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Weak Convergence and Weak* Convergence[1]

Keiko Narita
Hirosaki-city
Aomori, Japan

Noboru Endou
Gifu National College of Technology
Gifu, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we deal with weak convergence on sequences in real normed spaces, and weak* convergence on sequences in dual spaces of real normed spaces. In the first section, we proved some topological properties of dual spaces of real normed spaces. We used these theorems for proofs of Section 3. In Section 2, we defined weak convergence and weak* convergence, and proved some properties. By `RNS_Real` Mizar functor, real normed spaces as real number spaces already defined in the article [18], we regarded sequences of real numbers as sequences of `RNS_Real`. So we proved the last theorem in this section using the theorem (8) from [25]. In Section 3, we defined weak sequential compactness of real normed spaces. We showed some lemmas for the proof and proved the theorem of weak sequential compactness of reflexive real Banach spaces. We referred to [36], [23], [24] and [3] in the formalization.

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The notation and terminology used in this paper have been introduced in the following articles: [4], [19], [17], [18], [28], [5], [6], [21], [30], [26], [25], [29], [1], [22], [16], [2], [7], [34], [35], [37], [31], [27], [14], [33], and [8].

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1. Some Properties about Dual Spaces of Real Normed Spaces

Let X be a non empty set, F be a sequence of $X^{\mathbb{N}}$, and k be a natural number. One can check that the functor $F(k)$ yields a sequence of X. Now we state the propositions:

(1) Let us consider a strict real normed space X, and a non empty subset A of X. Suppose for every point f of DualSp X such that for every point x of X such that $x \in A$ holds $(\text{Bound2Lipschitz}(f, X))(x) = 0$ holds $\text{Bound2Lipschitz}(f, X) = 0_{\text{DualSp}\,X}$. Then $\text{ClNLin}(A) = X$.
Proof: Set $M = \text{ClNLin}(A)$. Consider Z being a subset of X such that $Z =$ the carrier of $\text{Lin}(A)$ and $M = \langle \overline{Z}, \text{Zero}(\overline{Z}, X), \text{Add}(\overline{Z}, X), \text{Mult}(\overline{Z}, X),$ the norm of $\overline{Z}$ induced by $X\rangle$. Reconsider $Y =$ the carrier of M as a non empty subset of X. $Y =$ the carrier of X by [18, (2)], [32, (15)], [16, (4)], [17, (25)]. □

(2) Let us consider a strict real normed space X. If DualSp X is separable, then X is separable.
Proof: Set $Y = \text{DualSp}\,X$. Consider Y_1 being a sequence of Y such that $\text{rng}\,Y_1$ is dense. Define $\mathcal{P}[\text{natural number}, \text{point of } X] \equiv \|Y_1(\$_1)\|/_2 \leqslant |Y_1(\$_1)(\$_2)|$ and $\|\$_2\| \leqslant 1$. For every element n of $\mathbb{N}$, there exists a point x of X such that $\mathcal{P}[n, x]$ by [4, (46)], [15, (45)], [17, (24)]. Consider X_2 being a function from $\mathbb{N}$ into the carrier of X such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, X_2(n)]$ from [6, Sch. 3]. For every natural number n, $\|Y_1(n)\|/_2 \leqslant |Y_1(n)(X_2(n))|$ and $\|X_2(n)\| \leqslant 1$. Consider X_2 being a sequence of X such that for every natural number n, $\|Y_1(n)\|/_2 \leqslant |Y_1(n)(X_2(n))|$ and $\|X_2(n)\| \leqslant 1$. Set $X_1 = \text{rng}\,X_2$. For every point f of Y such that for every point x of X such that $x \in X_1$ holds $(\text{Bound2Lipschitz}(f, X))(x) = 0$ holds $\text{Bound2Lipschitz}(f, X) = 0_Y$ by [17, (23)], [16, (14)], [22, (24)], [26, (20)]. $M = X$. □

(3) Let us consider a real number x, and a point x_1 of the real normed space of $\mathbb{R}$. If $x = x_1$, then $-x = -x_1$.

(4) Let us consider real numbers x, y, and points x_1, y_1 of the real normed space of $\mathbb{R}$. If $x = x_1$ and $y = y_1$, then $x - y = x_1 - y_1$. The theorem is a consequence of (3).

Let us consider a sequence s_2 of real numbers and a sequence s_3 of the real normed space of $\mathbb{R}$. Now we state the propositions:

(5) If $s_2 = s_3$, then s_2 is convergent iff s_3 is convergent. The theorem is a consequence of (4).

(6) If $s_2 = s_3$ and s_2 is convergent, then $\lim s_2 = \lim s_3$. The theorem is a consequence of (5) and (4).

(7) Let us consider a sequence s_3 of the real normed space of $\mathbb{R}$. If s_3 is Cauchy sequence by norm, then s_3 is convergent.
Proof: Reconsider $s_2 = s_3$ as a sequence of real numbers. For every real number s such that $0 < s$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|s_2(m) - s_2(n)| < s$ by [27, (8)], (4). $\square$

Let us note that the real normed space of $\mathbb{R}$ is complete.

Let X be a real normed space, g be a sequence of DualSp X, and x be a point of X. The functor $g\#x$ yielding a sequence of real numbers is defined by

(Def. 1) for every natural number i, $it(i) = g(i)(x)$.

2. Weak Convergence and Weak* Convergence

Let X be a real normed space and x be a sequence of X. We say that x is weakly convergent if and only if

(Def. 2) there exists a point x_0 of X such that for every Lipschitzian linear functional f in X, $f \cdot x$ is convergent and $\lim(f \cdot x) = f(x_0)$.

Now we state the proposition:

(8) Let us consider a real normed space X, and a sequence x of X. If $\operatorname{rng} x \subseteq \{0_X\}$, then x is weakly convergent.
Proof: Reconsider $x_0 = 0_X$ as a point of X. For every Lipschitzian linear functional f in X, $f \cdot x$ is convergent and $\lim(f \cdot x) = f(x_0)$ by [6, (4), (15)], [4, (44)]. $\square$

Let X be a real normed space and x be a sequence of X. Assume x is weakly convergent. The functor w-lim(x) yielding a point of X is defined by

(Def. 3) for every Lipschitzian linear functional f in X, $f \cdot x$ is convergent and $\lim(f \cdot x) = f(it)$.

Let us consider a real normed space X and a sequence x of X. Now we state the propositions:

(9) If x is convergent, then x is weakly convergent and w-lim$(x) = \lim x$.
Proof: Reconsider $x_0 = \lim x$ as a point of X. For every Lipschitzian linear functional f in X, $f \cdot x$ is convergent and $\lim(f \cdot x) = f(x_0)$ by [21, (19)], [20, (46)]. $\square$

(10) Suppose X is not trivial and x is weakly convergent. Then

(i) $\|x\|$ is bounded, and

(ii) $\|\text{w-lim}(x)\| \leqslant \liminf \|x\|$, and

(iii) w-lim$(x) \in \operatorname{ClNLin}(\operatorname{rng} x)$.

PROOF: Reconsider $x_0 = \text{w-lim}(x)$ as a point of X. For every point f of $\text{DualSp}\,X$, there exists a real number K_1 such that $0 \leqslant K_1$ and for every point y of X such that $y \in \text{rng}\,x$ holds $|f(y)| \leqslant K_1$ by [14, (3)], [20, (6)], [6, (15)]. Consider K being a real number such that $0 \leqslant K$ and for every point y of X such that $y \in \text{rng}\,x$ holds $\|y\| \leqslant K$. For every natural number n, $\|x\|(n) \leqslant K$ by [6, (4)]. For every natural number n, $|\|x\|(n)| < K+1$. For every point f of $\text{DualSp}\,X$, $|f(x_0)| \leqslant \liminf \|x\| \cdot \|f\|$ by [17, (26)], [6, (15)], [13, (12), (9)]. Consider Y being a non empty subset of $\mathbb{R}$ such that $Y = \{|(\text{Bound2Lipschitz}(F, X))(x_0)|$, where F is a point of $\text{DualSp}\,X : \|F\| \leqslant 1\}$ and $\|x_0\| = \sup Y$. $x_0 \in \text{ClNLin}(\text{rng}\,x)$ by [16, (29)], [18, (2)], [17, (23)], [32, (15)]. □

Let X be a real normed space and g be a sequence of $\text{DualSp}\,X$. We say that g is weakly* convergent if and only if

(Def. 4) there exists a point g_0 of $\text{DualSp}\,X$ such that for every point x of X, $g\#x$ is convergent and $\lim(g\#x) = g_0(x)$.

Assume g is weakly* convergent. The functor $\text{w*-lim}(g)$ yielding a point of $\text{DualSp}\,X$ is defined by

(Def. 5) for every point x of X, $g\#x$ is convergent and $\lim(g\#x) = it(x)$.

Now we state the proposition:

(11) Let us consider a real normed space X, and a sequence g of $\text{DualSp}\,X$. Suppose g is convergent. Then

(i) g is weakly* convergent, and

(ii) $\text{w*-lim}(g) = \lim g$.

PROOF: Reconsider $g_0 = \lim g$ as a point of $\text{DualSp}\,X$. For every point x of X, $g\#x$ is convergent and $\lim(g\#x) = g_0(x)$ by [17, (33), (26)]. □

Let us consider a real normed space X and a sequence f of $\text{DualSp}\,X$. Now we state the propositions:

(12) If f is weakly convergent, then f is weakly* convergent.
PROOF: Reconsider $f_0 = \text{w-lim}(f)$ as a point of $\text{DualSp}\,X$. For every point x of X, $f\#x$ is convergent and $\lim(f\#x) = f_0(x)$ by [6, (15)]. □

(13) If X is reflexive, then f is weakly convergent iff f is weakly* convergent.
PROOF: If f is weakly* convergent, then f is weakly convergent by [18, (21)], [6, (15)]. □

(14) Let us consider a real Banach space X, and a subset T of $\text{DualSp}\,X$. Suppose for every point x of X, there exists a real number K such that $0 \leqslant K$ and for every point f of $\text{DualSp}\,X$ such that $f \in T$ holds $|f(x)| \leqslant K$. Then there exists a real number L such that

(i) $0 \leqslant L$, and

(ii) for every point f of DualSp X such that $f \in T$ holds $\|f\| \leqslant L$.

Proof: Reconsider $T_1 = T$ as a subset of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$. For every point x of X, there exists a real number K such that $0 \leqslant K$ and for every point f of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$ such that $f \in T_1$ holds $\|f(x)\| \leqslant K$. Consider L being a real number such that $0 \leqslant L$ and for every point f of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$ such that $f \in T_1$ holds $\|f\| \leqslant L$. For every point f of DualSp X such that $f \in T$ holds $\|f\| \leqslant L$ by [18, (18)]. □

(15) Let us consider a real Banach space X, and a sequence f of DualSp X. Suppose f is weakly* convergent. Then

(i) $\|f\|$ is bounded, and

(ii) $\|\text{w*-lim}(f)\| \leqslant \liminf \|f\|$.

Proof: Reconsider $f_0 = \text{w*-lim}(f)$ as a point of DualSp X. For every point x of X, there exists a real number K such that $0 \leqslant K$ and for every point g of DualSp X such that $g \in \operatorname{rng} f$ holds $|g(x)| \leqslant K$ by [6, (11)], [13, (12)], [4, (46)]. Consider L being a real number such that $0 \leqslant L$ and for every point g of DualSp X such that $g \in \operatorname{rng} f$ holds $\|g\| \leqslant L$. For every natural number n, $|\|f\|(n)| < L + 1$ by [6, (4)]. For every point x of X, $|f_0(x)| \leqslant \liminf \|f\| \cdot \|x\|$ by [13, (12), (9)], [17, (26)], [25, (1)]. □

(16) Let us consider a real normed space X, a point x of X, a sequence v of DualSp X, and a sequence v_1 of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$. If $v = v_1$, then $v \# x = v_1 \# x$.

(17) Let us consider a real Banach space X, a subset X_1 of X, and a sequence v of DualSp X. Suppose $\|v\|$ is bounded and X_1 is dense and for every point x of X such that $x \in X_1$ holds $v \# x$ is convergent. Then v is weakly* convergent.

Proof: Reconsider $v_1 = v$ as a sequence of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$. Reconsider $X_2 = X_1$ as a subset of LinearTopSpaceNorm X. For every point x of X such that $x \in X_2$ holds $v_1 \# x$ is convergent. For every point x of X, there exists a real number K such that $0 \leqslant K$ and for every natural number n, $\|(v_1 \# x)(n)\| \leqslant K$ by [14, (3)], [17, (26)], (16). Consider t being a point of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$ such that for every point x of X, $v_1 \# x$ is convergent and $t(x) = \lim(v_1 \# x)$ and $\|t(x)\| \leqslant \liminf \|v_1\| \cdot \|x\|$ and $\|t\| \leqslant \liminf \|v_1\|$.

Reconsider $g_0 = t$ as a point of DualSp X. For every point x of X, $v\#x$ is convergent and $\lim(v\#x) = g_0(x)$. $\square$

(18) Let us consider a real Banach space X, and a sequence f of DualSp X. Then f is weakly* convergent if and only if $\|f\|$ is bounded and there exists a subset X_1 of X such that X_1 is dense and for every point x of X such that $x \in X_1$ holds $f\#x$ is convergent. The theorem is a consequence of (15) and (17).

3. Weak Sequential Compactness of Real Banach Spaces

Let X be a real normed space and X_1 be a non empty subset of X. We say that X_1 is weakly sequentially compact if and only if

(Def. 6) for every sequence s_2 of X_1, there exists a sequence s_3 of X such that s_3 is subsequence of s_2 and weakly convergent and w-lim$(s_3) \in X$.

Now we state the proposition:

(19) Let us consider a real normed space X, and a sequence x of X. Suppose X is reflexive. Then x is weakly convergent if and only if BidualFunc $X \cdot x$ is weakly* convergent.

Proof: Set $f =$ BidualFunc $X \cdot x$. Consider f_0 being a point of DualSp DualSp X such that for every point h of DualSp X, $f\#h$ is convergent and $\lim(f\#h) = f_0(h)$. Consider x_0 being a point of X such that for every point g of DualSp X, $f_0(g) = g(x_0)$. For every Lipschitzian linear functional g in X, $g \cdot x$ is convergent and $\lim(g \cdot x) = g(x_0)$ by [6, (15)]. $\square$

Let us consider a real normed space X, a sequence f of DualSp X, and a point x of X.

Let us assume that $\|f\|$ is bounded. Now we state the propositions:

(20) There exists a sequence f_0 of DualSp X such that

(i) f_0 is a subsequence of f, and

(ii) $\|f_0\|$ is bounded, and

(iii) $f_0\#x$ is convergent.

Proof: Consider r_0 being a real number such that $0 < r_0$ and for every natural number m, $|\|f\|(m)| < r_0$. Set $r = r_0 \cdot \|x\| + 1$. For every natural number m, $|(f\#x)(m)| < r$ by [17, (26)]. Reconsider $s_2 = f\#x$ as a sequence of real numbers. Consider s_3 being a sequence of real numbers such that s_3 is subsequence of s_2 and convergent. Consider N being an increasing sequence of $\mathbb{N}$ such that $s_3 = s_2 \cdot N$. Set $f_0 = f \cdot N$. For every natural number k, $(f_0\#x)(k) = s_3(k)$ by [6, (15)]. For every natural number n, $|\|f_0\|(n)| < r_0$ by [6, (15)]. $\square$

(21) There exists a sequence f_0 of DualSp X such that

(i) f_0 is a subsequence of f, and

(ii) $\|f_0\|$ is bounded, and

(iii) $f_0\#x$ is convergent and subsequence of $f\#x$.

PROOF: Consider r_0 being a real number such that $0 < r_0$ and for every natural number m, $|\|f\|(m)| < r_0$. Set $r = r_0 \cdot \|x\| + 1$. For every natural number m, $|(f\#x)(m)| < r$ by [17, (26)]. Reconsider $s_2 = f\#x$ as a sequence of real numbers. Consider s_3 being a sequence of real numbers such that s_3 is subsequence of s_2 and convergent. Consider N being an increasing sequence of $\mathbb{N}$ such that $s_3 = s_2 \cdot N$. Reconsider $f_0 = f \cdot N$ as a sequence of DualSp X. For every natural number n, $|\|f_0\|(n)| < r_0$ by [6, (15)]. □

(22) There exists a sequence f_0 of DualSp X and there exists an increasing sequence N of $\mathbb{N}$ such that f_0 is a subsequence of f and $\|f_0\|$ is bounded and $f_0\#x$ is convergent and subsequence of $f\#x$ and $f_0 = f \cdot N$. The theorem is a consequence of (21).

Let us consider a real normed space X, a sequence f of DualSp X, and a sequence x of X.

Let us assume that $\|f\|$ is bounded. Now we state the propositions:

(23) There exists a sequence F of (the carrier of DualSp $X)^{\mathbb{N}}$ such that

(i) $F(0)$ is a subsequence of f, and

(ii) $F(0)\#x(0)$ is convergent, and

(iii) for every natural number k, $F(k+1)$ is a subsequence of $F(k)$, and

(iv) for every natural number k, $F(k+1)\#x(k+1)$ is convergent.

PROOF: Set $D =$ (the carrier of DualSp $X)^{\mathbb{N}}$. Consider f_0 being a sequence of DualSp X such that f_0 is a subsequence of f and $\|f_0\|$ is bounded and $f_0\#x(0)$ is convergent. Reconsider $A = f_0$ as an element of D. Define $\mathcal{P}[\text{natural number}, \text{sequence of DualSp } X, \text{sequence of DualSp } X] \equiv$ if $\|\$_2\|$ is bounded, then $\$_3$ is a subsequence of $\$_2$ and $\|\$_3\|$ is bounded and $\$_3\#x(\$_1+1)$ is convergent. For every natural number n and for every element z of D, there exists an element y of D such that $\mathcal{P}[n, z, y]$ by (20), [6, (8)]. Consider F being a sequence of D such that $F(0) = A$ and for every natural number n, $\mathcal{P}[n, F(n), F(n+1)]$ from [10, Sch. 2]. Define $\mathcal{Q}[\text{natural number}] \equiv F(\$_1+1)$ is a subsequence of $F(\$_1)$ and $\|F(\$_1+1)\|$ is bounded and $F(\$_1+1)\#x(\$_1+1)$ is convergent. For every natural number n, $\mathcal{Q}[n]$ from [1, Sch. 2]. □

(24) There exists a sequence F of (the carrier of $\operatorname{DualSp} X)^{\mathbb{N}}$ and there exists a sequence N of $\mathbb{N}^{\mathbb{N}}$ such that $F(0)$ is a subsequence of f and $F(0)\#x(0)$ is convergent and $N(0)$ is an increasing sequence of $\mathbb{N}$ and $F(0) = f \cdot N(0)$ and for every natural number k, $F(k+1)$ is a subsequence of $F(k)$ and for every natural number k, $F(k+1)\#x(k+1)$ is convergent and for every natural number k, $F(k+1)\#x(k+1)$ is a subsequence of $F(k)\#x(k+1)$ and for every natural number k, $N(k+1)$ is an increasing sequence of $\mathbb{N}$ and for every natural number k, $F(k+1) = F(k) \cdot N(k+1)$.

PROOF: Consider f_0 being a sequence of $\operatorname{DualSp} X$ such that f_0 is a subsequence of f and $\|f_0\|$ is bounded and $f_0\#x(0)$ is convergent and subsequence of $f\#x(0)$. Consider N_0 being an increasing sequence of $\mathbb{N}$ such that $f_0 = f \cdot N_0$. Set $D_1 =$ (the carrier of $\operatorname{DualSp} X)^{\mathbb{N}}$. Set $D_2 = \mathbb{N}^{\mathbb{N}}$. Reconsider $A = f_0$ as an element of D_1. Reconsider $B = N_0$ as an element of D_2. Define $\mathcal{P}[\text{natural number}, \text{sequence of } \operatorname{DualSp} X, \text{sequence of } \mathbb{N}, \text{sequence of } \operatorname{DualSp} X, \text{sequence of } \mathbb{N}] \equiv$ if $\|\$_2\|$ is bounded, then $\$_4$ is a subsequence of $\$_2$ and $\|\$_4\|$ is bounded and $\$_4\#x(\$_1+1)$ is convergent and subsequence of $\$_2\#x(\$_1+1)$ and $\$_5$ is an increasing sequence of $\mathbb{N}$ and $\$_4 = \$_2 \cdot \$_5$. For every natural number n and for every element z of D_1 and for every element y of D_2, there exists an element z_1 of D_1 and there exists an element y_1 of D_2 such that $\mathcal{P}[n, z, y, z_1, y_1]$ by (22), [6, (8)]. Consider F being a sequence of D_1, N being a sequence of D_2 such that $F(0) = A$ and $N(0) = B$ and for every natural number n, $\mathcal{P}[n, F(n), N(n), F(n+1), N(n+1)]$ from [11, Sch. 3]. Define $\mathcal{Q}[\text{natural number}] \equiv F(\$_1+1)$ is a subsequence of $F(\$_1)$ and $\|F(\$_1+1)\|$ is bounded and $F(\$_1+1)\#x(\$_1+1)$ is convergent and subsequence of $F(\$_1)\#x(\$_1+1)$ and $N(\$_1+1)$ is an increasing sequence of $\mathbb{N}$ and $F(\$_1+1) = F(\$_1) \cdot N(\$_1+1)$. For every natural number n, $\mathcal{Q}[n]$ from [1, Sch. 2]. □

(25) There exists a sequence M of $\operatorname{DualSp} X$ such that

 (i) M is a subsequence of f, and
 (ii) for every natural number k, $M\#x(k)$ is convergent.

PROOF: Consider F being a sequence of (the carrier of $\operatorname{DualSp} X)^{\mathbb{N}}$, N being a sequence of $\mathbb{N}^{\mathbb{N}}$ such that $F(0)$ is a subsequence of f and $F(0)\#x(0)$ is convergent and $N(0)$ is an increasing sequence of $\mathbb{N}$ and $F(0) = f \cdot N(0)$ and for every natural number k, $F(k+1)$ is a subsequence of $F(k)$ and for every natural number k, $F(k+1)\#x(k+1)$ is convergent and for every natural number k, $F(k+1)\#x(k+1)$ is a subsequence of $F(k)\#x(k+1)$ and for every natural number k, $N(k+1)$ is an increasing sequence of $\mathbb{N}$ and for every natural number k, $F(k+1) = F(k) \cdot N(k+1)$. Define $\mathcal{F}(\text{element of } \mathbb{N}) = F(\$_1)(\$_1)$. Consider M being a function from $\mathbb{N}$ into

DualSp X such that for every element k of $\mathbb{N}$, $M(k) = \mathcal{F}(k)$ from [6, Sch. 4]. For every natural number k, $M(k) = F(k)(k)$. Set $D = \mathbb{N}^{\mathbb{N}}$. Reconsider $A = N(0)$ as an element of D. Define $\mathcal{P}[\text{natural number}, \text{sequence of } \mathbb{N}, \text{sequence of } \mathbb{N}] \equiv \$_3 = \$_2 \cdot N(\$_1 + 1)$. For every natural number n and for every element x of D, there exists an element y of D such that $\mathcal{P}[n, x, y]$ by [6, (8)]. Consider J being a sequence of D such that $J(0) = A$ and for every natural number n, $\mathcal{P}[n, J(n), J(n+1)]$ from [10, Sch. 2]. Define $\mathcal{Q}[\text{natural number}] \equiv J(\$_1)$ is an increasing sequence of $\mathbb{N}$. For every natural number n such that $\mathcal{Q}[n]$ holds $\mathcal{Q}[n+1]$. For every natural number n, $\mathcal{Q}[n]$ from [1, Sch. 2]. Define $\mathcal{R}[\text{natural number}] \equiv F(\$_1) = f \cdot J(\$_1)$. For every natural number n such that $\mathcal{R}[n]$ holds $\mathcal{R}[n+1]$ by [34, (36)]. For every natural number n, $\mathcal{R}[n]$ from [1, Sch. 2]. Define $\mathcal{H}(\text{element of } \mathbb{N}) = J(\$_1)(\$_1)$. Consider L being a function from $\mathbb{N}$ into $\mathbb{N}$ such that for every element k of $\mathbb{N}$, $L(k) = \mathcal{H}(k)$ from [6, Sch. 4]. For every natural number k, $L(k) = J(k)(k)$. Reconsider $L_0 = L$ as a sequence of real numbers. For every natural number k, $L_0(k) < L_0(k+1)$ by [6, (7), (15)], [12, (14), (1)]. For every natural number k, $M(k) = (f \cdot L)(k)$ by [6, (15)]. For every natural number k, $M\#x(k)$ is convergent by [1, (6), (11)], [12, (14)], [30, (3)]. $\square$

Now we state the propositions:

(26) Let us consider a real Banach space X, and a sequence f of DualSp X. Suppose X is separable and $\|f\|$ is bounded. Then there exists a sequence f_0 of DualSp X such that f_0 is subsequence of f and weakly* convergent.
PROOF: Consider x_0 being a sequence of X such that $\operatorname{rng} x_0$ is dense. Consider f_0 being a sequence of DualSp X such that f_0 is a subsequence of f and for every natural number n, $f_0\#x_0(n)$ is convergent. For every point x of X, there exists a real number K such that $0 \leqslant K$ and for every natural number n, $|(f\#x)(n)| \leqslant K$ by [14, (3)], [17, (26)]. Set $T = \operatorname{rng} f_0$. Consider N being an increasing sequence of $\mathbb{N}$ such that $f_0 = f \cdot N$. For every point x of X, there exists a real number K such that $0 \leqslant K$ and for every point g of DualSp X such that $g \in T$ holds $|g(x)| \leqslant K$ by [6, (15), (11)]. Consider L being a real number such that $0 \leqslant L$ and for every point g of DualSp X such that $g \in T$ holds $\|g\| \leqslant L$. Set $M = L + 1$. For every Lipschitzian linear functional g in X such that $g \in T$ for every points x, y of X, $|g(x) - g(y)| \leqslant M \cdot \|x - y\|$ by [31, (16)], [17, (26)]. For every point x of X, $f_0\#x$ is convergent by [9, (8), (16)], [22, (6)], [16, (17)]. Define $\mathcal{X}[\text{element of the carrier of } X, \text{object}] \equiv \$_2 = \lim(f_0\#\$_1)$. For every element x of the carrier of X, there exists an element y of $\mathbb{R}$ such that $\mathcal{X}[x, y]$. Consider f_1 being a function from the carrier of X into $\mathbb{R}$ such that for every element x of the carrier of X, $\mathcal{X}[x, f_1(x)]$ from [6,

Sch. 3]. f_1 is additive by [13, (7)], [14, (6)]. f_1 is homogeneous by [13, (9)], [14, (8)]. Consider M being a real number such that $0 < M$ and for every natural number n, $|\|f\|(n)| < M$. □

(27) Let us consider a real Banach space X, and a sequence x of X. Suppose X is reflexive and $\|x\|$ is bounded. Then there exists a sequence x_0 of X such that x_0 is subsequence of x and weakly convergent.

PROOF: Set $L = \mathrm{ClNLin}(\mathrm{rng}\, x)$. For every object z such that $z \in \mathrm{rng}\, x$ holds $z \in$ the carrier of L by [32, (15)], [16, (4)]. □

(28) Let us consider a real Banach space X, and a non empty subset X_1 of X. Suppose X is non trivial and reflexive. Then X_1 is weakly sequentially compact if and only if there exists a non empty subset S of $\mathbb{R}$ such that $S = \{\|x\|$, where x is a point of $X : x \in X_1\}$ and S is upper bounded.

PROOF: For every sequence s_2 of X_1, there exists a sequence s_3 of X such that s_3 is subsequence of s_2 and weakly convergent and w-lim$(s_3) \in X$ by [6, (7)], (27). □

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