{R}^n , $t \in y$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in](f_i)_1, (f_i)_2]$.

PROOF: $\text{MeasurableRectangle LeftOpenIntervals}(n)$ is a family of subsets of \mathcal{R}^n . Reconsider $y = x$ as a subset of \mathcal{R}^n . Consider g being a function such that $x = \prod g$ and $g \in \prod \text{LeftOpenIntervals}(n)$. Define $\mathcal{P}[\text{natural number, set}] \equiv$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that $\$2 = x$ and $g(\$1) =](x)_1, (x)_2]$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$. There exists a finite sequence f_1 of elements of $\mathbb{R} \times \mathbb{R}$ such that $\text{len } f_1 = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, f_{1i}]$ from [25, Sch. 1]. Consider f_1 being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\text{len } f_1 = n$ and for every natural number i such that $i \in \text{Seg } n$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that $f_{1i} = x$ and $g(i) =](x)_1, (x)_2]$. For every natural number i such that $i \in \text{Seg } n$ holds $g(i) =](f_{1i})_1, (f_{1i})_2]$. For every element t of \mathcal{R}^n such that $t \in y$ for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in](f_{1i})_1, (f_{1i})_2]$. For every element t of \mathcal{R}^n such that for every natural

number i such that $i \in \text{Seg } n$ holds $t(i) \in](f_{1i})_1, (f_{1i})_2]$ holds $t \in y$ by [6, (93)]. \square

- (40) There exists a subset y of \mathcal{R}^n and there exist elements a, b of \mathcal{R}^n such that $x = y$ and for every object $s, s \in y$ iff there exists an element t of \mathcal{R}^n such that $s = t$ and for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in]a(i), b(i)]$. The theorem is a consequence of (39) and (16).

Now we state the proposition:

- (41) Let us consider a set x . Suppose $x \in \text{MeasurableRectangleLeftOpenIntervals}(n)$. Then there exist elements a, b of \mathcal{R}^n such that for every element t of $\mathcal{R}^n, t \in x$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in]a(i), b(i)]$. The theorem is a consequence of (39) and (16).

7. FINITE PRODUCT OF RIGHT OPEN INTERVALS

Let n be a non zero natural number. The functor $\text{RightOpenIntervals}(n)$ yielding a classical semiring family of $\mathbb{R}(n)$ is defined by the term

- (Def. 10) $\text{Seg } n \longmapsto$ (the set of all right open real bounded intervals).

From now on n denotes a non zero natural number.

Now we state the propositions:

- (42) $\text{RightOpenIntervals}(n) = \text{Seg } n \longmapsto$ the set of all $[a, b[$ where a, b are real numbers.
- (43) $\text{MeasurableRectangleRightOpenIntervals}(n)$ is a semiring of \mathcal{R}^n . The theorem is a consequence of (8).

Let us consider an object x .

Let us assume that $x \in \text{MeasurableRectangleRightOpenIntervals}(n)$. Now we state the propositions:

- (44) There exists a subset y of \mathcal{R}^n such that
- (i) $x = y$, and
 - (ii) for every natural number i such that $i \in \text{Seg } n$ there exist real numbers a, b such that for every element t of \mathcal{R}^n such that $t \in y$ holds $t(i) \in [a, b[$.

The theorem is a consequence of (43).

- (45) There exists a subset y of \mathcal{R}^n and there exists an n -element finite sequence f of elements of $\mathbb{R} \times \mathbb{R}$ such that $x = y$ and for every element t of $\mathcal{R}^n, t \in y$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in [(f_i)_1, (f_i)_2[$.

PROOF: $\text{MeasurableRectangleRightOpenIntervals}(n)$ is a family of subsets of \mathcal{R}^n . Reconsider $y = x$ as a subset of \mathcal{R}^n . Consider g being a function

such that $x = \prod g$ and $g \in \prod \text{RightOpenIntervals}(n)$. Define $\mathcal{P}[\text{natural number, set}] \equiv$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that $\$2 = x$ and $g(\$1) = [(x)_1, (x)_2[$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$. There exists a finite sequence f_1 of elements of $\mathbb{R} \times \mathbb{R}$ such that $\text{len } f_1 = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, f_{1_i}]$ from [25, Sch. 1]. Consider f_1 being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\text{len } f_1 = n$ and for every natural number i such that $i \in \text{Seg } n$ there exists an element x of $\mathbb{R} \times \mathbb{R}$ such that $f_{1_i} = x$ and $g(i) = [(x)_1, (x)_2[$. For every natural number i such that $i \in \text{Seg } n$ holds $g(i) = [(f_{1_i})_1, (f_{1_i})_2[$. For every element t of \mathcal{R}^n such that $t \in y$ for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in [(f_{1_i})_1, (f_{1_i})_2[$. For every element t of \mathcal{R}^n such that for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in [(f_{1_i})_1, (f_{1_i})_2[$ holds $t \in y$ by [6, (93)]. \square

- (46) There exists a subset y of \mathcal{R}^n and there exist elements a, b of \mathcal{R}^n such that $x = y$ and for every object $s, s \in y$ iff there exists an element t of \mathcal{R}^n such that $s = t$ and for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (45) and (16).

Now we state the proposition:

- (47) Let us consider a set x . Suppose $x \in \text{MeasurableRectangle RightOpenIntervals}(n)$. Then there exist elements a, b of \mathcal{R}^n such that for every element t of $\mathcal{R}^n, t \in x$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (45) and (16).

8. n -DIMENSIONAL PRODUCT OF SUBSET FAMILY

In the sequel n denotes a natural number, X denotes a set, and S denotes a family of subsets of X .

Let us consider n and X . The functor $\text{Product}(n, X)$ yielding a set is defined by

- (Def. 11) for every object $f, f \in it$ iff there exists a function g such that $f = \prod g$ and $g \in \prod(\text{Seg } n \mapsto X)$.

Now we state the propositions:

- (48) $\text{Product}(n, X) \subseteq 2^{(\bigcup(\text{Seg } n \mapsto X))^{\text{dom}(\text{Seg } n \mapsto X)}}$.
- (49) $\text{Product}(n, S)$ is a family of subsets of $\prod(\text{Seg } n \mapsto X)$.

PROOF: Reconsider $S_1 = \text{Product}(n, S)$ as a subset of $2^{(\bigcup(\text{Seg } n \mapsto S))^{\text{dom}(\text{Seg } n \mapsto S)}}$. $S_1 \subseteq 2^{\prod(\text{Seg } n \mapsto X)}$ by [1, (9)], [24, (13), (7)], [9, (77), (81)]. \square

(50) Let us consider a non empty family S of subsets of X . Then $\text{Product}(n, S) =$ the set of all $\prod f$ where f is an n -tuple of S .

PROOF: $\text{Product}(n, S) \subseteq$ the set of all $\prod f$ where f is an n -tuple of S by (10), [6, (131)]. the set of all $\prod f$ where f is an n -tuple of $S \subseteq \text{Product}(n, S)$ by [6, (131)], (10). \square

(51) Let us consider a non zero natural number n . Then $\text{Product}(n, X) \subseteq 2^{(\bigcup X)^{\text{Seg } n}}$.

Let us consider a non zero natural number n , a non empty set X , and a non empty family S of subsets of X .

Let us assume that $S \neq \{\emptyset\}$. Now we state the propositions:

(52) $\text{Product}(n, S) \subseteq 2^{X^{\text{Seg } n}}$. The theorem is a consequence of (51) and (5).

(53) $\bigcup \text{Product}(n, S) \subseteq X^{\text{Seg } n}$. The theorem is a consequence of (52).

Let n be a natural number and X be a non empty set. Let us note that $\text{Product}(n, X)$ is non empty.

Now we state the proposition:

(54) Let us consider a non empty set X , a non empty family S of subsets of X , and a subset x of $X^{\text{Seg } n}$. Then x is an element of $\text{Product}(n, S)$ if and only if there exists an n -tuple s of S such that for every element t of $X^{\text{Seg } n}$, for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in s(i)$ iff $t \in x$.

9. THE SET OF ALL CLOSED REAL BOUNDED INTERVALS

The set of all closed real bounded intervals yielding a family of subsets of \mathbb{R} is defined by the term

(Def. 12) the set of all $[a, b]$ where a, b are real numbers.

Now we state the proposition:

(55) The set of all closed real bounded intervals = $\{I, \text{ where } I \text{ is a subset of } \mathbb{R} : I \text{ is closed interval}\}$.

Let us note that the set of all closed real bounded intervals is non empty.

Now we state the propositions:

(56) The set of all closed real bounded intervals is \cap -closed and has the empty element and countable cover.

PROOF: The set of all closed real bounded intervals is \cap -closed. There exists a countable subset X of the set of all closed real bounded intervals such that X is a cover of \mathbb{R} by [27, (4)], [14, (58)], [2, (2)], [12, (8)]. \square

(57) Let us consider a natural number n . Then $\text{Seg } n \mapsto$ (the set of all closed real bounded intervals) is an n -element finite sequence.

10. THE SET OF ALL OPEN REAL BOUNDED INTERVALS

The set of all open real bounded intervals yielding a family of subsets of \mathbb{R} is defined by the term

(Def. 13) the set of all $]a, b[$ where a, b are real numbers.

Now we state the proposition:

(58) The set of all open real bounded intervals $\subseteq \{I, \text{ where } I \text{ is a subset of } \mathbb{R} : I \text{ is open interval}\}$.

Let us observe that the set of all open real bounded intervals is non empty.

Now we state the propositions:

(59) The set of all open real bounded intervals is \cap -closed and has the empty element and countable cover.

PROOF: The set of all open real bounded intervals is \cap -closed. There exists a countable subset X of the set of all open real bounded intervals such that X is a cover of \mathbb{R} by [27, (4)], [14, (58)], [2, (2)], [12, (8)]. \square

(60) Let us consider a natural number n . Then $\text{Seg } n \mapsto$ (the set of all open real bounded intervals) is an n -element finite sequence.

11. n -DIMENSIONAL SUBSET FAMILY OF \mathbb{R}

From now on n denotes a natural number and S denotes a family of subsets of \mathbb{R} .

Now we state the proposition:

(61) $\text{Product}(n, S)$ is a family of subsets of \mathcal{R}^n . The theorem is a consequence of (49) and (8).

Let us consider n and S . One can check that the functor $\text{Product}(n, S)$ yields a family of subsets of \mathcal{R}^n . Now we state the propositions:

(62) Let us consider a non empty family S of subsets of \mathbb{R} , and a subset x of \mathcal{R}^n . Then x is an element of $\text{Product}(n, S)$ if and only if there exists an n -tuple s of S such that for every element t of \mathcal{R}^n , for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in s(i)$ iff $t \in x$.

PROOF: If x is an element of $\text{Product}(n, S)$, then there exists an n -tuple s of S such that for every element t of \mathcal{R}^n , for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in s(i)$ iff $t \in x$ by [6, (93)]. If there exists an n -tuple s of S such that for every element t of \mathcal{R}^n , for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in s(i)$ iff $t \in x$, then x is an element of $\text{Product}(n, S)$ by [6, (93)]. \square

- (63) Let us consider a non zero natural number n , and an n -tuple s of the set of all closed real bounded intervals. Then there exist elements a, b of \mathcal{R}^n such that for every natural number i such that $i \in \text{Seg } n$ holds $s(i) = [a(i), b(i)]$.

PROOF: $s \in$ (the set of all closed real bounded intervals) $^{\text{Seg } n}$. Consider f being a function such that $s = f$ and $\text{dom } f = \text{Seg } n$ and $\text{rng } f \subseteq$ the set of all closed real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$_2$ and $s(\$_1) = [(f)_1, (f)_2]$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\text{len } f = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \text{Seg } n$ holds $(z)_1(i) = (f_i)_1$ and $(z)_2(i) = (f_i)_2$. Reconsider $a = (z)_1, b = (z)_2$ as an element of \mathcal{R}^n . For every natural number i such that $i \in \text{Seg } n$ holds $s(i) = [a(i), b(i)]$. \square

- (64) Let us consider a non zero natural number n , and an element x of $\text{Product}(n, \text{the set of all closed real bounded intervals})$. Then there exist elements a, b of \mathcal{R}^n such that for every element t of $\mathcal{R}^n, t \in x$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in [a(i), b(i)]$. The theorem is a consequence of (62) and (63).

Let us consider a non zero natural number n , a subset x of \mathcal{R}^n , and elements a, b of \mathcal{R}^n . Now we state the propositions:

- (65) Suppose for every element t of $\mathcal{R}^n, t \in x$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in [a(i), b(i)]$. Then x is an element of $\text{Product}(n, \text{the set of all closed real bounded intervals})$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$1 = n$ and $\$2 = [a(n), b(n)]$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of the set of all closed real bounded intervals such that $\mathcal{P}[i, d]$. There exists a finite sequence g of elements of the set of all closed real bounded intervals such that $\text{len } g = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, g_i]$ from [25, Sch. 1]. Consider g being a finite sequence of elements of the set of all closed real bounded intervals such that $\text{len } g = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, g_i]$. For every natural number i such that $i \in \text{Seg } n$ holds $g(i) = [a(i), b(i)]$. There exists a function g such that $x = \prod g$ and $g \in \prod(\text{Seg } n \mapsto \text{(the set of all closed real bounded intervals)})$ by [4, (89)], [24, (13), (7)], [1, (9)]. \square

- (66) Suppose for every element t of $\mathcal{R}^n, t \in x$ iff for every natural number

i such that $i \in \text{Seg } n$ holds $t(i) \in]a(i), b(i)[$. Then x is an element of $\text{Product}(n, \text{the set of all left open real bounded intervals})$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$1 = n$ and $\$2 =]a(n), b(n)[$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of the set of all left open real bounded intervals such that $\mathcal{P}[i, d]$. There exists a finite sequence g of elements of the set of all left open real bounded intervals such that $\text{len } g = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, g_i]$ from [25, Sch. 1]. Consider g being a finite sequence of elements of the set of all left open real bounded intervals such that $\text{len } g = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, g_i]$. For every natural number i such that $i \in \text{Seg } n$ holds $g(i) =]a(i), b(i)[$. There exists a function g such that $x = \prod g$ and $g \in \prod(\text{Seg } n \mapsto (\text{the set of all left open real bounded intervals}))$ by [4, (89)], [24, (13), (7)], [1, (9)]. \square

- (67) Suppose for every element t of \mathcal{R}^n , $t \in x$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in [a(i), b(i)[$. Then x is an element of $\text{Product}(n, \text{the set of all right open real bounded intervals})$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$1 = n$ and $\$2 = [a(n), b(n)[$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of the set of all right open real bounded intervals such that $\mathcal{P}[i, d]$. There exists a finite sequence g of elements of the set of all right open real bounded intervals such that $\text{len } g = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, g_i]$ from [25, Sch. 1]. Consider g being a finite sequence of elements of the set of all right open real bounded intervals such that $\text{len } g = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, g_i]$. For every natural number i such that $i \in \text{Seg } n$ holds $g(i) = [a(i), b(i)[$. There exists a function g such that $x = \prod g$ and $g \in \prod(\text{Seg } n \mapsto (\text{the set of all right open real bounded intervals}))$ by [4, (89)], [24, (13), (7)], [1, (9)]. \square

Now we state the propositions:

- (68) Let us consider a non zero natural number n , and an n -tuple s of the set of all left open real bounded intervals. Then there exist elements a, b of \mathcal{R}^n such that for every natural number i such that $i \in \text{Seg } n$ holds $s(i) =]a(i), b(i)[$.

PROOF: $s \in (\text{the set of all left open real bounded intervals})^{\text{Seg } n}$. Consider f being a function such that $s = f$ and $\text{dom } f = \text{Seg } n$ and $\text{rng } f \subseteq$ the set of all left open real bounded intervals. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element f of $\mathbb{R} \times \mathbb{R}$ such that $f = \$2$ and $s(\$1) = |(f)_1, (f)_2|$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of

elements of $\mathbb{R} \times \mathbb{R}$ such that $\text{len } f = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \text{Seg } n$ holds $(z)_1(i) = (f_i)_1$ and $(z)_2(i) = (f_i)_2$. Reconsider $a = (z)_1, b = (z)_2$ as an element of \mathcal{R}^n . For every natural number i such that $i \in \text{Seg } n$ holds $s(i) =]a(i), b(i)[$. \square

(69) Let us consider a non zero natural number n , and an element x of $\text{Product}(n, \text{the set of all left open real bounded intervals})$. Then there exist elements a, b of \mathcal{R}^n such that for every element t of $\mathcal{R}^n, t \in x$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in]a(i), b(i)[$. The theorem is a consequence of (62) and (68).

(70) Let us consider a non zero natural number n , and an n -tuple s of the set of all right open real bounded intervals. Then there exist elements a, b of \mathcal{R}^n such that for every natural number i such that $i \in \text{Seg } n$ holds $s(i) = [a(i), b(i)[$.

PROOF: $s \in (\text{the set of all right open real bounded intervals})^{\text{Seg } n}$. Consider f being a function such that $s = f$ and $\text{dom } f = \text{Seg } n$ and $\text{rng } f \subseteq \text{the set of all right open real bounded intervals}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \text{there exists an element } f \text{ of } \mathbb{R} \times \mathbb{R} \text{ such that } f = \$2 \text{ and } s(\$1) = [(f)_1, (f)_2[$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\text{len } f = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \text{Seg } n$ holds $(z)_1(i) = (f_i)_1$ and $(z)_2(i) = (f_i)_2$. Reconsider $a = (z)_1, b = (z)_2$ as an element of \mathcal{R}^n . For every natural number i such that $i \in \text{Seg } n$ holds $s(i) = [a(i), b(i)[$. \square

(71) Let us consider a non zero natural number n , and an element x of $\text{Product}(n, \text{the set of all right open real bounded intervals})$. Then there exist elements a, b of \mathcal{R}^n such that for every element t of $\mathcal{R}^n, t \in x$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (62) and (70).

12. CLOSED/OPEN/LEFT-OPEN/RIGHT-OPEN – HYPER INTERVAL

From now on n denotes a natural number and a, b, c, d denote elements of \mathcal{R}^n .

Let us consider $n, a,$ and b . The functor $\text{ClosedHyperInterval}(a, b)$ yielding a subset of \mathcal{R}^n is defined by

(Def. 14) for every object x , $x \in it$ iff there exists an element y of \mathcal{R}^n such that $x = y$ and for every natural number i such that $i \in \text{Seg } n$ holds $y(i) \in [a(i), b(i)]$.

The functor $\text{OpenHyperInterval}(a, b)$ yielding a subset of \mathcal{R}^n is defined by

(Def. 15) for every object x , $x \in it$ iff there exists an element y of \mathcal{R}^n such that $x = y$ and for every natural number i such that $i \in \text{Seg } n$ holds $y(i) \in]a(i), b(i)[$.

The functor $\text{LeftOpenHyperInterval}(a, b)$ yielding a subset of \mathcal{R}^n is defined by

(Def. 16) for every object x , $x \in it$ iff there exists an element y of \mathcal{R}^n such that $x = y$ and for every natural number i such that $i \in \text{Seg } n$ holds $y(i) \in]a(i), b(i)[$.

The functor $\text{RightOpenHyperInterval}(a, b)$ yielding a subset of \mathcal{R}^n is defined by

(Def. 17) for every object x , $x \in it$ iff there exists an element y of \mathcal{R}^n such that $x = y$ and for every natural number i such that $i \in \text{Seg } n$ holds $y(i) \in [a(i), b(i)[$.

Now we state the proposition:

$$(72) \quad \text{ClosedHyperInterval}(a, a) = \{a\}.$$

PROOF: $\text{ClosedHyperInterval}(a, a) \subseteq \{a\}$ by [6, (124)].

$$\{a\} \subseteq \text{ClosedHyperInterval}(a, a). \quad \square$$

Let us consider n and a . Let us observe that $\text{ClosedHyperInterval}(a, a)$ is trivial.

Now we state the proposition:

- (73) (i) $\text{OpenHyperInterval}(a, b) \subseteq \text{LeftOpenHyperInterval}(a, b)$, and
(ii) $\text{OpenHyperInterval}(a, b) \subseteq \text{RightOpenHyperInterval}(a, b)$, and
(iii) $\text{LeftOpenHyperInterval}(a, b) \subseteq \text{ClosedHyperInterval}(a, b)$, and
(iv) $\text{RightOpenHyperInterval}(a, b) \subseteq \text{ClosedHyperInterval}(a, b)$.

Let us consider n , a , and b . We say that $a \leq b$ if and only if

(Def. 18) for every natural number i such that $i \in \text{Seg } n$ holds $a(i) \leq b(i)$.

One can verify that the predicate is reflexive.

Now we state the propositions:

$$(74) \quad \text{If } a \leq b \leq c, \text{ then } a \leq c.$$

$$(75) \quad \text{If } a \leq c \text{ and } d \leq b,$$

$$\text{then } \text{ClosedHyperInterval}(c, d) \subseteq \text{ClosedHyperInterval}(a, b).$$

$$(76) \quad \text{If } a \leq b, \text{ then } \text{ClosedHyperInterval}(a, b) \text{ is not empty. The theorem is a consequence of (75) and (72).}$$

Let us consider n , a , and b . We say that $a < b$ if and only if

(Def. 19) for every natural number i such that $i \in \text{Seg } n$ holds $a(i) < b(i)$.

Now we state the propositions:

(77) If $a < b < c$, then $a < c$.

(78) If $b < a$ and n is not zero, then $\text{ClosedHyperInterval}(a, b)$ is empty.

(79) $n \mapsto r$ is an element of \mathcal{R}^n .

PROOF: Set $f = n \mapsto r$. $f \in \mathbb{R}^{\text{Seg } n}$ by [6, (112), (93)]. \square

Let us consider n and r . Note that the functor $n \mapsto r$ yields an element of \mathcal{R}^n . One can check that the functor $\langle r \rangle$ yields an element of \mathcal{R}^1 . Now we state the propositions:

(80) Let us consider a non zero natural number n , and a point e of \mathcal{E}^n . Then there exists an element a of \mathcal{R}^n such that

(i) $a = e$, and

(ii) $\text{OpenHypercube}(e, r) = \text{OpenHyperInterval}(a - n \mapsto r, a + n \mapsto r)$.

PROOF: Reconsider $a = e$ as an element of \mathcal{R}^n . Reconsider $p = e$ as a point of \mathcal{E}_T^n . Consider e_0 being a point of \mathcal{E}^n such that $p = e_0$ and $\text{OpenHypercube}(e_0, r) = \text{OpenHypercube}(p, r)$. $\text{OpenHypercube}(e, r) \subseteq \text{OpenHyperInterval}(a - n \mapsto r, a + n \mapsto r)$ by [8, (27)], [6, (57)], [8, (11)], [18, (4)]. $\text{OpenHyperInterval}(a - n \mapsto r, a + n \mapsto r) \subseteq \text{OpenHypercube}(e, r)$ by [10, (22)], [8, (27)], [6, (57)], [8, (11)]. \square

(81) Let us consider a point p of \mathcal{E}_T^n . Then there exists an element a of \mathcal{R}^n such that

(i) $a = p$, and

(ii) $\text{ClosedHypercube}(p, b) = \text{ClosedHyperInterval}(a - b, a + b)$.

PROOF: Reconsider $a = p$ as an element of \mathcal{R}^n . $\text{ClosedHypercube}(p, b) \subseteq \text{ClosedHyperInterval}(a - b, a + b)$ by [10, (22)], [8, (11), (27)]. $\text{ClosedHyperInterval}(a - b, a + b) \subseteq \text{ClosedHypercube}(p, b)$ by [10, (22)], [8, (11), (27)]. \square

13. CORRESPONDANCE BETWEEN INTERVAL AND 1-DIMENSIONAL HYPER INTERVAL

Let us consider a real number x . Now we state the propositions:

(82) $x \in [r, s]$ if and only if $1 \mapsto x \in \text{ClosedHyperInterval}(\langle r \rangle, \langle s \rangle)$.

PROOF: Set $a_1 = \langle r \rangle$. Set $b_1 = \langle s \rangle$. For every real number x such that $x \in [r, s]$ holds $1 \mapsto x \in \text{ClosedHyperInterval}(a_1, b_1)$ by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in \text{ClosedHyperInterval}(a_1, b_1)$ holds $x \in [r, s]$ by [24, (7)]. \square

(83) $x \in]r, s[$ if and only if $1 \mapsto x \in \text{OpenHyperInterval}(\langle r \rangle, \langle s \rangle)$.

PROOF: Set $a_1 = \langle r \rangle$. Set $b_1 = \langle s \rangle$. For every real number x such that $x \in]r, s[$ holds $1 \mapsto x \in \text{OpenHyperInterval}(a_1, b_1)$ by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in \text{OpenHyperInterval}(a_1, b_1)$ holds $x \in]r, s[$ by [24, (7)]. \square

(84) $x \in]r, s]$ if and only if $1 \mapsto x \in \text{LeftOpenHyperInterval}(\langle r \rangle, \langle s \rangle)$.

PROOF: Set $a_1 = \langle r \rangle$. Set $b_1 = \langle s \rangle$. For every real number x such that $x \in]r, s]$ holds $1 \mapsto x \in \text{LeftOpenHyperInterval}(a_1, b_1)$ by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in \text{LeftOpenHyperInterval}(a_1, b_1)$ holds $x \in]r, s]$ by [24, (7)]. \square

(85) $x \in [r, s[$ if and only if $1 \mapsto x \in \text{RightOpenHyperInterval}(\langle r \rangle, \langle s \rangle)$.

PROOF: Set $a_1 = \langle r \rangle$. Set $b_1 = \langle s \rangle$. For every real number x such that $x \in [r, s[$ holds $1 \mapsto x \in \text{RightOpenHyperInterval}(a_1, b_1)$ by [4, (2)], [24, (7)]. For every real number x such that $1 \mapsto x \in \text{RightOpenHyperInterval}(a_1, b_1)$ holds $x \in [r, s[$ by [24, (7)]. \square

14. CORRESPONDANCE BETWEEN MEASURABLE RECTANGLE AND PRODUCT

From now on n denotes a non zero natural number.

Now we state the propositions:

(86) Let us consider an n -tuple s of the set of all open real bounded intervals. Then there exist elements a, b of \mathcal{R}^n such that for every natural number i such that $i \in \text{Seg } n$ holds $s(i) =]a(i), b(i)[$.

PROOF: $s \in (\text{the set of all open real bounded intervals})^{\text{Seg } n}$. Consider f being a function such that $s = f$ and $\text{dom } f = \text{Seg } n$ and $\text{rng } f \subseteq \text{the set of all open real bounded intervals}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \text{there exists an element } f \text{ of } \mathbb{R} \times \mathbb{R} \text{ such that } f = \$_2 \text{ and } s(\$_1) =](f)_1, (f)_2[$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\text{len } f = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \text{Seg } n$ holds $(z)_1(i) = (f_i)_1$ and $(z)_2(i) = (f_i)_2$. Reconsider $a = (z)_1, b = (z)_2$ as an element of \mathcal{R}^n . For every natural number i such that $i \in \text{Seg } n$ holds $s(i) =]a(i), b(i)[$. \square

(87) Let us consider an element x of $\text{Product}(n, \text{the set of all open real bounded intervals})$. Then there exist elements a, b of \mathcal{R}^n such that for every element t of $\mathcal{R}^n, t \in x$ iff for every natural number i such that

$i \in \text{Seg } n$ holds $t(i) \in]a(i), b(i)[$. The theorem is a consequence of (62) and (86).

- (88) Let us consider an n -tuple s of the set of all left open real bounded intervals. Then there exist elements a, b of \mathcal{R}^n such that for every natural number i such that $i \in \text{Seg } n$ holds $s(i) =]a(i), b(i)[$.

PROOF: $s \in (\text{the set of all left open real bounded intervals})^{\text{Seg } n}$. Consider f being a function such that $s = f$ and $\text{dom } f = \text{Seg } n$ and $\text{rng } f \subseteq \text{the set of all left open real bounded intervals}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \text{there exists an element } f \text{ of } \mathbb{R} \times \mathbb{R} \text{ such that } f = \$_2 \text{ and } s(\$_1) = [(f)_1, (f)_2]$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\text{len } f = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \text{Seg } n$ holds $(z)_1(i) = (f_i)_1$ and $(z)_2(i) = (f_i)_2$. Reconsider $a = (z)_1, b = (z)_2$ as an element of \mathcal{R}^n . For every natural number i such that $i \in \text{Seg } n$ holds $s(i) =]a(i), b(i)[$. \square

- (89) Let us consider an element x of $\text{Product}(n, \text{the set of all left open real bounded intervals})$. Then there exist elements a, b of \mathcal{R}^n such that for every element t of $\mathcal{R}^n, t \in x$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in]a(i), b(i)[$. The theorem is a consequence of (62) and (88).

- (90) Let us consider an n -tuple s of the set of all right open real bounded intervals. Then there exist elements a, b of \mathcal{R}^n such that for every natural number i such that $i \in \text{Seg } n$ holds $s(i) = [a(i), b(i)[$.

PROOF: $s \in (\text{the set of all right open real bounded intervals})^{\text{Seg } n}$. Consider f being a function such that $s = f$ and $\text{dom } f = \text{Seg } n$ and $\text{rng } f \subseteq \text{the set of all right open real bounded intervals}$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \text{there exists an element } f \text{ of } \mathbb{R} \times \mathbb{R} \text{ such that } f = \$_2 \text{ and } s(\$_1) = [(f)_1, (f)_2]$. For every natural number i such that $i \in \text{Seg } n$ there exists an element d of $\mathbb{R} \times \mathbb{R}$ such that $\mathcal{P}[i, d]$ by [7, (3)]. Consider f being a finite sequence of elements of $\mathbb{R} \times \mathbb{R}$ such that $\text{len } f = n$ and for every natural number i such that $i \in \text{Seg } n$ holds $\mathcal{P}[i, f_i]$ from [25, Sch. 1]. Consider z being an element of $\mathcal{R}^n \times \mathcal{R}^n$ such that for every natural number i such that $i \in \text{Seg } n$ holds $(z)_1(i) = (f_i)_1$ and $(z)_2(i) = (f_i)_2$. Reconsider $a = (z)_1, b = (z)_2$ as an element of \mathcal{R}^n . For every natural number i such that $i \in \text{Seg } n$ holds $s(i) = [a(i), b(i)[$. \square

- (91) Let us consider an element x of $\text{Product}(n, \text{the set of all right open real bounded intervals})$. Then there exist elements a, b of \mathcal{R}^n such that

for every element t of \mathcal{R}^n , $t \in x$ iff for every natural number i such that $i \in \text{Seg } n$ holds $t(i) \in [a(i), b(i)[$. The theorem is a consequence of (62) and (90).

- (92) $\text{MeasurableRectangleLeftOpenIntervals}(n) = \text{Product}(n, \text{the set of all left open real bounded intervals})$. The theorem is a consequence of (40) and (66).
- (93) $\text{MeasurableRectangleRightOpenIntervals}(n) = \text{Product}(n, \text{the set of all right open real bounded intervals})$. The theorem is a consequence of (46) and (67).

15. CHEBYSHEV DISTANCE

In the sequel n denotes a non zero natural number and x, y, z denote elements of \mathcal{R}^n .

Let us consider n . The functor $D_{\text{Chebyshev}}^n$ yielding a function from $\mathcal{R}^n \times \mathcal{R}^n$ into \mathbb{R} is defined by

(Def. 20) for every elements x, y of \mathcal{R}^n , $it(x, y) = \sup \text{rng}|x - y|$.

Now we state the propositions:

- (94) (i) the set of all $|x(i) - y(i)|$ where i is an element of $\text{Seg } n$ is real-membered, and

(ii) the set of all $|x(i) - y(i)|$ where i is an element of $\text{Seg } n = \text{rng}|x - y|$.

PROOF: Set S = the set of all $|x(i) - y(i)|$ where i is an element of $\text{Seg } n$. $S \subseteq \text{rng}|x - y|$ by [8, (27)], [6, (124)]. For every object t such that $t \in \text{rng}|x - y|$ holds $t \in S$ by [6, (124)], [8, (27)]. \square

- (95) There exists an extended real-membered set S such that

(i) S = the set of all $|x(i) - y(i)|$ where i is an element of $\text{Seg } n$, and

(ii) $(D_{\text{Chebyshev}}^n)(x, y) = \sup S$.

The theorem is a consequence of (94).

- (96) $(D_{\text{Chebyshev}}^n)(x, y) = |x - y|(\text{max-diff-index}(x, y))$.

PROOF: $(D_{\text{Chebyshev}}^n)(x, y) \leq |x - y|(\text{max-diff-index}(x, y))$ by [15, (5)]. \square

- (97) $(D_{\text{Chebyshev}}^n)(x, y) = 0$ if and only if $x = y$.

PROOF: Consider S being an extended real-membered set such that $S =$ the set of all $|x(i) - y(i)|$ where i is an element of $\text{Seg } n$ and

$(D_{\text{Chebyshev}}^n)(x, y) = \sup S$. $S = \{0\}$ by [19, (2)], [3, (53)], [4, (1)]. \square

- (98) $(D_{\text{Chebyshev}}^n)(x, y) = (D_{\text{Chebyshev}}^n)(y, x)$. The theorem is a consequence of (1).

(99) $(D_{\text{Chebyshev}}^n)(x, y) \leq (D_{\text{Chebyshev}}^n)(x, z) + (D_{\text{Chebyshev}}^n)(z, y).$

PROOF: Reconsider $s_1 = \sup \text{rng}|x - y|$, $s_2 = \sup \text{rng}|x - z|$, $s_3 = \sup \text{rng}|z - y|$ as a real number. $s_1 \leq s_2 + s_3$ by [8, (27)], [5, (56)], [6, (124)], (2). \square

(100) $D_{\text{Chebyshev}}^n$ is a metric of \mathcal{R}^n . The theorem is a consequence of (97), (98), and (99).

(101) $\rho^2([0, 0], [1, 1]) = \sqrt{2}.$

(102) $(D_{\text{Chebyshev}}^2)([0, 0], [1, 1]) = 1.$

PROOF: Consider S being an extended real-membered set such that $S =$ the set of all $|[0, 0](i) - [1, 1](i)|$ where i is an element of Seg 2 and $(D_{\text{Chebyshev}}^2)([0, 0], [1, 1]) = \sup S$. $S = \{0 - 1\}$ by [4, (2), (44)]. \square

Let us consider elements x, y of \mathcal{R}^1 . Now we state the propositions:

(103) $(D_{\text{Chebyshev}}^1)(x, y) = |x(1) - y(1)|.$

PROOF: Consider S being an extended real-membered set such that $S =$ the set of all $|x(i) - y(i)|$ where i is an element of Seg 1 and $(D_{\text{Chebyshev}}^1)(x, y) = \sup S$. $S = \{|x(1) - y(1)|\}$ by [4, (2)]. \square

(104) $\rho^1(x, y) = |x(1) - y(1)|.$

Now we state the propositions:

(105) $\rho^1 = D_{\text{Chebyshev}}^1$. The theorem is a consequence of (104) and (103).

(106) $\rho^2 \neq D_{\text{Chebyshev}}^2$. The theorem is a consequence of (101) and (102).

Let n be a non zero natural number. The functor $L_\infty(n)$ yielding a strict metric space is defined by the term

(Def. 21) $\langle \mathcal{R}^n, D_{\text{Chebyshev}}^n \rangle.$

Let us observe that $L_\infty(n)$ is non empty.

The functor $\mathcal{E}_\infty^n(n)$ yielding a strict real linear topological structure is defined by

(Def. 22) the topological structure of $it = (L_\infty(n))_{\text{top}}$ and the RLS structure of $it = \mathbb{R}_{\mathbb{R}}^{\text{Seg } n}.$

Now we state the proposition:

(107) The RLS structure of $\mathcal{E}_{\mathbb{T}}^n =$ the RLS structure of $\mathcal{E}_\infty^n(n).$

Let n be a non zero natural number. Let us note that $\mathcal{E}_\infty^n(n)$ is non empty.

Now we state the propositions:

(108) Let us consider an element x of \mathcal{R}^0 . Then

(i) $\text{Intervals}(x, r)$ is empty, and

(ii) $\prod \text{Intervals}(x, r) = \{\emptyset\}.$

(109) If $r \leq 0$, then $\prod \text{Intervals}(x, r)$ is empty.

In the sequel p denotes an element of $L_\infty(n)$.

Let n be a non zero natural number and p be an element of $L_\infty(n)$. The functor ${}^{\textcircled{a}}p$ yielding an element of \mathcal{R}^n is defined by the term

(Def. 23) p .

Now we state the propositions:

(110) $\text{Ball}(p, r) = \prod \text{Intervals}({}^{\textcircled{a}}p, r)$. The theorem is a consequence of (109), (95), and (96).

(111) Let us consider a point e of \mathcal{E}^n . If $e = p$, then $\text{Ball}(p, r) = \text{OpenHypercube}(e, r)$. The theorem is a consequence of (110).

Let n be a non zero natural number, p be an element of $L_\infty(n)$, and r be a negative real number. Let us note that $\overline{\text{Ball}}(p, r)$ is empty.

Now we state the propositions:

(112) Let us consider an object t . Then $t \in \overline{\text{Ball}}(p, r)$ if and only if there exists a function f such that $t = f$ and $\text{dom } f = \text{Seg } n$ and for every natural number i such that $i \in \text{Seg } n$ holds $f(i) \in [({}^{\textcircled{a}}p)(i) - r, ({}^{\textcircled{a}}p)(i) + r]$. The theorem is a consequence of (95).

(113) Let us consider a point p of \mathcal{E}_T^n , and an element q of $L_\infty(n)$. Suppose $q = p$. Then $\overline{\text{Ball}}(q, r) = \text{ClosedHypercube}(p, n \mapsto r)$.

PROOF: For every object x such that $x \in \overline{\text{Ball}}(q, r)$ holds $x \in \text{ClosedHypercube}(p, n \mapsto r)$ by (112), [6, (57), (93)], [10, (22)]. For every object x such that $x \in \text{ClosedHypercube}(p, n \mapsto r)$ holds $x \in \overline{\text{Ball}}(q, r)$ by [10, (22)], [6, (131), (124), (57)]. \square

(114) $\text{Ball}(p, r) = \text{OpenHyperInterval}({}^{\textcircled{a}}p - n \mapsto r, {}^{\textcircled{a}}p + n \mapsto r)$. The theorem is a consequence of (80) and (110).

(115) $\overline{\text{Ball}}(p, r) = \text{ClosedHyperInterval}({}^{\textcircled{a}}p - n \mapsto r, {}^{\textcircled{a}}p + n \mapsto r)$. The theorem is a consequence of (81) and (113).

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