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Cousin's Lemma

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Summary. We formalize, in two different ways, that "the n-dimensional Euclidean metric space is a complete metric space" (version 1. with the results obtained in [13], [26], [25] and version 2., the results obtained in [13], [14], (*registrations*) [24]).

With the Cantor's theorem - in complete metric space (proof by Karol Pąk in [22]), we formalize "The Nested Intervals Theorem in 1-dimensional Euclidean metric space".

Pierre Cousin's proof in 1892 [18] the lemma, published in 1895 [9] states that:

> "Soit, sur le plan YOX, une aire connexe S limitée par un contour fermé simple ou complexe; on suppose qu'à chaque point de S ou de son périmètre correspond un cercle, de rayon non nul, ayant ce point pour centre : il est alors toujours possible de subdiviser S en régions, en nombre fini et assez petites pour que chacune d'elles soit complétement intérieure au cercle correspondant à un point convenablement choisi dans S ou sur son périmètre."

(In the plane YOX let S be a connected area bounded by a closed contour, simple or complex; one supposes that at each point of S or its perimeter there is a circle, of non-zero radius, having this point as its centre; it is then always possible to subdivide S into regions, finite in number and sufficiently small for each one of them to be entirely inside a circle corresponding to a suitably chosen point in S or on its perimeter) [23].

Cousin's Lemma, used in Henstock and Kurzweil integral [29] (generalized Riemann integral), state that: "for any gauge δ, there exists at least one δ-fine tagged partition". In the last section, we formalize this theorem. We use the suggestions given to the Cousin's Theorem p.11 in [5] and with notations: [4], [29], [19], [28] and [12].

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1. Preliminaries

Now we state the proposition:

(1) Let us consider non empty, increasing finite sequences p, q of elements of $\mathbb{R}$. Suppose $p(\operatorname{len} p) < q(1)$. Then $p \frown q$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$.

Let us consider real numbers a, b. Now we state the propositions:

(2) If $1 < a$ and $0 < b < 1$, then $\log_a b < 0$.

(3) If $1 < a$ and $1 < b$, then $0 < \log_a b$.

Let us consider a finite sequence p and a natural number i.

Let us assume that $i \in \operatorname{dom} p$. Now we state the propositions:

(4) (i) $i = 1$, or

(ii) $1 < i$.

(5) (i) $i = \operatorname{len} p$, or

(ii) $i < \operatorname{len} p$.

Now we state the propositions:

(6) Let us consider an object x. Then $\prod\langle\{x\}\rangle = \{\langle x\rangle\}$.

(7) Let us consider an element x of $\mathcal{R}^1$. Then there exists a real number r_3 such that $x = \langle r_3\rangle$.

(8) Let us consider a real number a. Then $\langle a\rangle$ is a point of $\mathcal{E}^1$.

(9) Let us consider real numbers a, b. If $a \leqslant b$, then $a \leqslant \frac{a+b}{2} \leqslant b$.

(10) Let us consider real numbers a, b, c. If $a \leqslant b < c$, then $a < \frac{b+c}{2}$.

Let us consider real numbers a, b. Now we state the propositions:

(11) If $a < b$, then $\frac{a+b}{2} < b$.

(12) If $a \leqslant b$, then $[a, b]$ is a non empty, compact subset of $\mathbb{R}$.

(13) Let us consider a finite sequence f. Suppose $2 \leqslant \operatorname{len} f$. Then $f_{\restriction 1}(\operatorname{len} f_{\restriction 1}) = f(\operatorname{len} f)$.

2. $\mathcal{E}^n$ is Complete - Proof Version 1

From now on n denotes a natural number, s_1 denotes a sequence of $\mathcal{E}^n$, and s_2 denotes a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

Now we state the propositions:

(14) Let us consider elements x, y of $\mathcal{E}^n$, and points g, h of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $x = g$ and $y = h$, then $\rho(x, y) = \|g - h\|$.

(15) (i) s_1 is a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and

(ii) s_2 is a sequence of $\mathcal{E}^n$.

Proof: s_1 is a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ by [10, (67), (22)]. s_2 is a sequence of $\mathcal{E}^n$ by [10, (22), (67)]. □

Let us assume that $s_1 = s_2$. Now we state the propositions:

(16) s_1 is Cauchy if and only if s_2 is Cauchy sequence by norm. The theorem is a consequence of (14).

(17) s_1 is convergent if and only if s_2 is convergent. The theorem is a consequence of (14).

(18) Let us consider a sequence S_1 of $\mathcal{E}^n$. If S_1 is Cauchy, then S_1 is convergent. The theorem is a consequence of (15), (16), and (17).

(19) $\mathcal{E}^n$ is complete.

3. $\mathcal{E}^n$ is Complete - Proof Version 2

Now we state the propositions:

(20) The distance by norm of $\langle \mathcal{E}^n, \|\cdot\| \rangle = \rho^n$. The theorem is a consequence of (14).

(21) MetricSpaceNorm$\langle \mathcal{E}^n, \|\cdot\| \rangle = \mathcal{E}^n$. The theorem is a consequence of (20).

(22) $\mathcal{E}^n$ is complete. The theorem is a consequence of (21).

Let n be a natural number. Let us note that $\mathcal{E}^n$ is complete.

4. The Nested Intervals Theorem (1-dimensional Euclidean Space)

Let a, b be sequences of real numbers. The functor IntervalSeq(a, b) yielding a sequence of subsets of $\mathcal{R}^1$ is defined by

(Def. 1) for every natural number i, $it(i) = \prod \langle [a(i), b(i)] \rangle$.

Now we state the propositions:

(23) Let us consider sequences a, b of real numbers, and a natural number i. Then $(\text{IntervalSeq}(a, b))(i) = \prod \langle [a(i), b(i)] \rangle$.

(24) Let us consider sequences a, b of real numbers. Then $\mathrm{IntervalSeq}(a, b)$ is a sequence of subsets of $\mathcal{E}^1$.

(25) $\prod\langle\mathbb{R}\rangle = \mathcal{R}^1$.

(26) Let us consider real numbers a, b, and points x_1, x_2 of $\mathcal{E}^1$. Suppose $x_1 = \langle a \rangle$ and $x_2 = \langle b \rangle$. Then $\rho(x_1, x_2) = |a - b|$.

(27) Let us consider real numbers a, b, and a subset S of $\mathcal{E}^1$. Suppose $a \leqslant b$ and $S = \prod\langle[a, b]\rangle$. Let us consider points x, y of $\mathcal{E}^1$. If x, $y \in S$, then $\rho(x, y) \leqslant b - a$.
PROOF: Set $s = \prod\langle[a, b]\rangle$. For every points x, y of $\mathcal{E}^1$ such that x, $y \in s$ holds $\rho(x, y) \leqslant b - a$ by (6), [10, (67), (22)], (7). □

(28) Let us consider real numbers a, b, and a subset S of $\mathcal{E}^1$. If $a \leqslant b$ and $S = \prod\langle[a, b]\rangle$, then S is bounded.
PROOF: Set $s = \prod\langle[a, b]\rangle$. There exists a real number r such that $0 < r$ and for every points x, y of $\mathcal{E}^1$ such that x, $y \in s$ holds $\rho(x, y) \leqslant r$ by (6), [10, (67), (22)], (7). □

Let us consider sequences a, b of real numbers.

Let us assume that for every natural number i, $a(i) \leqslant b(i)$ and $a(i) \leqslant a(i+1)$ and $b(i+1) \leqslant b(i)$. Now we state the propositions:

(29) $\mathrm{IntervalSeq}(a, b)$ is a non-empty, pointwise bounded, closed sequence of subsets of $\mathcal{E}^1$.
PROOF: Reconsider $s = \mathrm{IntervalSeq}(a, b)$ as a sequence of subsets of $\mathcal{E}^1$. s is non-empty by (23), [1, (26)], [3, (2)]. s is pointwise bounded by (23), (6), [10, (67), (22)]. s is closed by (23), [10, (67), (22)], (25). □

(30) $\mathrm{IntervalSeq}(a, b)$ is non ascending. The theorem is a consequence of (23).

(31) Let us consider real numbers a, b, x. If $a \leqslant x \leqslant b$, then $\langle x \rangle \in \prod\langle[a, b]\rangle$.
PROOF: Reconsider $P = \langle x \rangle$ as a point of $\mathcal{E}^1$. There exists a function g such that $g = P$ and $\mathrm{dom}\, g = \mathrm{dom}\langle[a, b]\rangle$ and for every object y such that $y \in \mathrm{dom}\langle[a, b]\rangle$ holds $g(y) \in \langle[a, b]\rangle(y)$ by [3, (2)]. □

(32) Let us consider real numbers a, b, and a subset S of $\mathcal{E}^1$. If $a \leqslant b$ and $S = \prod\langle[a, b]\rangle$, then $\varnothing S = b - a$. The theorem is a consequence of (28), (31), (27), (8), and (26).

(33) Let us consider sequences a, b of real numbers. Suppose for every natural number i, $a(i) \leqslant b(i)$ and a is non-decreasing and b is non-increasing. Then

 (i) a is convergent, and

 (ii) b is convergent.

(34) Let us consider sequences a, b of real numbers. Suppose $a(0) \leqslant b(0)$ and for every natural number i, $a(i + 1) = a(i)$ and $b(i + 1) = \frac{a(i)+b(i)}{2}$ or

$a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Let us consider a natural number i. Then $a(i) \leqslant b(i)$.

Proof: Define $\mathcal{P}$[object] $\equiv$ there exists a natural number i such that $\$_1 = i$ and $a(i) \leqslant b(i)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

Let us consider sequences a, b of real numbers, a sequence S of subsets of $\mathcal{E}^1$, and a natural number i. Now we state the propositions:

(35) Suppose $a(0) \leqslant b(0)$ and $S = \text{IntervalSeq}(a, b)$ and for every natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then

(i) $a(i) \leqslant b(i)$, and

(ii) $a(i) \leqslant a(i+1)$, and

(iii) $b(i+1) \leqslant b(i)$, and

(iv) $(\varnothing S)(i) = b(i) - a(i)$.

The theorem is a consequence of (34), (9), (24), (23), and (32).

(36) Suppose $a(0) = b(0)$ and $S = \text{IntervalSeq}(a, b)$ and for every natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then

(i) $a(i) = a(0)$, and

(ii) $b(i) = b(0)$, and

(iii) $(\varnothing S)(i) = 0$.

Proof: Define $\mathcal{P}$[natural number] $\equiv a(\$_1) = a(0)$ and $b(\$_1) = b(0)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

(37) Let us consider sequences a, b of real numbers. Suppose for every natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Let us consider a natural number i, and a real number r. If $r = 2^i$ and $r \neq 0$, then $b(i) - a(i) \leqslant \frac{b(0)-a(0)}{r}$.

Proof: Define $\mathcal{P}$[object] $\equiv$ there exists a natural number i and there exists a real number r such that $\$_1 = i$ and $r = 2^i$ and $r \neq 0$ and $b(i) - a(i) \leqslant \frac{b(0)-a(0)}{r}$. $\mathcal{P}[0]$ by [17, (4)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [17, (87), (6)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. Consider i_1 being a natural number, r_1 being a real number such that $i = i_1$ and $r_1 = 2^{i_1}$ and $r_1 \neq 0$ and $b(i_1) - a(i_1) \leqslant \frac{b(0)-a(0)}{r_1}$. □

(38) Let us consider sequences a, b of real numbers, and a sequence S of subsets of $\mathcal{E}^1$. Suppose $a(0) \leqslant b(0)$ and $S = \text{IntervalSeq}(a, b)$ and for every

natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then

(i) $\varnothing S$ is convergent, and

(ii) $\lim \varnothing S = 0$.

The theorem is a consequence of (36), (35), (34), (33), (3), and (37).

(39) Let us consider sequences a, b of real numbers. Suppose $a(0) \leqslant b(0)$ and for every natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then $\bigcap \mathrm{IntervalSeq}(a, b)$ is not empty. The theorem is a consequence of (24), (35), (29), (30), and (38).

(40) Let us consider a real number r, and sequences a, b of real numbers. Suppose $0 < r$ and $a(0) \leqslant b(0)$ and for every natural number i, $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then there exists a real number c such that

(i) for every natural number j, $a(j) \leqslant c \leqslant b(j)$, and

(ii) there exists a natural number k such that $c-r < a(k)$ and $b(k) < c+r$.

The theorem is a consequence of (39), (23), (24), (35), (29), and (38).

5. Tagged Partition

Now we state the propositions:

(41) Let us consider a non empty, closed interval subset I of $\mathbb{R}$. Then there exist real numbers a, b such that

(i) $a \leqslant b$, and

(ii) $I = [a, b]$.

(42) Let us consider non empty, closed interval subsets I_1, I_2 of $\mathbb{R}$. Suppose $\sup I_1 = \inf I_2$. Then there exist real numbers a, b, c such that

(i) $a \leqslant c \leqslant b$, and

(ii) $I_1 = [a, c]$, and

(iii) $I_2 = [c, b]$.

The theorem is a consequence of (41).

Let A be a non empty, closed interval subset of $\mathbb{R}$ and D be a partition of A. The set of tagged partitions of D yielding a subset of $\mathbb{R}^*$ is defined by

(Def. 2) for every object x, $x \in it$ iff there exists a non empty, non-decreasing finite sequence s of elements of $\mathbb{R}$ such that $x = s$ and $\mathrm{dom}\, s = \mathrm{dom}\, D$ and for every natural number i such that $i \in \mathrm{dom}\, s$ holds $s(i) \in \mathrm{divset}(D, i)$.

Now we state the propositions:

(43) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a partition D of A. Then $D \in$ the set of tagged partitions of D.
PROOF: For every natural number i such that $i \in \operatorname{dom} D$ holds $D(i) \in \operatorname{divset}(D, i)$ by [15, (19)], (4). $\square$

(44) Let us consider real numbers a, b, and a non empty, closed interval subset I_4 of $\mathbb{R}$. If $I_4 = [a, b]$, then $\langle b \rangle$ is a partition of I_4.
PROOF: $\langle b \rangle$ is a partition of I_4 by [3, (39)], [15, (19)]. $\square$

Let I be a non empty, closed interval subset of $\mathbb{R}$ and φ be a positive yielding function from I into $\mathbb{R}$.

A tagged partition of I and φ is defined by

(Def. 3) there exists a partition D of I and there exists an element T of the set of tagged partitions of D such that $it = \langle D, T \rangle$.

Let T_1 be a tagged partition of I and φ. We say that T_1 is δ-fine if and only if

(Def. 4) there exists a partition D of I and there exists an element T of the set of tagged partitions of D such that $T_1 = \langle D, T \rangle$ and for every natural number i such that $i \in \operatorname{dom} D$ holds $\operatorname{vol}(\operatorname{divset}(D, i)) \leqslant \varphi(T(i))$.

6. PARTITION COMPOSITION

Let us consider a real number r. Now we state the propositions:

(45) (i) $\sup\{r\} = r$, and

(ii) $\inf\{r\} = r$.

(46) $\operatorname{vol}(\{r\}) = 0$. The theorem is a consequence of (45).

(47) Let us consider non empty, closed interval subsets I_1, I_2 of $\mathbb{R}$, and a positive yielding function φ from I_1 into $\mathbb{R}$. Suppose $I_2 \subseteq I_1$. Then $\varphi \restriction I_2$ is a positive yielding function from I_2 into $\mathbb{R}$.

(48) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, and a real number c. Suppose $c \in I$. Then

(i) $[\inf I, c]$ is a non empty, closed interval subset of $\mathbb{R}$, and

(ii) $[c, \sup I]$ is a non empty, closed interval subset of $\mathbb{R}$, and

(iii) $\sup[\inf I, c] = \inf[c, \sup I]$.

The theorem is a consequence of (41).

Let I_5, I_6 be non empty, closed interval subsets of $\mathbb{R}$, D_4 be a partition of I_5, and D_6 be a partition of I_6. Assume $\sup I_5 \leqslant \inf I_6$. The functor $D_4 \cdot D_6$

yielding a non empty, increasing finite sequence of elements of $\mathbb{R}$ is defined by the term

(Def. 5) $\begin{cases} D_4 \frown D_6, & \textbf{if } D_6(1) \neq \sup I_5, \\ D_4 \frown D_{6\restriction 1}, & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(49) Let us consider non empty, closed interval subsets I_5, I_6 of $\mathbb{R}$, a partition D_4 of I_5, and a partition D_6 of I_6. Suppose $\sup I_5 = \inf I_6$ and $\operatorname{len} D_6 = 1$ and $D_6(1) = \inf I_6$. Then $D_4 \cdot D_6 = D_4$.

(50) Let us consider non empty, closed interval subsets I_1, I_2, I of $\mathbb{R}$. Suppose $\sup I_1 \leqslant \inf I_2$ and $\inf I \leqslant \inf I_1$ and $\sup I_2 \leqslant \sup I$. Then $I_1 \cup I_2 \subseteq I$.

(51) Let us consider non empty, closed interval subsets I_1, I_2, I of $\mathbb{R}$, a partition D_1 of I_1, and a partition D_2 of I_2. Suppose $\sup I_1 \leqslant \inf I_2$ and $I = [\inf I_1, \sup I_2]$. Then $D_1 \cdot D_2$ is a partition of I. The theorem is a consequence of (50).

(52) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, and a partition D of I. Then the set of tagged partitions of D is not empty.

(53) Let us consider a non empty, increasing finite sequence s of elements of $\mathbb{R}$, and a real number r. Suppose $s(\operatorname{len} s) < r$. Then $s \frown \langle r \rangle$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$. The theorem is a consequence of (1).

(54) Let us consider non empty, increasing finite sequences s_1, s_2 of elements of $\mathbb{R}$, and a real number r. Suppose $s_1(\operatorname{len} s_1) < r < s_2(1)$. Then $(s_1 \frown \langle r \rangle) \frown s_2$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$. The theorem is a consequence of (53) and (1).

(55) Let us consider non empty, closed interval subsets I_1, I_2, I of $\mathbb{R}$. Suppose $\sup I_1 = \inf I_2$ and $I = I_1 \cup I_2$. Then

 (i) $\inf I = \inf I_1$, and

 (ii) $\sup I = \sup I_2$.

(56) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, and a partition D of I. Then

 (i) $\operatorname{divset}(D, 1) = [\inf I, D(1)]$, and

 (ii) for every natural number j such that $j \in \operatorname{dom} D$ and $j \neq 1$ holds $\operatorname{divset}(D, j) = [D(j-1), D(j)]$.

PROOF: For every natural number j such that $j \in \operatorname{dom} D$ and $j \neq 1$ holds $\operatorname{divset}(D, j) = [D(j-1), D(j)]$ by [12, (4)]. $\square$

(57) Let us consider a real number r, and finite sequences p, q of elements of $\mathbb{R}$. Then $\operatorname{len}((p \frown \langle r \rangle) \frown q) = \operatorname{len} p + \operatorname{len} q + 1$.

(58) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, and a partition D of I. Then every element of the set of tagged partitions of D is not empty. The theorem is a consequence of (43).

(59) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, a partition D of I, and an element T of the set of tagged partitions of D. Then $\operatorname{rng} T \subseteq \mathbb{R}$. The theorem is a consequence of (43).

Let I be a non empty, closed interval subset of $\mathbb{R}$, φ be a positive yielding function from I into $\mathbb{R}$, and T_1 be a tagged partition of I and φ. The functor T_1-partition yielding a partition of I is defined by

(Def. 6) there exists a partition D of I and there exists an element T of the set of tagged partitions of D such that $it = D$ and $T_1 = \langle D, T \rangle$.

7. Examples of Partitions

In the sequel r, s denote real numbers.

Now we state the proposition:

(60) Let us consider a function φ from $[r, s]$ into $]0, +\infty[$. Suppose $r \leqslant s$. Then the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$ is a family of subsets of $[r, s]_{\mathrm{T}}$.

Let us consider a function φ from $[r, s]$ into $]0, +\infty[$ and a family S of subsets of $[r, s]_{\mathrm{T}}$.

Let us assume that $r \leqslant s$ and $S =$ the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$. Now we state the propositions:

(61) S is a cover of $[r, s]_{\mathrm{T}}$.
Proof: $[r, s] \subseteq \bigcup S$ by [8, (3)]. □

(62) S is open.
Proof: For every subset P of $[r, s]_{\mathrm{T}}$ such that $P \in S$ holds P is open by [11, (17)], [20, (35)], [11, (15), (9), (10)]. □

(63) Suppose $S =$ the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$. Then S is connected.
Proof: For every subset X of $[r, s]_{\mathrm{T}}$ such that $X \in S$ holds X is connected by [16, (43)]. □

(64) Let us consider a function φ from $[r, s]$ into $]0, +\infty[$, and a family S of subsets of $[r, s]_{\mathrm{T}}$. Suppose $r \leqslant s$ and $S =$ the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$. Let us consider an interval cover I of S. Then

(i) I is a finite sequence of elements of $2^{\mathbb{R}}$, and

(ii) $\operatorname{rng} I \subseteq S$, and

(iii) $\bigcup \operatorname{rng} I = [r, s]$, and

(iv) for every natural number n such that $1 \leqslant n$ holds if $n \leqslant \operatorname{len} I$, then I_n is not empty and if $n + 1 \leqslant \operatorname{len} I$, then $\inf I_n \leqslant \inf I_{n+1}$ and $\sup I_n \leqslant \sup I_{n+1}$ and $\inf I_{n+1} < \sup I_n$ and if $n + 2 \leqslant \operatorname{len} I$, then $\sup I_n \leqslant \inf I_{n+2}$, and

(v) if $[r, s] \in S$, then $I = \langle [r, s] \rangle$, and

(vi) if $[r, s] \notin S$, then there exists a real number p such that $r < p \leqslant s$ and $I(1) = [r, p[$ and there exists a real number p such that $r \leqslant p < s$ and $I(\operatorname{len} I) =]p, s]$ and for every natural number n such that $1 < n < \operatorname{len} I$ there exist real numbers p, q such that $r \leqslant p < q \leqslant s$ and $I(n) =]p, q[$.

The theorem is a consequence of (61), (62), and (63).

(65) Let us consider real numbers r, s, t, x. Then

(i) if $r \leqslant x - t$ and $x + t \leqslant s$, then $]x - t, x + t[\cap [r, s] =]x - t, x + t[$, and

(ii) if $r \leqslant x - t$ and $s < x + t$, then $]x - t, x + t[\cap [r, s] =]x - t, s]$, and

(iii) if $x - t < r$ and $x + t \leqslant s$, then $]x - t, x + t[\cap [r, s] = [r, x + t[$, and

(iv) if $x - t < r$ and $s < x + t$, then $]x - t, x + t[\cap [r, s] = [r, s]$.

(66) Let us consider real numbers r, s, t, x, and a subset X_1 of $\mathbb{R}$. Suppose $0 < t$ and $r \leqslant x \leqslant s$ and $X_1 =]x - t, x + t[\cap [r, s]$. Then

(i) if $r \leqslant x - t$ and $x + t \leqslant s$, then $\inf X_1 = x - t$ and $\sup X_1 = x + t$, and

(ii) if $r \leqslant x - t$ and $s < x + t$, then $\inf X_1 = x - t$ and $\sup X_1 = s$, and

(iii) if $x - t < r$ and $x + t \leqslant s$, then $\inf X_1 = r$ and $\sup X_1 = x + t$, and

(iv) if $x - t < r$ and $s < x + t$, then $\inf X_1 = r$ and $\sup X_1 = s$.

The theorem is a consequence of (65).

Let us consider real numbers a, b, c, non empty, compact subsets I_5, I_6 of $\mathbb{R}$, a partition D_4 of I_5, a partition D_6 of I_6, and natural numbers i, j.

Let us assume that $a \leqslant c \leqslant b$ and $I_5 = [a, c]$ and $I_6 = [c, b]$. Now we state the propositions:

(67) Suppose $i \in \operatorname{dom} D_4$ and $j \in \operatorname{dom} D_6$. Then

(i) if $i < \operatorname{len} D_4$, then $D_4(i) < D_6(j)$, and

(ii) if $i = \operatorname{len} D_4$ and $c < D_6(1)$, then $D_4(i) < D_6(j)$, and

(iii) if $D_6(1) = c$, then $D_4(\operatorname{len} D_4) = D_6(1)$.

PROOF: If $i < \operatorname{len} D_4$, then $D_4(i) < D_6(j)$ by [3, (3)]. If $i = \operatorname{len} D_4$ and $c < D_6(1)$, then $D_4(i) < D_6(j)$ by [7, (6)], [3, (91)]. □

(68) If $i \in \operatorname{dom} D_4$ and $j \in \operatorname{dom} D_6$, then if $c < D_6(1)$, then $D_4(i) < D_6(j)$. The theorem is a consequence of (67).

(69) Let us consider real numbers a, b, c, and non empty, compact subsets I_4, I_5, I_6 of $\mathbb{R}$. Suppose $a \leqslant c \leqslant b$ and $I_4 = [a, b]$ and $I_5 = [a, c]$ and $I_6 = [c, b]$. Let us consider a partition D_4 of I_5, and a partition D_6 of I_6. Suppose $c < D_6(1)$. Then $D_4 \frown D_6$ is a partition of I_4.

PROOF: Set $D_5 = D_4 \frown D_6$. For every extended reals e_1, e_2 such that e_1, $e_2 \in \operatorname{dom} D_5$ and $e_1 < e_2$ holds $D_5(e_1) < D_5(e_2)$ by [3, (25)], (68), [2, (11)], [3, (1)]. $\operatorname{rng} D_5 \subseteq I_4$ by [3, (31)]. $D_5(\operatorname{len} D_5) = \sup I_4$ by [3, (3), (22)], [15, (19)]. □

(70) Let us consider real numbers a, b, and a non empty, closed interval subset I_4 of $\mathbb{R}$. Suppose $a \leqslant b$ and $I_4 = [a, b]$. Let us consider a partition D_3 of I_4. If $\operatorname{len} D_3 = 1$, then $D_3 = \langle b \rangle$.

(71) Let us consider real numbers a, b, a non empty, compact subset I_4 of $\mathbb{R}$, and a partition D_3 of I_4. Suppose $2 \leqslant \operatorname{len} D_3$. Then $D_{3\lfloor 1}$ is a partition of I_4.

PROOF: Set $D = D_{3\lfloor 1}$. D is a non empty, increasing finite sequence of elements of $\mathbb{R}$ by [3, (60)]. $\operatorname{rng} D \subseteq I_4$ by [7, (33)]. $D(\operatorname{len} D) = \sup I_4$ by [3, (3)]. □

(72) Let us consider real numbers a, b. Suppose $a < b$. Then $\langle a, b \rangle$ is a non empty, increasing finite sequence of elements of $\mathbb{R}$.

PROOF: Set $s = \langle a, b \rangle$. s is increasing by [3, (44), (2)]. □

(73) Let us consider real numbers a, b, and a non empty, closed interval subset I_4 of $\mathbb{R}$. Suppose $a < b$ and $I_4 = [a, b]$. Then $\langle a, b \rangle$ is a partition of I_4.

PROOF: $\langle a, b \rangle$ is a partition of I_4 by (72), [6, (127)], [3, (44)], [15, (19)]. □

8. Cousin's Lemma

Now we state the proposition:

(74) Let us consider real numbers a, b, and a positive yielding function φ from $[a, b]$ into $\mathbb{R}$. Suppose $a \leqslant b$. Then there exists a non empty, increasing finite sequence x of elements of $\mathbb{R}$ and there exists a non empty finite sequence t of elements of $\mathbb{R}$ such that $x(1) = a$ and $x(\operatorname{len} x) = b$ and $t(1) = a$ and $\operatorname{dom} x = \operatorname{dom} t$ and for every natural number i such that $i - 1$, $i \in \operatorname{dom} t$ holds $t(i) - \varphi(t(i)) \leqslant x(i-1) \leqslant t(i)$ and for every natural number i such that $i \in \operatorname{dom} t$ holds $t(i) \leqslant x(i) \leqslant t(i) + \varphi(t(i))$.

Proof: Define $\mathcal{P}[\text{object}] \equiv$ there exists a non empty, increasing finite sequence x of elements of $\mathbb{R}$ and there exists a non empty finite sequence t of elements of $\mathbb{R}$ such that $x(1) = a$ and $x(\operatorname{len} x) = \$_1$ and $t(1) = a$ and $\operatorname{dom} x = \operatorname{dom} t$ and for every natural number i such that $i-1$, $i \in \operatorname{dom} t$ holds $t(i) - \varphi(t(i)) \leqslant x(i-1) \leqslant t(i)$ and for every natural number i such that $i \in \operatorname{dom} t$ holds $t(i) \leqslant x(i) \leqslant t(i) + \varphi(t(i))$. Consider C being a set such that for every object x, $x \in C$ iff $x \in [a, b]$ and $\mathcal{P}[x]$. For every object x such that $x \in C$ holds x is real. Reconsider $c = \sup C$ as a real number. $c \in [a, b]$. Consider d being an element of $\overline{\mathbb{R}}$ such that $d \in C$ and $c - \varphi(c) < d$. Consider D_0 being a non empty, increasing finite sequence of elements of $\mathbb{R}$, T_0 being a non empty finite sequence of elements of $\mathbb{R}$ such that $D_0(1) = a$ and $D_0(\operatorname{len} D_0) = d$ and $T_0(1) = a$ and $\operatorname{dom} D_0 = \operatorname{dom} T_0$ and for every natural number i such that $i-1$, $i \in \operatorname{dom} T_0$ holds $T_0(i) - \varphi(T_0(i)) \leqslant D_0(i-1) \leqslant T_0(i)$ and for every natural number i such that $i \in \operatorname{dom} T_0$ holds $T_0(i) \leqslant D_0(i) \leqslant T_0(i) + \varphi(T_0(i))$. $c \in C$ and $\mathcal{P}[c]$ by (1), [27, (32)], [3, (22), (39), (1)]. $c = b$ by (1), [27, (32)], [3, (22), (39), (1)]. □

(75) Cousin's Lemma:

Let us consider a non empty, closed interval subset I of $\mathbb{R}$, and a positive yielding function φ from I into $\mathbb{R}$. Then there exists a tagged partition T_1 of I and φ such that T_1 is δ-fine.

Proof: Consider a, b being real numbers such that $a \leqslant b$ and $I = [a, b]$. Reconsider $r = \frac{1}{2}$ as a positive real number. Reconsider $\phi = r \cdot \varphi$ as a positive yielding function from I into $\mathbb{R}$. Consider x being a non empty, increasing finite sequence of elements of $\mathbb{R}$, t being a non empty finite sequence of elements of $\mathbb{R}$ such that $x(1) = a$ and $x(\operatorname{len} x) = b$ and $t(1) = a$ and $\operatorname{dom} x = \operatorname{dom} t$ and for every natural number i such that $i-1, i \in \operatorname{dom} t$ holds $t(i) - \phi(t(i)) \leqslant x(i-1) \leqslant t(i)$ and for every natural number i such that $i \in \operatorname{dom} t$ holds $t(i) \leqslant x(i) \leqslant t(i) + \phi(t(i))$. Reconsider $D = x$ as a partition of I. Reconsider $T = t$ as an element of the set of tagged partitions of D. Reconsider $T_1 = \langle D, T \rangle$ as a tagged partition of I and φ. T_1 is δ-fine by [15, (19)], (4), [8, (3)], [21, (20)]. □

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