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Topology from Neighbourhoods

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Summary. Using Mizar [9], and the formal topological space structure (`FMT_Space_Str`) [19], we introduce the three `U-FMT` conditions (`U-FMT filter`, `U-FMT with point` and `U-FMT local`) similar to those V_I, V_{II}, V_{III} and V_{IV} of the proposition 2 in [10]:

> If to each element x of a set X there corresponds a set $\mathcal{B}(x)$ of subsets of X such that the properties V_I, V_{II}, V_{III} and V_{IV} are satisfied, then there is a unique topological structure on X such that, for each $x \in X$, $\mathcal{B}(x)$ is the set of neighborhoods of x in this topology.

We present a correspondence between a topological space and a space defined with the formal topological space structure with the three `U-FMT` conditions called the topology from neighbourhoods. For the formalization, we were inspired by the works of Bourbaki [11] and Claude Wagschal [31].

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The notation and terminology used in this paper have been introduced in the following articles: [24], [16], [1], [30], [17], [19], [12], [13], [27], [2], [34], [25], [28], [4], [14], [23], [32], [33], [22], [29], [5], [6], [8], [18], [26], and [15].

1. Preliminaries

From now on X denotes a non empty set.

Now we state the propositions:

(1) Let us consider families B, Y of subsets of X. If $Y \subseteq \text{UniCl}(B)$, then $\bigcup Y \in \text{UniCl}(B)$.

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(2) Let us consider an empty family B of subsets of X. Suppose for every elements B_1, B_2 of B, there exists a subset B_3 of B such that $B_1 \cap B_2 = \bigcup B_3$ and $X = \bigcup B$. Then FinMeetCl(B) $\subseteq$ UniCl(B).
PROOF: FinMeetCl(B) $\subseteq$ UniCl(B) by [22, (1)]. □

(3) Let us consider a non empty family B of subsets of X. Suppose for every elements B_1, B_2 of B, there exists a subset B_3 of B such that $B_1 \cap B_2 = \bigcup B_3$ and $X = \bigcup B$. Then FinMeetCl(B) $\subseteq$ UniCl(B).
PROOF: Reconsider $x_0 = x$ as a subset of X. Consider Y being a family of subsets of X such that $Y \subseteq B$ and Y is finite and $x_0 =$ Intersect(Y). Define $\mathcal{P}$[natural number] $\equiv$ for every family Y of subsets of X for every subset x of X such that $Y \subseteq B$ and $\overline{\overline{Y}} = \$_1$ and $x =$ Intersect(Y) holds $x \in$ UniCl(B). $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [20, (24)], [22, (10), (9)], [15, (2)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. □

(4) Let us consider a family B of subsets of X. Suppose for every elements B_1, B_2 of B, there exists a subset B_3 of B such that $B_1 \cap B_2 = \bigcup B_3$ and $X = \bigcup B$. Then
   (i) UniCl(B) = UniCl(FinMeetCl(B)), and
   (ii) $\langle X, \text{UniCl}(B)\rangle$ is topological space-like.

PROOF: UniCl(B) = UniCl(FinMeetCl(B)) by [24, (4)], (2), (3), [7, (15)]. □

(5) Let us consider a non empty formal topological space R. Then there exists a relational structure S such that for every element x of R, $U_F(x)$ is a subset of S.

Let T be a non empty topological space. One can verify that NeighSp T is filled.

2. Open, Neighborhood and Conditions for Topological Space from Neighborhoods

Let E be a non empty, strict formal topological space and O be a subset of E. We say that O is open if and only if

(Def. 1) for every element x of E such that $x \in O$ holds $O \in U_F(x)$.

We say that E is U-FMT filter if and only if

(Def. 2) for every element x of E, $U_F(x)$ is a filter of the carrier of E.

We say that E is U-FMT with point if and only if

(Def. 3) for every element x of E and for every element V of $U_F(x)$, $x \in V$.

We say that E is U-FMT local if and only if

(Def. 4) for every element x of E and for every element V of $U_F(x)$, there exists an element W of $U_F(x)$ such that for every element y of E such that y is an element of W holds V is an element of $U_F(y)$.

Now we state the proposition:

(6) Let us consider a non empty, strict formal topological space E. Suppose E is U-FMT filter. Let us consider an element x of E. Then $U_F(x)$ is not empty.

Let us consider a non empty, strict formal topological space E. Now we state the propositions:

(7) If E is U-FMT with point, then E is filled.

(8) If E is filled and for every element x of E, $U_F(x)$ is not empty, then E is U-FMT with point.

(9) If E is filled and U-FMT filter, then E is U-FMT with point. The theorem is a consequence of (8).

Observe that there exists a non empty, strict formal topological space which is U-FMT local, U-FMT with point, and U-FMT filter.

Now we state the proposition:

(10) Let us consider a U-FMT filter, non empty, strict formal topological space E, and an element x of E. Then the carrier of $E \in U_F(x)$.

Let E be a U-FMT filter, non empty, strict formal topological space and x be an element of E.

A neighbourhood of x is a subset of E and is defined by

(Def. 5) $it \in U_F(x)$.

Let us observe that there exists a neighbourhood of x which is open.

Let A be a subset of E.

A neighbourhood of A is a subset of E and is defined by

(Def. 6) for every element x of E such that $x \in A$ holds $it \in U_F(x)$.

Note that there exists a neighbourhood of A which is open.

Now we state the proposition:

(11) Let us consider a U-FMT filter, non empty, strict formal topological space E, a subset A of E, a neighbourhood C of A, and a subset B of E. If $C \subseteq B$, then B is a neighbourhood of A.

Let E be a U-FMT filter, non empty, strict formal topological space and A be a subset of E. The functor Neighborhood A yielding a family of subsets of E is defined by the term

(Def. 7) the set of all N where N is a neighbourhood of A.

Now we state the proposition:

(12) Let us consider a U-FMT filter, non empty, strict formal topological space E, and a non empty subset A of E. Then Neighborhood A is a filter of the carrier of E. The theorem is a consequence of (10).

Let E be a non empty, strict formal topological space. We say that E is U-FMT filter base if and only if

(Def. 8) for every element x of the carrier of E, $U_F(x)$ is a filter base of the carrier of E.

Let E be a non empty formal topological space. The functor $[E]$ yielding a function from the carrier of E into $2^{2^{(\text{the carrier of } E)}}$ is defined by

(Def. 9) for every element x of the carrier of E, $it(x) = [U_F(x)]$.

Let E be a non empty, strict formal topological space. The functor gen-filter E yielding a non empty, strict formal topological space is defined by the term

(Def. 10) $\langle$the carrier of $E, [E]\rangle$.

Now we state the proposition:

(13) Let us consider a non empty, strict formal topological space E. Suppose E is U-FMT filter base. Then gen-filter E is U-FMT filter.

PROOF: For every element x of gen-filter E, $U_F(x)$ is a filter of the carrier of gen-filter E by [16, (25)]. □

3. Topology from Neighborhoods: a Definition

A topology from neighbourhoods is a U-FMT local, U-FMT with point, U-FMT filter, non empty, strict formal topological space. Let E be a topology from neighbourhoods and x be an element of E. We introduce the notation the neighborhood system of x as a synonym of $U_F(x)$.

Let us note that there exists a subset of E which is open.

The functor the open set family of E yielding a non empty family of subsets of the carrier of E is defined by the term

(Def. 11) the set of all O where O is an open subset of E.

Now we state the propositions:

(14) Let us consider a topology from neighbourhoods E. Then

(i) $\emptyset$, the carrier of $E \in$ the open set family of E, and

(ii) for every family a of subsets of E such that $a \subseteq$ the open set family of E holds $\bigcup a \in$ the open set family of E, and

(iii) for every subsets a, b of E such that $a, b \in$ the open set family of E holds $a \cap b \in$ the open set family of E.

Proof: $\emptyset \in$ the open set family of E. The carrier of $E \in$ the open set family of E by [30, (5)]. For every family a of subsets of E such that $a \subseteq$ the open set family of E holds $\bigcup a \in$ the open set family of E by [15, (74)]. For every subsets a, b of E such that $a, b \in$ the open set family of E holds $a \cap b \in$ the open set family of E. □

(15) Let us consider a topology from neighbourhoods E, an element a of E, and a neighbourhood V of a. Then there exists an open subset O of E such that

(i) $a \in O$, and

(ii) $O \subseteq V$.

The theorem is a consequence of (6).

(16) Let us consider a topology from neighbourhoods E, a non empty subset A of E, and a subset V of E. Then V is a neighbourhood of A if and only if there exists an open subset O of E such that $A \subseteq O \subseteq V$.

Proof: If V is a neighbourhood of A, then there exists an open subset O of E such that $A \subseteq O \subseteq V$ by (15), (14), [13, (4)]. If there exists an open subset O of E such that $A \subseteq O \subseteq V$, then V is a neighbourhood of A. □

(17) Let us consider a topology from neighbourhoods E, and a non empty subset A of E. Then Neighborhood A is a filter of the carrier of E.

Let E be a topology from neighbourhoods and A be a non empty subset of E. The open neighbourhoods of A yielding a family of subsets of the carrier of E is defined by the term

(Def. 12) the set of all N where N is an open neighbourhood of A.

Now we state the propositions:

(18) Let us consider a topology from neighbourhoods E, a filter $\mathcal{F}$ of the carrier of E, a non empty subset $\mathcal{S}$ of $\mathcal{F}$, and a non empty subset A of E. Suppose $\mathcal{F} =$ Neighborhood A and $\mathcal{S} =$ the open neighbourhoods of A. Then $\mathcal{S}$ is filter basis. The theorem is a consequence of (16).

(19) Let us consider a non empty topological space T. Then there exists a topology from neighbourhoods E such that

(i) the carrier of $T =$ the carrier of E, and

(ii) the open set family of $E =$ the topology of T.

Proof: There exists a non empty, strict formal topological space E such that E is U-FMT filter, U-FMT with point, and U-FMT local and the carrier of $T =$ the carrier of E and there exists a topology from neighbourhoods T_1 such that $T_1 = E$ and the open set family of $T_1 =$ the topology of T by (13), [23, (1)], [21, (3), (7)]. Consider E being a non empty, strict formal topological space such that the carrier of T = the carrier of E and E is U-FMT filter, U-FMT with point, and U-FMT local and there exists a topology from neighbourhoods T_1 such that $T_1 = E$ and the open set family of T_1 = the topology of T. Consider T_1 being a topology from neighbourhoods such that $T_1 = E$ and the open set family of T_1 = the topology of T. $\square$

(20) Let us consider a non empty topological space T, and a topology from neighbourhoods E. Suppose the carrier of T = the carrier of E and the open set family of E = the topology of T. Let us consider an element x of E. Then $U_F(x) = \{V$, where V is a subset of E : there exists a subset O of T such that $O \in$ the topology of T and $x \in O$ and $O \subseteq V\}$. The theorem is a consequence of (15).

4. Basis

Let E be a topology from neighbourhoods and F be a family of subsets of E. We say that F is quasi basis if and only if

(Def. 13) the open set family of $E \subseteq \mathrm{UniCl}(F)$.

Note that the open set family of E is quasi basis and there exists a family of subsets of E which is quasi basis.

Let S be a family of subsets of E. We say that S is open if and only if

(Def. 14) $S \subseteq$ the open set family of E.

One can check that there exists a family of subsets of E which is open and there exists a family of subsets of E which is open and quasi basis.

A basis of E is an open, quasi basis family of subsets of E. Now we state the propositions:

(21) Let us consider a topology from neighbourhoods E, and a basis B of E. Then the open set family of $E = \mathrm{UniCl}(B)$. The theorem is a consequence of (14).

(22) Let us consider a non empty family B of subsets of X. Suppose for every elements B_1, B_2 of B, there exists a subset B_3 of B such that $B_1 \cap B_2 = \bigcup B_3$ and $X = \bigcup B$. Then there exists a topology from neighbourhoods E such that

   (i) the carrier of $E = X$, and

   (ii) B is a basis of E.

   The theorem is a consequence of (4) and (19).

(23) Let us consider a topology from neighbourhoods E, and a basis B of E. Then

(i) for every elements B_1, B_2 of B, there exists a subset B_3 of B such that $B_1 \cap B_2 = \bigcup B_3$, and

(ii) the carrier of $E = \bigcup B$.

Proof: For every elements B_1, B_2 of B, there exists a subset B_3 of B such that $B_1 \cap B_2 = \bigcup B_3$ by [7, (16)], (14). The carrier of $X \in$ the open set family of X. Consider Y being a family of subsets of X such that $Y \subseteq B$ and the carrier of $X = \bigcup Y$. $\square$

5. Correspondence between Topological Space and Topology from Neighborhoods

Let T be a non empty topological space. The functor TopSpace2FMT T yielding a topology from neighbourhoods is defined by

(Def. 15) the carrier of *it* = the carrier of T and the open set family of *it* = the topology of T.

Let E be a topology from neighbourhoods. The functor FMT2TopSpace E yielding a strict topological space is defined by

(Def. 16) the carrier of *it* = the carrier of E and the open set family of E = the topology of *it*.

Let us observe that FMT2TopSpace E is non empty.

Now we state the propositions:

(24) Let us consider a non empty, strict topological space T. Then T = FMT2TopSpace TopSpace2FMT T.

(25) Let us consider a topology from neighbourhoods E. Then E = TopSpace2FMT FMT2TopSpace E.

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