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Altitude, Orthocenter of a Triangle and Triangulation

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Summary. We introduce the altitudes of a triangle (the cevians perpendicular to the opposite sides). Using the generalized Ceva's Theorem, we prove the existence and uniqueness of the orthocenter of a triangle [7]. Finally, we formalize in Mizar [1] some formulas [2] to calculate distance using triangulation.

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1. Preliminaries

From now on n denotes a natural number, i denotes an integer, r, s, t denote real numbers, A_1, B_1, C_1, D_1 denote points of $\mathcal{E}^n_{\rm T}$, L_1, L_2 denote elements of $\mathrm{Lines}(\mathcal{R}^n)$, and A, B, C denote points of $\mathcal{E}^2_{\rm T}$.

Now we state the propositions:

(1) If $0 < i \cdot r < r$, then $i = 1$.

(2) Let us consider an integer i. If $\frac{-3}{2} < i < \frac{1}{2}$, then $i = 0$ or $i = -1$.

(3) Suppose r is not zero and s is not zero and t is not zero. Then $(\frac{-r}{-s}) \cdot (\frac{-t}{-r}) \cdot (\frac{-s}{-t}) = 1$.

(4) If $0 < r < 2 \cdot \pi$, then $\sin(\frac{r}{2}) \neq 0$. The theorem is a consequence of (1).

(5) If $-2 \cdot \pi < r < 0$, then $\sin(\frac{r}{2}) \neq 0$. The theorem is a consequence of (4).

(6) $\tan(2 \cdot \pi - r) = -\tan r$.

(7) If $A_1 \in \mathrm{Line}(B_1, C_1)$ and $A_1 \neq C_1$, then $\mathrm{Line}(B_1, C_1) = \mathrm{Line}(A_1, C_1)$.

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(8) If $A_1 \neq C_1$ and $A_1 \in \mathrm{Line}(B_1, C_1)$, then $B_1 \in \mathrm{Line}(A_1, C_1)$.

(9) Suppose $A_1 \neq B_1$ and $A_1 \neq C_1$ and $|(A_1 - B_1, A_1 - C_1)| = 0$ and $L_1 = \mathrm{Line}(A_1, B_1)$ and $L_2 = \mathrm{Line}(A_1, C_1)$. Then $L_1 \perp L_2$.

(10) If $B_1 \neq C_1$ and $|(B_1 - A_1, B_1 - C_1)| = 0$, then $A_1 \neq C_1$.

(11) $|(A_1 - B_1, A_1 - C_1)| = |(B_1 - A_1, C_1 - A_1)|$.

(12) Suppose $B_1 \neq C_1$ and $r = -(\frac{|(B_1,C_1)|-|(C_1,C_1)|-|(A_1,B_1)|+|(A_1,C_1)|}{|(B_1-C_1,B_1-C_1)|})$ and $D_1 = r \cdot B_1 + (1 - r) \cdot C_1$. Then $|(D_1 - A_1, D_1 - C_1)| = 0$.

(13) If $A_1 \neq B_1$ and $C_1 = r \cdot A_1 + (1 - r) \cdot B_1$ and $C_1 = B_1$, then $r = 0$.

(14) (i) $|(B_1, C_1)| - |(C_1, C_1)| - |(A_1, B_1)| + |(A_1, C_1)| = |(C_1 - A_1, B_1 - C_1)|$, and

(ii) $|(B_1 - C_1, B_1 - C_1)| + |(C_1 - A_1, B_1 - C_1)| = |(B_1 - C_1, B_1 - A_1)|$.

(15) $|(A_1 - B_1, A_1 - C_1)| = -|(A_1 - B_1, C_1 - A_1)|$.

(16) $|(B_1 - A_1, C_1 - A_1)| = |(A_1 - B_1, A_1 - C_1)|$.

(17) $|(B_1 - A_1, C_1 - A_1)| = -|(B_1 - A_1, A_1 - C_1)|$. The theorem is a consequence of (16) and (15).

(18) Suppose $B_1 \neq C_1$ and $C_1 \neq A_1$ and $A_1 \neq B_1$ and $|(C_1 - A_1, B_1 - C_1)|$ is not zero and $|(B_1 - C_1, A_1 - B_1)|$ is not zero and $|(C_1 - A_1, A_1 - B_1)|$ is not zero and $r = -(\frac{|(B_1,C_1)|-|(C_1,C_1)|-|(A_1,B_1)|+|(A_1,C_1)|}{|(B_1-C_1,B_1-C_1)|})$ and $s = -(\frac{|(C_1,A_1)|-|(A_1,A_1)|-|(B_1,C_1)|+|(B_1,A_1)|}{|(C_1-A_1,C_1-A_1)|})$ and

$t = -(\frac{|(A_1,B_1)|-|(B_1,B_1)|-|(C_1,A_1)|+|(C_1,B_1)|}{|(A_1-B_1,A_1-B_1)|})$. Then $\frac{(\frac{(\frac{r}{1-r}) \cdot s}{1-s}) \cdot t}{1-t} = 1$. The theorem is a consequence of (14), (15), and (3).

(19) If $C_1 = r \cdot A_1 + (1 - r) \cdot B_1$ and $r = 1$, then $C_1 = A_1$.

(20) If $C_1 = r \cdot A_1 + (1 - r) \cdot B_1$ and $r = 0$, then $C_1 = B_1$.

(21) If $|(B_1 - C_1, B_1 - C_1)| = -|(C_1 - A_1, B_1 - C_1)|$, then $|(B_1 - C_1, A_1 - B_1)| = 0$. The theorem is a consequence of (15).

(22) Suppose $B_1 \neq C_1$ and $r = -(\frac{|(B_1,C_1)|-|(C_1,C_1)|-|(A_1,B_1)|+|(A_1,C_1)|}{|(B_1-C_1,B_1-C_1)|})$ and $r = 1$. Then $|(B_1 - C_1, A_1 - B_1)| = 0$. The theorem is a consequence of (14) and (21).

(23) If $A \neq B$ and $A \neq C$, then $|A - B| + |A - C| \neq 0$.

(24) If A, B, C form a triangle, then $A \notin \mathrm{Line}(B, C)$.

(25) If $A \neq B$ and $B \neq C$ and $|(B - A, B - C)| = 0$, then $\measuredangle(A, B, C) = \frac{\pi}{2}$ or $\measuredangle(A, B, C) = (\frac{3}{2}) \cdot \pi$.

(26) If A, B, C form a triangle, then $\sin(\frac{\measuredangle(A,B,C)}{2}) > 0$.

(27) If $\measuredangle(B, A, C) \neq \measuredangle(C, B, A)$, then $\sin(\frac{\measuredangle(B,A,C)-\measuredangle(C,B,A)}{2}) \neq 0$. The theorem is a consequence of (5) and (4).

(28) If A, B, C form a triangle, then $\sin \measuredangle(A, B, C) \neq 0$.

Let us assume that A, C, B form a triangle and $\measuredangle(A, C, B) < \pi$. Now we state the propositions:

(29) $\measuredangle(A, C, B) = \pi - (\measuredangle(C, B, A) + \measuredangle(B, A, C))$.

(30) $\measuredangle(B, A, C) + \measuredangle(C, B, A) = \pi - \measuredangle(A, C, B)$. The theorem is a consequence of (29).

Let us assume that A, B, C form a triangle. Now we state the propositions:

(31) $\measuredangle(B, A, C) - \measuredangle(C, B, A) \neq \pi$.

(32) $\measuredangle(B, A, C) - \measuredangle(C, B, A) \neq -\pi$.

Let us assume that A, B, C form a triangle. Now we state the propositions:

(33) $(-2) \cdot \pi < \measuredangle(B, A, C) - \measuredangle(C, B, A) < 2 \cdot \pi$.

(34) $-\pi < \frac{\measuredangle(B,A,C) - \measuredangle(C,B,A)}{2} < \pi$. The theorem is a consequence of (33).

Let us assume that A, B, C form a triangle and $\measuredangle(B, A, C) < \pi$. Now we state the propositions:

(35) $-\pi < \measuredangle(B, A, C) - \measuredangle(C, B, A) < \pi$.

(36) $-(\frac{\pi}{2}) < \frac{\measuredangle(B,A,C) - \measuredangle(C,B,A)}{2} < \frac{\pi}{2}$. The theorem is a consequence of (35).

2. ORTHOCENTER

From now on D denotes a point of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d denote real numbers.

Let A, B, C be points of $\mathcal{E}^2_{\mathrm{T}}$. Assume $B \neq C$. The functor $\mathrm{Altit}\triangle(A, B, C)$ yielding an element of $\mathrm{Lines}(\mathcal{R}^2)$ is defined by

(Def. 1) there exist elements L_1, L_2 of $\mathrm{Lines}(\mathcal{R}^2)$ such that $it = L_1$ and $L_2 = \mathrm{Line}(B, C)$ and $A \in L_1$ and $L_1 \perp L_2$.

Let us assume that $B \neq C$. Now we state the propositions:

(37) $A \in \mathrm{Altit}\triangle(A, B, C)$.

(38) $\mathrm{Altit}\triangle(A, B, C)$ is a line.

(39) $\mathrm{Altit}\triangle(A, B, C) = \mathrm{Altit}\triangle(A, C, B)$.

Now we state the propositions:

(40) If $B \neq C$ and $D \in \mathrm{Altit}\triangle(A, B, C)$, then $\mathrm{Altit}\triangle(D, B, C) = \mathrm{Altit}\triangle(A, B, C)$.

(41) If $B \neq C$ and $D \in \mathrm{Line}(B, C)$ and $D \neq C$, then $\mathrm{Altit}\triangle(A, B, C) = \mathrm{Altit}\triangle(A, D, C)$. The theorem is a consequence of (7).

Let A, B, C be points of $\mathcal{E}^2_{\mathrm{T}}$. Assume $B \neq C$. The functor $\mathrm{FootAltit}\triangle(A, B, C)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by

(Def. 2) there exists a point P of $\mathcal{E}^2_{\mathrm{T}}$ such that $it = P$ and $\mathrm{Altit}\,\triangle(A,B,C) \cap \mathrm{Line}(B,C) = \{P\}$.

Let us assume that $B \neq C$. Now we state the propositions:

(42) $\mathrm{FootAltit}\,\triangle(A,B,C) = \mathrm{FootAltit}\,\triangle(A,C,B)$. The theorem is a consequence of (39).

(43) (i) $\mathrm{FootAltit}\,\triangle(A,B,C) \in \mathrm{Line}(B,C)$, and

(ii) $\mathrm{FootAltit}\,\triangle(A,B,C) \in \mathrm{Altit}\,\triangle(A,B,C)$.

Now we state the propositions:

(44) If $B \neq C$ and $A \notin \mathrm{Line}(B,C)$, then $\mathrm{Altit}\,\triangle(A,B,C) = \mathrm{Line}(A, \mathrm{FootAltit}\,\triangle(A,B,C))$. The theorem is a consequence of (43).

(45) If $B \neq C$ and $A \in \mathrm{Line}(B,C)$, then $\mathrm{FootAltit}\,\triangle(A,B,C) = A$.

(46) If $B \neq C$ and $\mathrm{FootAltit}\,\triangle(A,B,C) = A$, then $A \in \mathrm{Line}(B,C)$.

Let us assume that $B \neq C$. Now we state the propositions:

(47) $|(A - \mathrm{FootAltit}\,\triangle(A,B,C), B - C)| = 0$. The theorem is a consequence of (44) and (45).

(48) $|(A - \mathrm{FootAltit}\,\triangle(A,B,C), B - \mathrm{FootAltit}\,\triangle(A,B,C))| = 0$. The theorem is a consequence of (43), (44), and (45).

(49) $|(A - \mathrm{FootAltit}\,\triangle(A,B,C), C - \mathrm{FootAltit}\,\triangle(A,B,C))| = 0$. The theorem is a consequence of (42) and (48).

Now we state the propositions:

(50) If $B \neq C$ and $B = \mathrm{FootAltit}\,\triangle(A,B,C)$, then $|(B - A, B - C)| = 0$. The theorem is a consequence of (49), (11), and (43).

(51) If $B \neq C$ and $D \in \mathrm{Line}(B,C)$ and $D \neq C$, then $\mathrm{FootAltit}\,\triangle(A,B,C) = \mathrm{FootAltit}\,\triangle(A,D,C)$. The theorem is a consequence of (7) and (41).

(52) If $B \neq C$ and $|(B - A, B - C)| = 0$, then $B = \mathrm{FootAltit}\,\triangle(A,B,C)$. The theorem is a consequence of (9) and (45).

(53) If $B \neq C$ and $B \neq A$ and $\measuredangle(A,B,C) = \frac{\pi}{2}$, then $\mathrm{FootAltit}\,\triangle(A,B,C) = B$. The theorem is a consequence of (11) and (52).

(54) If A, B, C form a triangle, then $A \neq \mathrm{FootAltit}\,\triangle(A,B,C)$. The theorem is a consequence of (43).

(55) If A, B, C form a triangle and $|(B - A, B - C)| \neq 0$, then $\mathrm{FootAltit}\,\triangle(A,B,C)$, B, A form a triangle.

PROOF: Set $p = \mathrm{FootAltit}\,\triangle(A,B,C)$. Consider P being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{FootAltit}\,\triangle(A,B,C) = P$ and $\mathrm{Altit}\,\triangle(A,B,C) \cap \mathrm{Line}(B,C) = \{P\}$. Consider L_1, L_2 being elements of $\mathrm{Lines}(\mathcal{R}^2)$ such that $\mathrm{Altit}\,\triangle(A,B,C) = L_1$ and $L_2 = \mathrm{Line}(B,C)$ and $A \in L_1$ and $L_1 \perp L_2$. $P \neq B$. $p \neq A$. p, B, A are mutually different. $P \in \mathrm{Line}(B,C)$. B, $C \in \mathrm{Line}(B,P)$. $\measuredangle(p,B,A) \neq \pi$

by [11, (11)], [12, (12)], (50), (8). $\measuredangle(B, A, p) \neq \pi$ by [11, (11)], [12, (12)]. $\measuredangle(A, p, B) \neq \pi$ by [11, (11)], [12, (12)], (8), (54). □

Let A, B, C be points of $\mathcal{E}^2_{\mathrm{T}}$. Assume $B \neq C$. The functor $|\operatorname{Altit}\triangle(A, B, C)|$ yielding a real number is defined by the term

(Def. 3) $|A - \operatorname{FootAltit}\triangle(A, B, C)|$.

Let us assume that $B \neq C$. Now we state the propositions:

(56) $0 \leqslant |\operatorname{Altit}\triangle(A, B, C)|$.

(57) $|\operatorname{Altit}\triangle(A, B, C)| = |\operatorname{Altit}\triangle(A, C, B)|$. The theorem is a consequence of (42).

Now we state the propositions:

(58) If $B \neq C$ and $|(B - A, B - C)| = 0$, then $|\operatorname{FootAltit}\triangle(A, B, C) - A| = |A - B|$. The theorem is a consequence of (52).

(59) Suppose $B \neq C$ and $r = -(\frac{|(B,C)|-|(C,C)|-|(A,B)|+|(A,C)|}{|(B-C,B-C)|})$ and $D = r \cdot B + (1 - r) \cdot C$ and $D \neq C$. Then $D = \operatorname{FootAltit}\triangle(A, B, C)$.
PROOF: $|(D - A, D - C)| = 0$. $D = \operatorname{FootAltit}\triangle(A, D, C)$. $D \in \operatorname{Line}(B, C)$ by [6, (4)]. □

(60) Suppose $B \neq C$ and $r = -(\frac{|(B,C)|-|(C,C)|-|(A,B)|+|(A,C)|}{|(B-C,B-C)|})$ and $D = r \cdot B + (1 - r) \cdot C$ and $D = C$. Then $C = \operatorname{FootAltit}\triangle(A, B, C)$. The theorem is a consequence of (13), (14), (15), (52), and (42).

(61) Suppose A, B, C form a triangle and $|(C - A, B - C)|$ is not zero and $|(B - C, A - B)|$ is not zero and $|(C - A, A - B)|$ is not zero. Then $\operatorname{Line}(A, \operatorname{FootAltit}\triangle(A, B, C))$, $\operatorname{Line}(C, \operatorname{FootAltit}\triangle(C, A, B))$, $\operatorname{Line}(B, \operatorname{FootAltit}\triangle(B, C, A))$ are concurrent. The theorem is a consequence of (60), (17), (47), (59), (18), and (22).

(62) If A, B, C form a triangle and $|(C - A, B - C)|$ is zero, then $\operatorname{FootAltit}\triangle(A, B, C) = C$ and $\operatorname{FootAltit}\triangle(B, C, A) = C$. The theorem is a consequence of (15), (52), and (42).

(63) Suppose A, B, C form a triangle and $C \in \operatorname{Altit}\triangle(A, B, C)$ and $C \in \operatorname{Altit}\triangle(B, C, A)$. Then $\operatorname{Altit}\triangle(A, B, C) \cap \operatorname{Altit}\triangle(B, C, A)$ is a point.
PROOF: Consider L_1, L_2 being elements of $\operatorname{Lines}(\mathcal{R}^2)$ such that $\operatorname{Altit}\triangle(A, B, C) = L_1$ and $L_2 = \operatorname{Line}(B, C)$ and $A \in L_1$ and $L_1 \perp L_2$. Consider L_3, L_4 being elements of $\operatorname{Lines}(\mathcal{R}^2)$ such that $\operatorname{Altit}\triangle(B, C, A) = L_3$ and $L_4 = \operatorname{Line}(C, A)$ and $B \in L_3$ and $L_3 \perp L_4$. $L_1 \nparallel L_3$ by [9, (41)], [6, (16)], [8, (108)], [12, (13)]. L_1 is not a point and L_3 is not a point. □

(64) Suppose B, C, A form a triangle and $C \in \operatorname{Altit}\triangle(B, C, A)$ and $C \in \operatorname{Altit}\triangle(C, A, B)$. Then $\operatorname{Altit}\triangle(B, C, A) \cap \operatorname{Altit}\triangle(C, A, B)$ is a point.
PROOF: Consider L_1, L_2 being elements of $\operatorname{Lines}(\mathcal{R}^2)$ such that

$\operatorname{Altit}\triangle(B,C,A) = L_1$ and $L_2 = \operatorname{Line}(C,A)$ and $B \in L_1$ and $L_1 \perp L_2$. Consider L_3, L_4 being elements of $\operatorname{Lines}(\mathcal{R}^2)$ such that $\operatorname{Altit}\triangle(C,A,B) = L_3$ and $L_4 = \operatorname{Line}(A,B)$ and $C \in L_3$ and $L_3 \perp L_4$. $L_1 \not\parallel L_3$ by [8, (71), (111)], [6, (16)], [9, (41)]. L_1 is not a point and L_3 is not a point. □

(65) Suppose C, A, B form a triangle and $C \in \operatorname{Altit}\triangle(C,A,B)$ and $C \in \operatorname{Altit}\triangle(A,B,C)$. Then $\operatorname{Altit}\triangle(C,A,B) \cap \operatorname{Altit}\triangle(A,B,C)$ is a point.

Proof: Consider L_1, L_2 being elements of $\operatorname{Lines}(\mathcal{R}^2)$ such that $\operatorname{Altit}\triangle(C,A,B) = L_1$ and $L_2 = \operatorname{Line}(A,B)$ and $C \in L_1$ and $L_1 \perp L_2$. Consider L_3, L_4 being elements of $\operatorname{Lines}(\mathcal{R}^2)$ such that $\operatorname{Altit}\triangle(A,B,C) = L_3$ and $L_4 = \operatorname{Line}(B,C)$ and $A \in L_3$ and $L_3 \perp L_4$. $L_1 \not\parallel L_3$ by [8, (71), (111)], [6, (16)], [9, (41)]. L_1 is not a point and L_3 is not a point. □

(66) Suppose A, B, C form a triangle and $|(C - A, B - C)| = 0$. Then

(i) $\operatorname{Altit}\triangle(A,B,C) \cap \operatorname{Altit}\triangle(B,C,A) = \{C\}$, and

(ii) $\operatorname{Altit}\triangle(B,C,A) \cap \operatorname{Altit}\triangle(C,A,B) = \{C\}$, and

(iii) $\operatorname{Altit}\triangle(C,A,B) \cap \operatorname{Altit}\triangle(A,B,C) = \{C\}$.

Proof: $A \notin \operatorname{Line}(B,C)$ and $B \notin \operatorname{Line}(C,A)$. $\operatorname{FootAltit}\triangle(A,B,C) = C$ and $\operatorname{FootAltit}\triangle(B,C,A) = C$. $\operatorname{Altit}\triangle(A,B,C) = \operatorname{Line}(A,C)$ and $\operatorname{Altit}\triangle(B,C,A) = \operatorname{Line}(B,C)$. $C \in \operatorname{Altit}\triangle(C,A,B)$. $\operatorname{Altit}\triangle(A,B,C) \cap \operatorname{Altit}\triangle(B,C,A) = \{C\}$ by [6, (22)], (63). $\operatorname{Altit}\triangle(B,C,A) \cap \operatorname{Altit}\triangle(C,A,B) = \{C\}$ by [12, (15)], (37), (64), [6, (22)]. $\operatorname{Altit}\triangle(C,A,B) \cap \operatorname{Altit}\triangle(A,B,C) = \{C\}$ by [12, (15)], (37), (65), [6, (22)]. □

(67) Suppose A, B, C form a triangle. Then there exists a point P of $\mathcal{E}^2_{\mathrm{T}}$ such that

(i) $\operatorname{Altit}\triangle(A,B,C) \cap \operatorname{Altit}\triangle(B,C,A) = \{P\}$, and

(ii) $\operatorname{Altit}\triangle(B,C,A) \cap \operatorname{Altit}\triangle(C,A,B) = \{P\}$, and

(iii) $\operatorname{Altit}\triangle(C,A,B) \cap \operatorname{Altit}\triangle(A,B,C) = \{P\}$.

The theorem is a consequence of (66), (61), (24), (44), and (38).

Let A, B, C be points of $\mathcal{E}^2_{\mathrm{T}}$. Assume A, B, C form a triangle. The functor $\operatorname{Orthocenter}\triangle(A,B,C)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by

(Def. 4) $\operatorname{Altit}\triangle(A,B,C) \cap \operatorname{Altit}\triangle(B,C,A) = \{it\}$ and $\operatorname{Altit}\triangle(B,C,A) \cap \operatorname{Altit}\triangle(C,A,B) = \{it\}$ and $\operatorname{Altit}\triangle(C,A,B) \cap \operatorname{Altit}\triangle(A,B,C) = \{it\}$.

3. Triangulation

Let us assume that $B \neq A$. Now we state the propositions:

(68) $(\sin\measuredangle(B,A,C) + \sin\measuredangle(C,B,A)) \cdot (|C - B| - |C - A|) = (\sin\measuredangle(B,A,C) - \sin\measuredangle(C,B,A)) \cdot (|C - B| + |C - A|)$.

(69) $\sin(\frac{\measuredangle(B,A,C)+\measuredangle(C,B,A)}{2}) \cdot \cos(\frac{\measuredangle(B,A,C)-\measuredangle(C,B,A)}{2}) \cdot (|C-B|-|C-A|) = \sin(\frac{\measuredangle(B,A,C)-\measuredangle(C,B,A)}{2}) \cdot \cos(\frac{\measuredangle(B,A,C)+\measuredangle(C,B,A)}{2}) \cdot (|C-B|+|C-A|)$. The theorem is a consequence of (68).

Now we state the proposition:

(70) Suppose A, B, C form a triangle and $\measuredangle(B,A,C) - \measuredangle(C,B,A) \neq \pi$ and $\measuredangle(B,A,C) - \measuredangle(C,B,A) \neq -\pi$. Then $\cos(\frac{\measuredangle(B,A,C)-\measuredangle(C,B,A)}{2}) \neq 0$. The theorem is a consequence of (2).

Let us assume that A, C, B form a triangle and $\measuredangle(A,C,B) < \pi$. Now we state the propositions:

(71) $\tan(\frac{\measuredangle(B,A,C)-\measuredangle(C,B,A)}{2}) = \cot(\frac{\measuredangle(A,C,B)}{2}) \cdot (\frac{|C-B|-|C-A|}{|C-B|+|C-A|})$.
Proof: $\measuredangle(B,A,C) - \measuredangle(C,B,A) \neq \pi$ and $\measuredangle(B,A,C) - \measuredangle(C,B,A) \neq -\pi$. Set $\alpha = \frac{\measuredangle(B,A,C)-\measuredangle(C,B,A)}{2}$. Set $\beta = \frac{\measuredangle(B,A,C)+\measuredangle(C,B,A)}{2}$. $\measuredangle(A,C,B) = \pi - (\measuredangle(C,B,A)+\measuredangle(B,A,C))$. Set $\alpha_1 = \frac{\measuredangle(A,C,B)}{2}$. $\sin\alpha_1 \neq 0$. $|C-B|+|C-A| \neq 0$ by [11, (42)]. $\sin\beta \cdot \cos\alpha \cdot (|C-B|-|C-A|) = \sin\alpha \cdot \cos\beta \cdot (|C-B|+|C-A|)$. $(|C-B|-|C-A|) \cdot \cos\alpha_1 \cdot 1 = (|C-B|+|C-A|) \cdot \sin\alpha_1 \cdot (\frac{\sin\alpha}{\cos\alpha})$. □

(72) $\frac{\measuredangle(B,A,C)-\measuredangle(C,B,A)}{2} = \arctan(\cot(\frac{\measuredangle(A,C,B)}{2}) \cdot (\frac{|C-B|-|C-A|}{|C-B|+|C-A|}))$. The theorem is a consequence of (71) and (36).

(73) $\measuredangle(B,A,C) - \measuredangle(C,B,A) = 2 \cdot \arctan(\cot(\frac{\measuredangle(A,C,B)}{2}) \cdot (\frac{|C-B|-|C-A|}{|C-B|+|C-A|}))$. The theorem is a consequence of (72).

(74) (i) $\measuredangle(B,A,C) = \arctan(\cot(\frac{\measuredangle(A,C,B)}{2}) \cdot (\frac{|C-B|-|C-A|}{|C-B|+|C-A|})) + (\frac{\pi}{2}) - (\frac{\measuredangle(A,C,B)}{2})$, and

(ii) $\measuredangle(C,B,A) = (\frac{\pi}{2}) - (\frac{\measuredangle(A,C,B)}{2}) - \arctan(\cot(\frac{\measuredangle(A,C,B)}{2}) \cdot (\frac{|C-B|-|C-A|}{|C-B|+|C-A|}))$.

The theorem is a consequence of (73) and (30).

(75) $|B-C| = \frac{|A-B| \cdot \sin\measuredangle(B,A,C)}{\sin(\measuredangle(B,A,C)+\measuredangle(C,B,A))}$.
Proof: $|B-C| = \frac{|A-B| \cdot \sin\measuredangle(B,A,C)}{\sin\measuredangle(A,C,B)}$ by [11, (6), (43)], (28). $\measuredangle(A,C,B) = \pi - (\measuredangle(C,B,A) + \measuredangle(B,A,C))$. □

(76) $|A-C| = \frac{|A-B| \cdot \sin\measuredangle(C,B,A)}{\sin(\measuredangle(B,A,C)+\measuredangle(C,B,A))}$.
Proof: $|A-C| = \frac{|A-B| \cdot \sin\measuredangle(C,B,A)}{\sin\measuredangle(A,C,B)}$ by [11, (6)], (28). $\measuredangle(A,C,B) = \pi - (\measuredangle(C,B,A) + \measuredangle(B,A,C))$ by [11, (20)], [10, (47)]. □

Now we state the propositions:

(77) Suppose A, C, B form a triangle and $\measuredangle(C,A,B) = \frac{\pi}{2}$.
Then $|\operatorname{Altit}\triangle(C,A,B)| = |A-B| \cdot \tan\measuredangle(A,B,C)$. The theorem is a consequence of (11) and (58).

(78) Suppose A, B, C form a triangle and $\measuredangle(C,A,B) = (\frac{3}{2}) \cdot \pi$.

Then $|\operatorname{Altit}\triangle(C,A,B)| = |A-B| \cdot \tan\measuredangle(C,B,A)$. The theorem is a consequence of (11) and (58).

(79) Suppose A, C, B form a triangle and $|(A-C, A-B)| = 0$. Then $|\operatorname{Altit}\triangle(C,A,B)| = |A-B| \cdot |\tan\measuredangle(A,B,C)|$. The theorem is a consequence of (11), (77), (56), (6), and (78).

(80) Suppose $B \neq C$ and $\operatorname{FootAltit}\triangle(A,B,C)$, B, A form a triangle. Then

(i) $|A-B| \cdot \sin\measuredangle(A,B,\operatorname{FootAltit}\triangle(A,B,C)) = |\operatorname{FootAltit}\triangle(A,B,C) - A|$, or

(ii) $|A-B| \cdot (-\sin\measuredangle(A,B,\operatorname{FootAltit}\triangle(A,B,C))) = |\operatorname{FootAltit}\triangle(A,B,C) - A|$.

The theorem is a consequence of (48).

(81) Suppose A, B, C form a triangle and $|(B-A, B-C)| \neq 0$. Then

(i) $|A-B| \cdot \sin\measuredangle(A,B,\operatorname{FootAltit}\triangle(A,B,C)) = |\operatorname{FootAltit}\triangle(A,B,C) - A|$, or

(ii) $|A-B| \cdot (-\sin\measuredangle(A,B,\operatorname{FootAltit}\triangle(A,B,C))) = |\operatorname{FootAltit}\triangle(A,B,C) - A|$.

The theorem is a consequence of (80) and (55).

(82) Suppose A, C, B form a triangle and $\measuredangle(A,C,B) < \pi$ and $|(A-C, A-B)| \neq 0$. Then $|\operatorname{Altit}\triangle(C,A,B)| = |A-B| \cdot |(\frac{\sin\measuredangle(C,B,A)}{\sin(\measuredangle(B,A,C)+\measuredangle(C,B,A))}) \cdot \sin\measuredangle(C,A,\operatorname{FootAltit}\triangle(C,A,B))|$. The theorem is a consequence of (76), (55), and (80).

(83) Suppose $0 < \measuredangle(B,A,D) < \pi$ and $0 < \measuredangle(D,A,C) < \pi$ and D, A, C are mutually different and B, A, D are mutually different. Then $\measuredangle(A,C,D) + \measuredangle(D,B,A) = 2 \cdot \pi - (\measuredangle(B,A,C) + \measuredangle(A,D,B) + \measuredangle(C,D,A))$.
PROOF: $\measuredangle(B,A,D) + \measuredangle(D,A,C) = \measuredangle(B,A,C)$ by [5, (2)], [11, (4)]. $\measuredangle(A,C,D) = \pi - (\measuredangle(C,D,A) + \measuredangle(D,A,C))$ by [10, (47)]. $\measuredangle(D,B,A) = \pi - (\measuredangle(A,D,B) + \measuredangle(B,A,D))$ by [10, (47)]. □

(84) Suppose A, C, B form a triangle and $\measuredangle(A,C,B) < \pi$ and A, D, B form a triangle and $\measuredangle(A,D,B) < \pi$ and $a = \measuredangle(C,B,A)$ and $b = \measuredangle(B,A,C)$ and $c = \measuredangle(D,B,A)$ and $d = \measuredangle(C,A,D)$. Then $|D-C|^{\mathbf{2}} = |A-B|^{\mathbf{2}} \cdot ((\frac{\sin a}{\sin(a+b)})^{\mathbf{2}} + (\frac{\sin c}{\sin(b+d+c)})^{\mathbf{2}} - 2 \cdot (\frac{\sin a}{\sin(b+a)}) \cdot (\frac{\sin c}{\sin(b+d+c)}) \cdot \cos d)$.
PROOF: Set $e = b + d$. $\sin(e+c) = \sin(\measuredangle(B,A,D) + \measuredangle(D,B,A))$ by [14, (79)]. □

(85) Suppose $\sin(2 \cdot s) \cdot \cos d = \cos(2 \cdot t)$. Then $(r \cdot \cos s)^{\mathbf{2}} + (r \cdot \sin s)^{\mathbf{2}} - 2 \cdot (r \cdot \cos s) \cdot (r \cdot \sin s) \cdot \cos d = 2 \cdot r^{\mathbf{2}} \cdot (\sin t)^{\mathbf{2}}$.

(86) Let us consider real numbers R, ϑ. Suppose $D \neq C$ and $0 \leqslant R$ and A, C, B form a triangle and $\measuredangle(A,C,B) < \pi$ and A, D, B form a triangle

and $\measuredangle(A,D,B) < \pi$ and $a = \measuredangle(C,B,A)$ and $b = \measuredangle(B,A,C)$ and $c = \measuredangle(D,B,A)$ and $d = \measuredangle(C,A,D)$ and $R \cdot \cos s = \frac{\sin a}{\sin(a+b)}$ and $R \cdot \sin s = \frac{\sin c}{\sin(b+d+c)}$ and $0 < \vartheta < \pi$ and $\sin(2 \cdot s) \cdot \cos d = \cos(2 \cdot \vartheta)$. Then $|D - C| = |A - B| \cdot \sqrt{2} \cdot R \cdot \sin\vartheta$.

PROOF: $|D-C|^{\mathbf{2}} = |A-B|^{\mathbf{2}} \cdot ((R\cdot\cos s)^{\mathbf{2}} + (R\cdot\sin s)^{\mathbf{2}} - 2\cdot(R\cdot\cos s)\cdot(R\cdot\sin s)\cdot\cos d)$. $|D - C| \neq -|A - B| \cdot \sqrt{2} \cdot R \cdot \sin\vartheta$ by [13, (25)], [11, (42)]. □

(87) Suppose A, C, B form a triangle and $\measuredangle(A,C,B) < \pi$ and D, A, C form a triangle and $\measuredangle(A,D,C) = \frac{\pi}{2}$. Then $|D - C| = (\frac{|A-B|\cdot\sin\measuredangle(C,B,A)}{\sin(\measuredangle(B,A,C)+\measuredangle(C,B,A))}) \cdot \sin\measuredangle(C,A,D)$. The theorem is a consequence of (76).

(88) Suppose B, C, A form a triangle and $\measuredangle(B,C,A) < \pi$ and D, C, A form a triangle and $\measuredangle(C,D,A) = \frac{\pi}{2}$. Then $|D - C| = (\frac{|A-B|\cdot\sin\measuredangle(A,B,C)}{\sin(\measuredangle(A,B,C)+\measuredangle(C,A,B))}) \cdot \sin\measuredangle(D,A,C)$. The theorem is a consequence of (75).

(89) Suppose A, C, B form a triangle and $\measuredangle(A,C,B) < \pi$ and D, A, C form a triangle and $\measuredangle(A,D,C) = \frac{\pi}{2}$ and $A \in \mathcal{L}(B,D)$ and $A \neq D$. Then $|D-C| = (\frac{|A-B|\cdot\sin\measuredangle(C,B,A)}{\sin(\measuredangle(C,A,D)-\measuredangle(C,B,A))}) \cdot \sin\measuredangle(C,A,D)$. The theorem is a consequence of (87).

(90) Suppose B, C, A form a triangle and $\measuredangle(B,C,A) < \pi$ and D, C, A form a triangle and $\measuredangle(C,D,A) = \frac{\pi}{2}$ and $A \in \mathcal{L}(D,B)$ and $A \neq D$. Then $|D - C| = (\frac{|A-B|\cdot\sin\measuredangle(A,B,C)}{\sin(\measuredangle(D,A,C)-\measuredangle(A,B,C))}) \cdot \sin\measuredangle(D,A,C)$.

PROOF: $\sin(\measuredangle(C,A,B) + \measuredangle(A,B,C)) = \sin(\measuredangle(D,A,C) - \measuredangle(A,B,C))$ by [4, (1)], [3, (8)]. □

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