Summable Family in a Commutative Group

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Summary. Hörlz et al. showed that it was possible to build “a generic theory of limits based on filters” in Isabelle/HOL [22], [7]. In this paper we present our formalization of this theory in Mizar [6].

First, we compare the notions of the limit of a family indexed by a directed set, or a sequence, in a metric space [30], a real normed linear space [29] and a linear topological space [14] with the concept of the limit of an image filter [16].

Then, following Bourbaki [9], [10] (TG.III, §5.1 Familles sommables dans un groupe commutatif), we conclude by defining the summable families in a commutative group (“additive notation” in [17]), using the notion of filters.

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The notation and terminology used in this paper have been introduced in the following articles: [26], [16], [1], [27], [4], [18], [34], [32], [30], [11], [12], [35], [17], [23], [29], [20], [37], [2], [13], [8], [28], [39], [14], [36], [19], [31], [38], [24], [3], [25], [5], [21], and [15].

1. Preliminaries

Now we state the propositions:

(1) Let us consider a set \( I \). Then \( \emptyset \) is an element of \( \text{Fin} I \).

(2) Let us consider sets \( I, J \). Suppose \( J \in \text{Fin} I \). Then there exists a finite sequence \( p \) of elements of \( I \) such that

\( i \) \( J = \text{rng} p \), and
(ii) $p$ is one-to-one.

(3) Let us consider a set $I$, a non empty set $Y$, a $Y$-valued many sorted set $x$ indexed by $I$, and a finite sequence $p$ of elements of $I$. Then $p \cdot x$ is a finite sequence of elements of $Y$.

(4) Let us consider non empty sets $I$, $X$, an $X$-valued many sorted set $x$ indexed by $I$, and finite sequences $p$, $q$ of elements of $I$. Then $(p \cdot q) \cdot x = p \cdot (q \cdot x)$.

**Proof:** For every object $t$ such that $t \in \text{dom}((p \cdot q) \cdot x)$ holds $(p \cdot q) \cdot x(t) = (p \cdot (q \cdot x))(t)$ by [33, (120)], [11, (13)], [4, (25)]. □

Let $I$ be a set, $Y$ be a non empty set, $x$ be a $Y$-valued many sorted set indexed by $I$, and $p$ be a finite sequence of elements of $I$. The functor $\#^p_x$ yielding a finite sequence of elements of $Y$ is defined by the term

(Def. 1) $p \cdot x$.

The functor $F(I)$ yielding a non empty, transitive, reflexive relational structure is defined by the term

(Def. 2) $\langle \text{Fin } I, \subseteq \rangle$.

Now we state the proposition:

(5) Let us consider a set $I$. Then $\Omega_{F(I)}$ is directed.

2. Convergence in Metric Spaces

Now we state the propositions:

(6) Let us consider a non empty metric space $M$, and a point $x$ of $M_{\text{top}}$. Then $\text{Balls } x$ is a generalized basis of $\text{BooleanFilterToFilter}(\text{the neighborhood system of } x)$.

(7) Let us consider a non empty metric space $M$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into the carrier of $M_{\text{top}}$, a point $x$ of $M_{\text{top}}$, and a generalized basis $B$ of $\text{BooleanFilterToFilter}(\text{the neighborhood system of } x)$. Suppose $\Omega_L$ is directed. Then $x \in \text{LimF}(f)$ if and only if for every element $b$ of $B$, there exists an element $i$ of $L$ such that for every element $j$ of $L$ such that $i \leq j$ holds $f(j) \in b$.

(8) Let us consider a non empty metric space $M$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into the carrier of $M_{\text{top}}$, and a point $x$ of $M_{\text{top}}$. Suppose $\Omega_L$ is directed. Then $x \in \text{LimF}(f)$ if and only if for every element $b$ of $\text{Balls } x$, there exists an element $n$ of $L$ such that for every element $m$ of $L$ such that $n \leq m$ holds $f(m) \in b$. The theorem is a consequence of (6).
(9) Let us consider a non empty metric space $M$, a sequence $s$ of the carrier of $M_{\text{top}}$, and a point $x$ of $M_{\text{top}}$. Then $x \in \lim F(s)$ if and only if for every element $b$ of Balls $x$, there exists a natural number $i$ such that for every natural number $j$ such that $i \leq j$ holds $s(j) \in b$. The theorem is a consequence of (6).

(10) Let us consider a non empty topological structure $T$, a sequence $s$ of $T$, and a point $x$ of $T$. Then $x \in \lim s$ if and only if for every subset $U_1$ of $T$ such that $U_1$ is open and $x \in U_1$ there exists a natural number $n$ such that for every natural number $m$ such that $n \leq m$ holds $s(m) \in U_1$.

Let us consider a non empty metric space $M$, a sequence $s$ of the carrier of $M_{\text{top}}$, and a point $x$ of $M_{\text{top}}$. Now we state the propositions:

(11) $x \in \lim s$ if and only if for every element $b$ of Balls $x$, there exists a natural number $n$ such that for every natural number $m$ such that $n \leq m$ holds $s(m) \in b$. The theorem is a consequence of (6) and (10).

(12) $x \in \lim F(s)$ if and only if $x \in \lim s$. The theorem is a consequence of (9) and (11).

3. Filter and Limit of a Sequence in Real Normed Space

Now we state the propositions:

(13) Let us consider a real normed space $N$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into the carrier of $(\text{MetricSpaceNorm } N)_{\text{top}}$, a point $x$ of $(\text{MetricSpaceNorm } N)_{\text{top}}$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Suppose $\Omega_L$ is directed. Then $x \in \lim F(f)$ if and only if for every element $b$ of $B$, there exists an element $i$ of $L$ such that for every element $j$ of $L$ such that $i \leq j$ holds $f(j) \in b$.

(14) Let us consider a real normed space $N$, and a point $x$ of $(\text{MetricSpaceNorm } N)_{\text{top}}$. Then Balls $x$ is a generalized basis of BooleanFilterToFilter(the neighborhood system of $x$).

(15) Let us consider a real normed space $N$, a sequence $s$ of the carrier of $(\text{MetricSpaceNorm } N)_{\text{top}}$, and a point $x$ of $(\text{MetricSpaceNorm } N)_{\text{top}}$. Then $x \in \lim F(s)$ if and only if for every element $b$ of Balls $x$, there exists a natural number $i$ such that for every natural number $j$ such that $i \leq j$ holds $s(j) \in b$.

(16) Let us consider a real normed space $N$, and a point $x$ of $(\text{MetricSpaceNorm } N)_{\text{top}}$. Then there exists a point $y$ of MetricSpaceNorm $N$ such that

(i) $y = x$, and
(ii) Balls\( x = \{\text{Ball}(y, \frac{1}{n}) \mid \text{where } n \text{ is a natural number : } n \neq 0\}.\)

(17) Let us consider a real normed space \( N \), a point \( x \) of \((\text{MetricSpaceNorm } N)_{\text{top}}\), a point \( y \) of \( \text{MetricSpaceNorm } N \), and a positive natural number \( n \). If \( x = y \), then \( \text{Ball}(y, \frac{1}{n}) \in \text{Balls } x \).

(18) Let us consider a real normed space \( N \), a point \( x \) of \( \text{MetricSpaceNorm } N \), and a natural number \( n \). Suppose \( n \neq 0 \). Then \( \text{Ball}(x, \frac{1}{n}) = \{q \mid q \text{ is a point of } N : \rho(x, q) < \frac{1}{n}\} \).

(19) Let us consider a real normed space \( N \), an element \( x \) of \( \text{MetricSpaceNorm } N \), and a natural number \( n \). Suppose \( n \neq 0 \). Then there exists a point \( y \) of \( N \) such that

(i) \( x = y \), and

(ii) \( \text{Ball}(x, \frac{1}{n}) = \{q \mid q \text{ is a point of } N : \|y - q\| < \frac{1}{n}\} \).

Let us consider a metric structure \( P_1 \). Now we state the propositions:

(20) \( P_{1\text{top}} = \langle \text{the carrier of } P_1, \text{the open set family of } P_1 \rangle \).

(21) The carrier of \( \langle \text{the carrier of } P_1, \text{the open set family of } P_1 \rangle \) = the carrier of \( P_1 \).

(22) The carrier of \( P_{1\text{top}} \) = the carrier of \( \langle \text{the carrier of } P_1, \text{the open set family of } P_1 \rangle \).

(23) The carrier of \( P_{1\text{top}} \) = the carrier of \( P_1 \).

Now we state the proposition:

(24) Let us consider a real normed space \( N \), a sequence \( s \) of the carrier of \((\text{MetricSpaceNorm } N)_{\text{top}}\), and a natural number \( j \). Then \( s(j) \) is an element of the carrier of \((\text{MetricSpaceNorm } N)_{\text{top}}\).

Let \( N \) be a real normed space and \( x \) be a point of \((\text{MetricSpaceNorm } N)_{\text{top}}\). The functor \( \# x \) yielding a point of \( N \) is defined by the term (Def. 3) \( x \).

Now we state the proposition:

(25) Let us consider a real normed space \( N \), a sequence \( s \) of the carrier of \((\text{MetricSpaceNorm } N)_{\text{top}}\), and a point \( x \) of \((\text{MetricSpaceNorm } N)_{\text{top}}\). Then \( x \in \text{LimF}(s) \) if and only if for every positive natural number \( n \), there exists a natural number \( i \) such that for every natural number \( j \) such that \( i \leq j \) holds \( \|\# x - \# s(j)\| < \frac{1}{n} \).

Proof: Reconsider \( x_1 = x \) as a point of \((\text{MetricSpaceNorm } N)_{\text{top}}\). Consider \( y_0 \) being a point of \( \text{MetricSpaceNorm } N \) such that \( y_0 = x_1 \) and \( \text{Balls } x_1 = \{\text{Ball}(y_0, \frac{1}{n}) \mid \text{where } n \text{ is a natural number : } n \neq 0\} \). If \( x \in \text{LimF}(s) \), then for every positive natural number \( n \), there exists a natural number \( i \) such that for every natural number \( j \) such that \( i \leq j \) holds
\[ \| \# x - \# s(j) \| < \frac{1}{n} \] by (9), [20] (2). If for every positive natural number \( n \), there exists a natural number \( i \) such that for every natural number \( j \) such that \( i \leq j \) holds \[ \| \# x - \# s(j) \| < \frac{1}{n}, \] then \( x \in \text{LimF}(s) \) by [20] (2), (9). □

4. Filter and Limit of a Sequence in Linear Topological Space

Now we state the propositions:

(26) Let us consider a non empty linear topological space \( X \). Then the neighborhood system of \( 0_X \) is a local base of \( X \).

(27) Let us consider a linear topological space \( X \), a local base \( O \) of \( X \), a point \( a \) of \( X \), and a family \( P \) of subsets of \( X \). Suppose \( P = \{ a + U, \text{ where } U \text{ is a subset of } X : U \in O \} \). Then \( P \) is a generalized basis of \( a \).

(28) Let us consider a non empty linear topological space \( X \), a point \( x \) of \( X \), and a local base \( O \) of \( X \). Then \( \{ x + U, \text{ where } U \text{ is a subset of } X : U \in O \text{ and } U \text{ is a neighbourhood of } 0_X \} = \{ x + U, \text{ where } U \text{ is a subset of } X : U \in O \text{ and } U \text{ is in the neighborhood system of } 0_X \} \).

Proof: Set \( F = \text{BooleanFilterToFilter}(\text{the neighborhood system of } x) \). \( F \subseteq [B] \) by [14] (9), [27] (3), [14] (8), [6]. \( [B] \subseteq F \) by [14] (37). □

(30) Let us consider a non empty linear topological space \( X \), a sequence \( s \) of the carrier of \( X \), a point \( x \) of \( X \), a local base \( V \) of \( X \), and a family \( B \) of subsets of \( X \). Suppose \( B = \{ x + U, \text{ where } U \text{ is a subset of } X : U \in V \text{ and } U \text{ is a neighbourhood of } 0_X \} \). Then \( x \in \text{LimF}(s) \) if and only if for every element \( v \) of \( B \), there exists a natural number \( i \) such that for every natural number \( j \) such that \( i \leq j \) holds \( s(j) \in v \). The theorem is a consequence of (29).

Proof: Set \( B = \{ x + U, \text{ where } U \text{ is a subset of } X : U \in V \text{ and } U \text{ is a neighbourhood of } 0_X \} \). \( B \) is a generalized basis of \( \text{BooleanFilterToFilter} \).
(the neighborhood system of $x$). For every element $b$ of $B$, there exists a natural number $i$ such that for every natural number $j$ such that $i \leq j$ holds $s(j) \in b$ by [5, (2)]. □

(32) Let us consider a non empty linear topological space $T$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into the carrier of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Suppose $\Omega_L$ is directed. Then $x \in \text{LimF}(f)$ if and only if for every element $b$ of $B$, there exists an element $i$ of $L$ such that for every element $j$ of $L$ such that $i \leq j$ holds $f(j) \in b$.

(33) Let us consider a non empty linear topological space $T$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into the carrier of $T$, a point $x$ of $T$, and a local base $V$ of $T$. Suppose $\Omega_L$ is directed. Then $x \in \text{LimF}(f)$ if and only if for every subset $v$ of $T$ such that $v \in V \cap$ (the neighborhood system of $0_T$) there exists an element $i$ of $L$ such that for every element $j$ of $L$ such that $i \leq j$ holds $f(j) \in x + v$.

5. Series in Abelian Group: a Definition

Let $I$ be a non empty set, $L$ be an Abelian group, $x$ be a (the carrier of $L$)-valued many sorted set indexed by $I$, and $J$ be an element of Fin$I$. The functor $\sum^J_{\kappa=0} x(\kappa)$ yielding an element of $L$ is defined by

(Def. 4) there exists a one-to-one finite sequence $p$ of elements of $I$ such that $\text{rng } p = J$ and $it = (\text{the addition of } L) \odot \#^p_x$.

Now we state the proposition:

(34) Let us consider a non empty set $I$, an Abelian group $L$, a (the carrier of $L$)-valued many sorted set $x$ indexed by $I$, an element $J$ of Fin$I$, and an element $e$ of Fin$I$. Suppose $e = \emptyset$. Then

(i) $\sum^e_{\kappa=0} x(\kappa) = 0_L$, and

(ii) for every elements $e$, $f$ of Fin$I$ such that $e$ misses $f$ holds $\sum^{e\cup f}_{\kappa=0} x(\kappa) = \sum^e_{\kappa=0} x(\kappa) + \sum^f_{\kappa=0} x(\kappa)$.

The theorem is a consequence of (4).

Let $I$ be a non empty set, $L$ be an Abelian group, and $x$ be a (the carrier of $L$)-valued many sorted set indexed by $I$. The functor $(\sum^\kappa_{\alpha=0} x(\alpha))_{\kappa \in \mathbb{N}}$ yielding a function from $\Omega_{\mathcal{F}(I)}$ into the carrier of $L$ is defined by

(Def. 5) for every element $j$ of Fin$I$, $it(j) = \sum^j_{\kappa=0} x(\kappa)$.
6. Product of Family as Limit in Commutative Topological Group

Let \( I \) be a non empty set, \( L \) be a commutative semi topological group, \( x \) be a \((\text{the carrier of } L)\)-valued many sorted set indexed by \( I \), and \( J \) be an element of \( \text{Fin} \, I \). The functor \( \text{Product}(x, J) \) yielding an element of \( L \) is defined by

(Def. 6) there exists a one-to-one finite sequence \( p \) of elements of \( I \) such that \( \text{rng} \, p = J \) and it \( = (\text{the multiplication of } L) \odot \#_p^p \).

(35) Let us consider a set \( I \), a semi topological group \( G \), a function \( f \) from \( \Omega_{\mathcal{F}}(I) \) into the carrier of \( G \), a point \( x \) of \( G \), and a generalized basis \( B \) of \( \text{BooleanFilterToFilter}(\text{the neighborhood system of } x) \). Then \( x \in \text{LimF}(f) \) if and only if for every element \( b \) of \( B \), there exists an element \( i \) of \( \mathcal{F}(I) \) such that for every element \( j \) of \( \mathcal{F}(I) \) such that \( i \leq j \) holds \( f(j) \in b \). The theorem is a consequence of (5).

(36) Let us consider a non empty set \( I \), a commutative semi topological group \( L \), a \((\text{the carrier of } L)\)-valued many sorted set \( x \) indexed by \( I \), an element \( J \) of \( \text{Fin} \, I \), and an element \( e \) of \( \text{Fin} \, I \). Suppose \( e = \emptyset \). Then

(i) \( \text{Product}(x, e) = 1_L \), and

(ii) for every elements \( e, f \) of \( \text{Fin} \, I \) such that \( e \) misses \( f \) holds \( \text{Product}(x, e \cup f) = \text{Product}(x, e) \cdot \text{Product}(x, f) \).

The theorem is a consequence of (4).

Let \( I \) be a non empty set, \( L \) be a commutative semi topological group, and \( x \) be a \((\text{the carrier of } L)\)-valued many sorted set indexed by \( I \). The functor the partial product of \( x \) yielding a function from \( \Omega_{\mathcal{F}}(I) \) into the carrier of \( L \) is defined by

(Def. 7) for every element \( j \) of \( \text{Fin} \, I \), \( it(j) = \text{Product}(x, j) \).

(37) Let us consider a non empty set \( I \), a commutative semi topological group \( G \), a \((\text{the carrier of } G)\)-valued many sorted set \( s \) indexed by \( I \), a point \( x \) of \( G \), and a generalized basis \( B \) of \( \text{BooleanFilterToFilter}(\text{the neighborhood system of } x) \). Then \( x \in \text{LimF}(\text{the partial product of } s) \) if and only if for every element \( b \) of \( B \), there exists an element \( i \) of \( \mathcal{F}(I) \) such that for every element \( j \) of \( \mathcal{F}(I) \) such that \( i \leq j \) holds \( (\text{the partial product of } s)(j) \in b \).

7. Summable Family in Commutative Topological Group

Let \( I \) be a non empty set, \( L \) be an Abelian semi additive topological group, \( x \) be a \((\text{the carrier of } L)\)-valued many sorted set indexed by \( I \), and \( J \) be an element of \( \text{Fin} \, I \). The functor \( \sum_{\kappa=0}^{J} x(\kappa) \) yielding an element of \( L \) is defined by

...
(Def. 8) there exists a one-to-one finite sequence $p$ of elements of $I$ such that $\text{rng } p = J$ and $it = \text{(the addition of } L) \odot \#^p_x$.

Now we state the propositions:

(38) Let us consider a set $I$, a semi additive topological group $G$, a function $f$ from $\Omega_{F(I)}$ into the carrier of $G$, a point $x$ of $G$, and a generalized basis $B$ of BooleanFilterToFilter(\text{the neighborhood system of } x). Then $x \in \text{LimF}(f)$ if and only if for every element $b$ of $B$, there exists an element $i$ of $F(I)$ such that for every element $j$ of $F(I)$ such that $i \leq j$ holds $f(j) \in b$. The theorem is a consequence of (5).

(39) Let us consider a non empty set $I$, an Abelian semi additive topological group $L$, a (the carrier of $L$)-valued many sorted set $x$ indexed by $I$, an element $J$ of Fin $I$, and an element $e$ of Fin $I$. Suppose $e = \emptyset$. Then

(i) $\sum_{\kappa=0}^{e} x(\kappa) = 0_L,$ and

(ii) for every elements $e, f$ of Fin $I$ such that $e$ misses $f$ holds $\sum_{\kappa=0}^{e \cup f} x(\kappa) = \sum_{\kappa=0}^{e} x(\kappa) + \sum_{\kappa=0}^{f} x(\kappa).$

The theorem is a consequence of (4).

Let $I$ be a non empty set, $L$ be an Abelian semi additive topological group, and $x$ be a (the carrier of $L$)-valued many sorted set indexed by $I$. The functor $(\sum_{\alpha=0}^{k} x(\alpha))_{\kappa \in \mathbb{N}}$ yielding a function from $\Omega_{F(I)}$ into the carrier of $L$ is defined by

(Def. 9) for every element $j$ of Fin $I$, $it(j) = \sum_{\kappa=0}^{j} x(\kappa)$.

Now we state the proposition:

(40) Let us consider a non empty set $I$, an Abelian semi additive topological group $G$, a (the carrier of $G$)-valued many sorted set $s$ indexed by $I$, a point $x$ of $G$, and a generalized basis $B$ of BooleanFilterToFilter(\text{the neighborhood system of } x). Then $x \in \text{LimF}((\sum_{\alpha=0}^{k} s(\alpha))_{\kappa \in \mathbb{N}})$ if and only if for every element $b$ of $B$, there exists an element $i$ of $F(I)$ such that for every element $j$ of $F(I)$ such that $i \leq j$ holds $(\sum_{\alpha=0}^{k} s(\alpha))_{\kappa \in \mathbb{N}}(j) \in b$.

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