Two Axiomatizations of Nelson Algebras

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Summary. Nelson algebras were first studied by Rasiowa and Białynicki-Birula [1] under the name N-lattices or quasi-pseudo-Boolean algebras. Later, in investigations by Monteiro and Brignole [3, 4], and [2] the name "Nelson algebras" was adopted – which is now commonly used to show the correspondence with Nelson's paper [14] on constructive logic with strong negation.

By a Nelson algebra we mean an abstract algebra

$$\langle L, \top, -, \neg, \rightarrow, \Rightarrow, \sqcup, \sqcap \rangle$$

where L is the carrier, $-$ is a quasi-complementation (Rasiowa used the sign $\sim$, but in Mizar "$-$" should be used to follow the approach described in [12] and [10]), $\neg$ is a weak pseudo-complementation, $\rightarrow$ is weak relative pseudo-complementation and $\Rightarrow$ is implicative operation. $\sqcup$ and $\sqcap$ are ordinary lattice binary operations of supremum and infimum.

In this article we give the definition and basic properties of these algebras according to [16] and [15]. We start with preliminary section on quasi-Boolean algebras (i.e. de Morgan bounded lattices). Later we give the axioms in the form of Mizar adjectives with names corresponding with those in [15]. As our main result we give two axiomatizations (non-equational and equational) and the full formal proof of their equivalence.

The second set of equations is rather long but it shows the logical essence of Nelson lattices. This formalization aims at the construction of algebraic model of rough sets [9] in our future submissions. Section 4 contains all items from Th. 1.2 and 1.3 (and the itemization is given in the text). In the fifth section we provide full formal proof of Th. 2.1 p. 75 [16].

The notation and terminology used in this paper have been introduced in the following articles: [5], [6], [7], [18], [11], [13], [17], and [8].

1. De Morgan and Quasi-Boolean Lattices

Let L be a non empty ortholattice structure. We say that L is de Morgan if and only if

(Def. 1)   for every elements x, y of L, $(x \sqcap y)^{\mathrm{c}} = x^{\mathrm{c}} \sqcup y^{\mathrm{c}}$.

One can verify that every non empty ortholattice structure which is de Morgan and involutive is also de Morgan and every non empty ortholattice structure which is de Morgan and involutive is also de Morgan.

Every non empty ortholattice structure which is trivial is also de Morgan and there exists a non empty ortholattice structure which is de Morgan, involutive, bounded, distributive, and lattice-like.

A de Morgan algebra is a de Morgan, involutive, distributive, lattice-like, non empty ortholattice structure.

A quasi-Boolean algebra is a bounded de Morgan algebra. From now on L denotes a quasi-Boolean algebra and x, y, z denote elements of L.

Now we state the propositions:

(1) $(x \sqcup y)^{\mathrm{c}} = x^{\mathrm{c}} \sqcap y^{\mathrm{c}}$.

(2) $(\top_L)^{\mathrm{c}} = \bot_L$.

(3) $(\bot_L)^{\mathrm{c}} = \top_L$.

(4) $x \sqcap (x \sqcap y) = x \sqcap y$.

(5) $x \sqcup (x \sqcup y) = x \sqcup y$.

2. The Structure and Operators in Nelson Algebras

We consider Nelson structures which extend ortholattice structures and are systems

$$\langle \text{a carrier, a unity, a complement operation, a weak pseudo-complementation,}$$

$$\text{a weak relative pseudo-complementation, an implicative operation,}$$

$$\text{a join operation, a meet operation} \rangle$$

where the carrier is a set, the unity is an element of the carrier, the complement operation and the weak pseudo-complementation are unary operations on the carrier, the weak relative pseudo-complementation and the implicative operation

and the join operation and the meet operation are binary operations on the carrier.

Note that there exists a Nelson structure which is strict and non empty and there exists a non empty Nelson structure which is trivial, de Morgan, involutive, bounded, distributive, and lattice-like.

Let L be a non empty Nelson structure and a, b be elements of L. The functor $a \rightarrow b$ yielding an element of L is defined by the term

(Def. 2)   (the weak relative pseudo-complementation of L)(a, b).

We say that $a < b$ if and only if

(Def. 3) $a \rightarrow b = \top_L$.

We say that $a \leqslant b$ if and only if

(Def. 4) $a = a \sqcap b$.

Let a be an element of L. The functor $\neg a$ yielding an element of L is defined by the term

(Def. 5)   (the weak pseudo-complementation of L)(a).

Let a, b be elements of L. The functor $a \Rightarrow b$ yielding an element of L is defined by the term

(Def. 6)   (the implicative operation of L)(a, b).

3. The Non-Equational Axiomatization

Let L be a non empty Nelson structure. We say that L has reflexive $<$ if and only if

(Def. 7)   for every element a of L, $a < a$.

We say that L has transitive $<$ if and only if

(Def. 8)   for every elements a, b, c of L such that $a < b < c$ holds $a < c$.

Let L be a bounded, lattice-like, non empty Nelson structure. We say that L is quasi-Boolean if and only if

(Def. 9) L is de Morgan, involutive, and distributive.

Let us note that every bounded, lattice-like, non empty Nelson structure which is quasi-Boolean is also de Morgan, involutive, and distributive.

Every bounded, lattice-like, non empty Nelson structure which is de Morgan, involutive, and distributive is also quasi-Boolean.

Let L be a non empty Nelson structure. We say that L satisfies (qpB$_3$) if and only if

(Def. 10)   for every elements x, a, b of L, $a \sqcap x < b$ iff $x < a \rightarrow b$.

We say that L satisfies (qpB$_4$) if and only if

(Def. 11)   for every elements a, b of L, $a \Rightarrow b = (a \rightarrow b) \sqcap (-b \rightarrow -a)$.

   We say that L satisfies (qpB$_5$) if and only if

(Def. 12)   for every elements a, b of L, $a \Rightarrow b = \top_L$ iff $a \sqcap b = a$.

   We say that L satisfies (qpB$_6$) if and only if

(Def. 13)   for every elements a, b, c of L such that $a < c$ and $b < c$ holds $a \sqcup b < c$.

   We say that L satisfies (qpB$_7$) if and only if

(Def. 14)   for every elements a, b, c of L such that $a < b$ and $a < c$ holds $a < b \sqcap c$.

   We say that L satisfies (qpB$_8$) if and only if

(Def. 15)   for every elements a, b of L, $a \sqcap -b < -(a \rightarrow b)$.

   We say that L satisfies (qpB$_9$) if and only if

(Def. 16)   for every elements a, b of L, $-(a \rightarrow b) < a \sqcap -b$.

   We say that L satisfies (qpB$_{10}$) if and only if

(Def. 17)   for every element a of L, $a < -\neg a$.

   We say that L satisfies (qpB$_{11}$) if and only if

(Def. 18)   for every element a of L, $-\neg a < a$.

   We say that L satisfies (qpB$_{12}$) if and only if

(Def. 19)   for every elements a, b of L, $a \sqcap -a < b$.

   We say that L satisfies (qpB$_{13}$) if and only if

(Def. 20)   for every element a of L, $\neg a = a \rightarrow -\top_L$.

   Let us observe that there exists a bounded, lattice-like, non empty Nelson structure which is quasi-Boolean and has reflexive $<$ and transitive $<$ and satisfies (qpB$_3$), (qpB$_4$), (qpB$_5$), (qpB$_6$), (qpB$_7$), (qpB$_8$), (qpB$_9$), (qpB$_{10}$), (qpB$_{11}$), (qpB$_{12}$), and (qpB$_{13}$).

   A Nelson algebra is a quasi-Boolean, bounded, lattice-like, non empty Nelson structure with reflexive $<$ and transitive $<$. Let L be a bounded, non empty Nelson structure and a, b be elements of L. Let us observe that the functor $a \Rightarrow b$ is defined by the term

(Def. 21) $(a \rightarrow b) \sqcap (-b \rightarrow -a)$.

   From now on L denotes a Nelson algebra and a, b, c, d, x, y, z denote elements of L.

   Now we state the propositions:

   (6) $a \sqsubseteq b$ if and only if $a \leqslant b$.

   (7) $a \leqslant b \leqslant a$ if and only if $a = b$.
        PROOF: If $a \leqslant b \leqslant a$, then $a = b$ by [18, (4), (8)]. $\square$

   (8) $a \sqcap b = \top_L$ if and only if $a = \top_L$ and $b = \top_L$.

(9) $a \leqslant b$ if and only if $a < b$ and $-b < -a$. The theorem is a consequence of (8).

(10) $a \sqcap b < a$. The theorem is a consequence of (9).

(11) $a < a \sqcup b$. The theorem is a consequence of (9).

(12) $a \leqslant b$ if and only if $a \Rightarrow b = \top_L$.

(13) $-(a \sqcap b) = -a \sqcup -b$. The theorem is a consequence of (1).

(14) $(a \sqcap -a) \sqcap (b \sqcup -b) = a \sqcap -a$. The theorem is a consequence of (1), (13), and (9).

(15) If $a \leqslant b \leqslant c$, then $a \leqslant c$.

(16) If $b \leqslant c$, then $a \sqcup b \leqslant a \sqcup c$ and $a \sqcap b \leqslant a \sqcap c$. The theorem is a consequence of (9), (1), and (13).

(17) $-a \sqcup b \leqslant a \rightarrow b$. The theorem is a consequence of (1), (2), (9), (10), (16), and (15).

(18) $(a \rightarrow b) \sqcap (-a \sqcup b) = -a \sqcup b$. The theorem is a consequence of (1), (13), (17), (10), (9), and (7).

(19) $-a \sqcup b < a \rightarrow b$. The theorem is a consequence of (18) and (9).

(20) $a \sqcap (a \rightarrow b) = a \sqcap (-a \sqcup b)$. The theorem is a consequence of (11), (10), (13), (1), (19), (9), and (7).

(21) If $-x < -y$, then $-(z \rightarrow x) < -(z \rightarrow y)$.

Let us assume that $x < y$. Now we state the propositions:

(22) $a \sqcap (a \rightarrow x) < y$. The theorem is a consequence of (20) and (10).

(23) $a \rightarrow x < a \rightarrow y$. The theorem is a consequence of (22).

(24) $a \rightarrow (b \sqcap c) = (a \rightarrow b) \sqcap (a \rightarrow c)$. The theorem is a consequence of (11), (13), (10), (23), (9), and (7).

4. Properties of Nelson Algebras

Now we state the propositions:

(25) [SEE ALSO [16] P. 69, TH. 1.2 (5)]:
$a \Rightarrow a = \top_L$.

(26) [SEE ALSO [16] P. 69, TH. 1.2 (6)]:
If $a \Rightarrow b = \top_L$ and $b \Rightarrow c = \top_L$, then $a \Rightarrow c = \top_L$.

(27) [SEE ALSO [16] P. 69, TH. 1.2 (7)]:
If $a \Rightarrow b = \top_L$ and $b \Rightarrow a = \top_L$, then $a = b$.

(28) [SEE ALSO [16] P. 69, TH. 1.2 (8)]:
$a \Rightarrow \top_L = \top_L$.

(29) [SEE ALSO [16] P. 69, TH. 1.3 (9)]:
      $a \rightarrow a = \top_L$.

(30) [SEE ALSO [16] P. 69, TH. 1.3 (10)]:
      If $a \rightarrow b = \top_L$ and $b \rightarrow c = \top_L$, then $a \rightarrow c = \top_L$.

(31) [SEE ALSO [16] P. 69, TH. 1.3 (11)]:
      If $b < c$, then $a \sqcup b < a \sqcup c$ and $a \sqcap b < a \sqcap c$.

(32) [SEE ALSO [16] P. 69, TH. 1.3 (12)]:
      If $a < b$ and $c < d$, then $a \sqcup c < b \sqcup d$ and $a \sqcap c < b \sqcap d$.

(33) [SEE ALSO [16] P. 69, TH. 1.3 (13)]:
      $a \sqcap (a \rightarrow b) < b$.

(34) [SEE ALSO [16] P. 69, TH. 1.3 (14)]:
      $a \rightarrow (b \rightarrow c) = (a \sqcap b) \rightarrow c$.

(35) [SEE ALSO [16] P. 69, TH. 1.3 (15)]:
      $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$.

(36) [SEE ALSO [16] P. 69, TH. 1.3 (16)]:
      $a < (a \rightarrow b) \rightarrow b$. The theorem is a consequence of (33).

(37) [SEE ALSO [16] P. 71, TH. 1.3 (50)]:
      $a < b \rightarrow (a \sqcap b)$. The theorem is a consequence of (9).

(38) [SEE ALSO [16] P. 69, TH. 1.3 (17)]:
      $a \sqcap -a \leqslant b \sqcup -b$. The theorem is a consequence of (1) and (9).

(39) [SEE ALSO [16] P. 70, TH. 1.3 (18)]:
      $a \leqslant b \Rightarrow a \sqcap b$. The theorem is a consequence of (37) and (9).

(40) [SEE ALSO [16] P. 70, TH. 1.3 (19)]:
      $a \rightarrow \neg b = b \rightarrow \neg a$. The theorem is a consequence of (35).

(41) [SEE ALSO [16] P. 70, TH. 1.3 (20)]:
      $a \rightarrow \top_L = \top_L$. The theorem is a consequence of (9).

(42) [SEE ALSO [16] P. 70, TH. 1.3 (21)]:
      $\bot_L \rightarrow a = \top_L$. The theorem is a consequence of (9).

(43) [SEE ALSO [16] P. 70, TH. 1.3 (22)]:
      $\top_L \rightarrow b = b$. The theorem is a consequence of (9), (33), and (7).

(44) [SEE ALSO [16] P. 70, TH. 1.3 (23)]:
      If $a = \top_L$ and $a \rightarrow b = \top_L$, then $b = \top_L$.

(45) [SEE ALSO [16] P. 70, TH. 1.3 (24)]:
      $a \rightarrow (b \rightarrow a) = \top_L$. The theorem is a consequence of (9).

(46) [SEE ALSO [16] P. 70, TH. 1.3 (25)]:
      $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = \top_L$. The theorem is a consequence of (33) and (35).

(47) [SEE ALSO [16] P. 70, TH. 1.3 (26)]:
$a \to (a \sqcup b) = \top_L$. The theorem is a consequence of (11).

(48) [SEE ALSO [16] P. 70, TH. 1.3 (27)]:
$b \to (a \sqcup b) = \top_L$. The theorem is a consequence of (11).

(49) [SEE ALSO [16] P. 70, TH. 1.3 (28)]:
$(a \to c) \to ((b \to c) \to ((a \sqcup b) \to c)) = \top_L$. The theorem is a consequence of (33) and (10).

(50) [SEE ALSO [16] P. 70, TH. 1.3 (29)]:
$(a \sqcap b) \to a = \top_L$. The theorem is a consequence of (10).

(51) [SEE ALSO [16] P. 70, TH. 1.3 (30)]:
$(a \sqcap b) \to b = \top_L$. The theorem is a consequence of (10).

(52) [SEE ALSO [16] P. 70, TH. 1.3 (31)]:
$(a \to b) \to ((a \to c) \to (a \to (b \sqcap c))) = \top_L$. The theorem is a consequence of (33).

(53) [SEE ALSO [16] P. 70, TH. 1.3 (32)]:
$(a \to \neg b) \to (b \to \neg a) = \top_L$. The theorem is a consequence of (40) and (29).

(54) [SEE ALSO [16] P. 70, TH. 1.3 (33)]:
$\neg(a \to a) \to b = \top_L$. The theorem is a consequence of (29), (2), (43), and (42).

(55) [SEE ALSO [16] P. 70, TH. 1.3 (34)]:
$-a \to (a \to b) = \top_L$.

(56) [SEE ALSO [16] P. 70, TH. 1.3 (35)]:
$(-(a \to b) \to (a \sqcap -b)) \sqcap ((a \sqcap -b) \to -(a \to b)) = \top_L$.

(57) [SEE ALSO [16] P. 70, TH. 1.3 (36)]:
$(-\neg a \to a) \sqcap (a \to -\neg a) = \top_L$.

(58) [SEE ALSO [16] P. 70, TH. 1.3 (37)]:
$--a = a$.

(59) [SEE ALSO [16] P. 70, TH. 1.3 (38)]:
$-(a \sqcup b) = -a \sqcap -b$.

(60) [SEE ALSO [16] P. 70, TH. 1.3 (39)]:
$-(a \sqcap b) = -a \sqcup -b$. The theorem is a consequence of (1).

(61) [SEE ALSO [16] P. 70, TH. 1.3 (40)]:
If $a < b$, then $b \to c < a \to c$ and $c \to a < c \to b$. The theorem is a consequence of (43), (46), (10), and (41).

(62) [SEE ALSO [16] P. 70, TH. 1.3 (41)]:
$(a \to b) \to ((c \to d) \to ((a \sqcap c) \to (b \sqcap d))) = \top_L$. The theorem is a consequence of (33).

(63)   [SEE ALSO [16] P. 70, TH. 1.3 (42)]:
       $(a \to b) \to ((c \to d) \to ((a \sqcup c) \to (b \sqcup d))) = \top_L$. The theorem is a consequence
       of (10).

(64)   [SEE ALSO [16] P. 70, TH. 1.3 (43)]:
       $(b \to a) \to ((c \to d) \to ((a \to c) \to (b \to d))) = \top_L$. The theorem is a consequ-
       ence of (33).

5. ALTERNATIVE EQUATIONAL AXIOMATICS BY RASIOWA

Let L be a non empty Nelson structure. We say that L satisfies (qpB$_0^*$) if
and only if

(Def. 22)   for every elements a, b of L, $a \leqslant b$ iff $a \Rightarrow b = \top_L$.

We say that L satisfies (qpB$_1^*$) if and only if

(Def. 23)   for every elements a, b of L, $a \to (b \to a) = \top_L$.

We say that L satisfies (qpB$_2^*$) if and only if

(Def. 24)   for every elements a, b, c of L, $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = \top_L$.

We say that L satisfies (qpB$_3^*$) if and only if

(Def. 25)   for every element a of L, $\top_L \to a = a$.

We say that L satisfies (qpB$_5^*$) if and only if

(Def. 26)   for every elements a, b of L, $(a \Rightarrow b) \to ((b \Rightarrow a) \to b) = (b \Rightarrow a) \to ((a \Rightarrow b) \to a)$.

We say that L satisfies (qpB$_6^*$) if and only if

(Def. 27)   for every elements a, b of L, $a \to (a \sqcup b) = \top_L$.

We say that L satisfies (qpB$_7^*$) if and only if

(Def. 28)   for every elements a, b of L, $b \to (a \sqcup b) = \top_L$.

We say that L satisfies (qpB$_8^*$) if and only if

(Def. 29)   for every elements a, b, c of L, $(a \to c) \to ((b \to c) \to ((a \sqcup b) \to c)) = \top_L$.

We say that L satisfies (qpB$_9^*$) if and only if

(Def. 30)   for every elements a, b of L, $(a \sqcap b) \to a = \top_L$.

We say that L satisfies (qpB$_{10}^*$) if and only if

(Def. 31)   for every elements a, b of L, $(a \sqcap b) \to b = \top_L$.

We say that L satisfies (qpB$_{11}^*$) if and only if

(Def. 32)   for every elements a, b, c of L, $(a \to b) \to ((a \to c) \to (a \to (b \sqcap c))) = \top_L$.

We say that L satisfies (qpB$_{12}^*$) if and only if

(Def. 33)   for every elements a, b of L, $(a \to \neg b) \to (b \to \neg a) = \top_L$.

We say that L satisfies (qpB$_{13}^*$) if and only if

(Def. 34)   for every elements a, b of L, $\neg(a \to a) \to b = \top_L$.

We say that L satisfies (qpB^*_{14}) if and only if

(Def. 35)   for every elements a, b of L, $-a \to (a \to b) = \top_L$.

We say that L satisfies (qpB^*_{15}) if and only if

(Def. 36)   for every elements a, b of L, $(-(a \to b) \to (a \sqcap -b)) \sqcap ((a \sqcap -b) \to -(a \to b)) = \top_L$.

We say that L satisfies (qpB^*_{17}) if and only if

(Def. 37)   for every elements a, b of L, $-(a \sqcup b) = -a \sqcap -b$.

We say that L satisfies (qpB^*_{19}) if and only if

(Def. 38)   for every element a of L, $(-\neg a \to a) \sqcap (a \to -\neg a) = \top_L$.

We introduce L satisfies (qpB^*_4) as a synonym of L satisfies (qpB_4) and L satisfies (qpB^*_{16}) as a synonym of L is de Morgan and L satisfies (qpB^*_{18}) as a synonym of L is involutive.

Note that every Nelson algebra satisfies (qpB^*_1), (qpB^*_2), (qpB^*_3), (qpB^*_4), (qpB^*_5), (qpB^*_6), (qpB^*_7), (qpB^*_8), (qpB^*_9), (qpB^*_{10}), (qpB^*_{11}), (qpB^*_{12}), (qpB^*_{13}), (qpB^*_{14}), (qpB^*_{15}), (qpB^*_{16}), (qpB^*_{17}), (qpB^*_{18}), and (qpB^*_{19}).

Now we state the proposition:

(65)   Let us consider a non empty Nelson structure L. Suppose L satisfies (qpB^*_0). Then L is a Nelson algebra if and only if L satisfies (qpB^*_1), (qpB^*_2), (qpB^*_3), (qpB^*_4), (qpB^*_5), (qpB^*_6), (qpB^*_7), (qpB^*_8), (qpB^*_9), (qpB^*_{10}), (qpB^*_{11}), (qpB^*_{12}), (qpB^*_{13}), (qpB^*_{14}), (qpB^*_{15}), (qpB^*_{16}), (qpB^*_{17}), (qpB^*_{18}), and (qpB^*_{19}).

PROOF: Reconsider $L_1 = L$ as a de Morgan, non empty Nelson structure. For every elements a, b of L_1 such that $a \sqcap b = \top_{L_1}$ holds $a = \top_{L_1}$ and $b = \top_{L_1}$. For every elements a, b of L_1, $a \leqslant b$ iff $a < b$ and $-b < -a$. Set $d = (\top_L)^c$. For every element a of L, $a \leqslant \top_L$. For every element a of L, $d \leqslant a$. For every element a of L, $d \sqcap a = d$. For every element a of L_1, $a \to \top_{L_1} = \top_{L_1}$. For every elements a, b, c of L_1 such that $a \to b = \top_{L_1}$ and $b \to c = \top_{L_1}$ holds $a \to c = \top_{L_1}$. L_1 has transitive $<$. L_1 satisfies (qpB_6). For every element a of L_1, $a \to a = \top_{L_1}$. L_1 satisfies (qpB_7). For every elements a, b of L_1, $a \sqcap b \leqslant a$. For every elements a, b of L_1, $a \leqslant a \sqcup b$. For every elements a, b of L_1, $b \leqslant a \sqcup b$. For every elements a, b of L_1, $a \sqcap b \leqslant b$. For every element a of L_1, $a \Rightarrow a = \top_{L_1}$. For every elements a, b of L_1, $a = b$ iff $a \Rightarrow b = \top_{L_1}$ and $b \Rightarrow a = \top_{L_1}$. For every elements a, b of L_1, $a \leqslant b \leqslant a$ iff $a = b$. L_1 has reflexive $<$. For every elements a, b, c of L_1 such that $a < b$ holds $b \to c < a \to c$ and $c \to a < c \to b$. For every elements a, b of L_1, $a \to (b \to (a \sqcap b)) = \top_{L_1}$. For every elements a, b, c of L_1 such that $a < b \to c$ holds $b < a \to c$. For every elements a, c of L_1, $a \to (a \to c) < a \to c$. L_1 satisfies (qpB_3). For every elements a, b, c of L_1

such that $b < c$ holds $a \sqcap b < a \sqcap c$. For every elements a, b, c of L_1 such that $b < c$ holds $a \sqcup b < a \sqcup c$. For every elements a, b, c of L_1 such that $a \leqslant c$ and $b \leqslant c$ holds $a \sqcup b \leqslant c$. For every elements a, b, c of L_1 such that $c \leqslant a$ and $c \leqslant b$ holds $c \leqslant a \sqcap b$. For every elements a, b of L_1, $b \sqcup a \leqslant a \sqcup b$. For every elements a, b of L_1, $a \sqcup b = b \sqcup a$. For every elements a, b of L_1, $a \sqcap b \leqslant b \sqcap a$. For every elements a, b of L_1, $a \sqcap b = b \sqcap a$. For every elements a, b, c of L_1 such that $a \leqslant b$ holds $a \sqcup c \leqslant b \sqcup c$. For every elements a, b of L_1, $b = (a \sqcap b) \sqcup b$. For every elements a, b of L_1, $a \sqcap (a \sqcup b) = a$. For every elements a, b, c of L_1 such that $b \leqslant c$ holds $a \sqcap b \leqslant a \sqcap c$. For every elements a, b, c of L_1 such that $a \leqslant b \leqslant c$ holds $a \leqslant c$. For every elements a, b, c of L_1, $a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c$. For every elements a, b, c of L_1, $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$. Set $c = \top_{L_1}$. For every element a of L_1, $c \sqcup a = c$ and $a \sqcup c = c$ by [18, (4)]. L_1 is distributive. L_1 satisfies (qpB$_5$). L_1 satisfies (qpB$_8$). L_1 satisfies (qpB$_9$). L_1 satisfies (qpB$_{10}$). L_1 satisfies (qpB$_{11}$). L_1 satisfies (qpB$_{12}$). For every elements a, b, c of L_1, $\neg \top_{L_1} = -\top_{L_1}$. For every elements a, b of L_1, $a \to \neg b = b \to \neg a$. L_1 satisfies (qpB$_{13}$). □

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