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Euler's Partition Theorem[1]

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Summary. In this article we prove the Euler's Partition Theorem which states that the number of integer partitions with odd parts equals the number of partitions with distinct parts. The formalization follows H.S. Wilf's lecture notes [28] (see also [1]).

Euler's Partition Theorem is listed as item `#45` from the "Formalizing 100 Theorems" list maintained by Freek Wiedijk at `http://www.cs.ru.nl/F.Wiedijk/100/` [27].

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The notation and terminology used in this paper have been introduced in the following articles: [22], [2], [3], [17], [7], [16], [19], [14], [15], [23], [9], [10], [24], [5], [18], [6], [11], [29], [12], [26], and [13].

1. Preliminaries

From now on x, y denote objects and i, j, k, m, n denote natural numbers.

Let r be an extended real number. One can verify that $\langle r \rangle$ is extended real-valued and $\langle r \rangle$ is decreasing, increasing, non-decreasing, and non-increasing.

Now we state the proposition:

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(1) Let us consider non-decreasing, extended real-valued finite sequences f, g. If $f(\operatorname{len} f) \leqslant g(1)$, then $f \frown g$ is non-decreasing.
PROOF: Set $f_3 = f \frown g$. For every extended reals e_1, e_2 such that e_1, $e_2 \in \operatorname{dom} f_3$ and $e_1 \leqslant e_2$ holds $f_3(e_1) \leqslant f_3(e_2)$ by [7, (25)], [25, (25)]. □

Let R be a binary relation. We say that R is odd-valued if and only if

(Def. 1) $\operatorname{rng} R \subseteq \mathbb{N}_{\mathrm{odd}}$.

(2) $n \in \mathbb{N}_{\mathrm{odd}}$ if and only if n is odd.

Let us note that every binary relation which is odd-valued is also non-zero and natural-valued.

Let F be a function. Observe that F is odd-valued if and only if the condition (Def. 2) is satisfied.

(Def. 2) for every x such that $x \in \operatorname{dom} F$ holds $F(x)$ is an odd natural number.

One can check that every binary relation which is empty is also odd-valued.

Let i be an odd natural number. Let us observe that $\langle i \rangle$ is odd-valued.

Let f, g be odd-valued finite sequences. Note that $f \frown g$ is odd-valued and every binary relation which is $\mathbb{N}_{\mathrm{odd}}$-valued is also odd-valued.

Let n be a natural number. A partition of n is a non-zero, non-decreasing, natural-valued finite sequence and is defined by

(Def. 3) $\sum it = n$.

Now we state the proposition:

(3) $\emptyset$ is a partition of 0.

Let n be a natural number. Observe that there exists a partition of n which is odd-valued and there exists a partition of n which is one-to-one.

Let us observe that sethood property holds for partitions of n.

Let f be an odd-valued finite sequence.

An odd organization of f is a valued reorganization of f and is defined by

(Def. 4) $2 \cdot n - 1 = f(it_{n,1})$ and ... and $2 \cdot n - 1 = f(it_{n,\operatorname{len}(it(n))})$.

(4) Let us consider an odd-valued finite sequence f, and a double reorganization o of $\operatorname{dom} f$. Suppose for every n, $2 \cdot n - 1 = f(o_{n,1})$ and ... and $2 \cdot n - 1 = f(o_{n,\operatorname{len}(o(n))})$. Then o is an odd organization of f.
PROOF: For every n, there exists x such that $x = f(o_{n,1})$ and ... and $x = f(o_{n,\operatorname{len}(o(n))})$. For every natural numbers n_1, n_2, i_1, i_2 such that $i_1 \in \operatorname{dom}(o(n_1))$ and $i_2 \in \operatorname{dom}(o(n_2))$ and $f(o_{n_1,i_1}) = f(o_{n_2,i_2})$ holds $n_1 = n_2$ by [25, (25)]. □

(5) Let us consider an odd-valued finite sequence f, a complex-valued finite sequence g, and double reorganizations o_1, o_2 of $\operatorname{dom} g$. Suppose o_1 is an odd organization of f and o_2 is an odd organization of f. Then $(\sum(g \odot o_1))(i) = (\sum(g \odot o_2))(i)$.

PROOF: For every double reorganizations o_1, o_2 of $\mathrm{dom}\, g$ such that o_1 is an odd organization of f and o_2 is an odd organization of f holds $\mathrm{rng}((f \odot o_1)(n)) \subseteq \mathrm{rng}((f \odot o_2)(n))$ by [19, (49), (1)], [25, (29), (25)]. □

(6) Let us consider a partition p of n. Then there exists an odd-valued finite sequence O and there exists a natural-valued finite sequence a such that $\mathrm{len}\, O = \mathrm{len}\, p = \mathrm{len}\, a$ and $p = O \cdot 2^a$ and $p(1) = O(1) \cdot 2^{a(1)}$ and ... and $p(\mathrm{len}\, p) = O(\mathrm{len}\, p) \cdot 2^{a(\mathrm{len}\, p)}$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every i and j such that $p(\$_1) = 2^i \cdot (2 \cdot j + 1)$ holds $\$_2 = \langle 2 \cdot j + 1, i \rangle$. For every k such that $k \in \mathrm{Seg}\, \mathrm{len}\, p$ there exists x such that $\mathcal{P}[k, x]$ by [20, (1)], [4, (4)]. Consider O_3 being a finite sequence such that $\mathrm{dom}\, O_3 = \mathrm{Seg}\, \mathrm{len}\, p$ and for every k such that $k \in \mathrm{Seg}\, \mathrm{len}\, p$ holds $\mathcal{P}[k, O_3(k)]$ from [7, Sch. 1]. Define $\mathcal{Q}(\text{object}) = O_3(\$_1)_{\mathbf{1}}$. Consider O being a finite sequence such that $\mathrm{len}\, O = \mathrm{len}\, p$ and for every k such that $k \in \mathrm{dom}\, O$ holds $O(k) = \mathcal{Q}(k)$ from [7, Sch. 2]. For every x such that $x \in \mathrm{dom}\, O$ holds $O(x)$ is an odd natural number by [20, (1)]. Define $\mathcal{T}(\text{object}) = O_3(\$_1)_{\mathbf{2}}$. Consider A being a finite sequence such that $\mathrm{len}\, A = \mathrm{len}\, p$ and for every k such that $k \in \mathrm{dom}\, A$ holds $A(k) = \mathcal{T}(k)$ from [7, Sch. 2]. For every x such that $x \in \mathrm{dom}\, A$ holds $A(x)$ is natural by [20, (1)]. Set $O_2 = O \cdot 2^A$. $p(1) = O(1) \cdot 2^{A(1)}$ and ... and $p(\mathrm{len}\, p) = O(\mathrm{len}\, p) \cdot 2^{A(\mathrm{len}\, p)}$ by [25, (25)], [20, (1)]. For every i such that $i \in \mathrm{dom}\, p$ holds $p(i) = O_2(i)$ by [25, (25)]. □

(7) Let us consider a finite set D, and a function f from D into $\mathbb{N}$. Then there exists a finite sequence K of elements of D such that for every element d of D, $\overline{\overline{\mathrm{Coim}(K, d)}} = f(d)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite set D such that $\overline{\overline{D}} = \$_1$ for every function f from D into $\mathbb{N}$, there exists a finite sequence K of elements of D such that for every element d of D, $\overline{\overline{\mathrm{Coim}(K, d)}} = f(d)$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [21, (55)], [8, (63)], [25, (57)], [13, (56)]. $\mathcal{P}[i]$ from [5, Sch. 2]. □

(8) Let us consider complex-valued finite sequences f_1, f_2, g_1, g_2. Suppose $\mathrm{len}\, f_1 = \mathrm{len}\, g_1$. Then $(f_1 \frown f_2) \cdot (g_1 \frown g_2) = (f_1 \cdot g_1) \frown (f_2 \cdot g_2)$.

(9) Let us consider natural-valued finite sequences f, K. Suppose for every i, $\overline{\overline{\mathrm{Coim}(K, i)}} = f(i)$. Then $\sum K = 1 \cdot f(1) + 2 \cdot f(2) + ((\mathrm{id}_{\mathrm{dom}\, f} \cdot f), 3) + \ldots$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every natural-valued finite sequences f, K such that $\mathrm{len}\, f = \$_1$ and for every i, $\overline{\overline{\mathrm{Coim}(K, i)}} = f(i)$ holds $\sum K = ((\mathrm{id}_{\mathrm{dom}\, f} \cdot f), 1) + \ldots$. $\mathcal{P}[0]$ by [25, (25)], [9, (72)], [19, (20), (22)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [25, (55)], [5, (13)], [7, (59)], [8, (51)]. $\mathcal{P}[i]$ from [5, Sch. 2]. □

(10) Let us consider a natural-valued finite sequence g, and a double reorgani-

zation s_1 of $\operatorname{dom} g$. Then there exists a $(2 \cdot \operatorname{len} s_1)$-element finite sequence K of elements of $\mathbb{N}$ such that for every j, $K(2 \cdot j) = 0$ and $K(2 \cdot j - 1) = g(s_{1j,1}) + ((g \odot s_1)(j), 2) + \ldots$. Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ if $\$_1 = 2 \cdot j - 1$, then $\$_2 = g(s_{1j,1}) + ((g \odot s_1)(j), 2) + \ldots$ and if $\$_1 = 2 \cdot j$, then $\$_2 = 0$. Set $S = \operatorname{Seg}(2 \cdot \operatorname{len} s_1)$. For every k such that $k \in S$ there exists x such that $\mathcal{P}[k, x]$ by [22, (9)]. Consider f being a finite sequence such that $\operatorname{dom} f = S$ and for every i such that $i \in S$ holds $\mathcal{P}[i, f(i)]$ from [7, Sch. 1]. $\operatorname{rng} f \subseteq \mathbb{N}$ by [22, (9)]. $f(2 \cdot i) = 0$. $f(2 \cdot i - 1) = g(s_{1i,1}) + ((g \odot s_1)(i), 2) + \ldots$ by [25, (25)], [5, (13)], [19, (15)]. $\square$

2. Euler Transformation

Now we state the proposition:

(11) Let us consider a one-to-one partition d of n. Then there exists an odd-valued partition e of n such that for every natural number j for every odd-valued finite sequence O_1 for every natural-valued finite sequence a_1 such that $\operatorname{len} O_1 = \operatorname{len} d = \operatorname{len} a_1$ and $d = O_1 \cdot 2^{a_1}$ for every double reorganization s_1 of $\operatorname{dom} d$ such that $1 = O_1(s_{11,1})$ and ... and $1 = O_1(s_{11,\operatorname{len}(s_1(1))})$ and $3 = O_1(s_{12,1})$ and ... and $3 = O_1(s_{12,\operatorname{len}(s_1(2))})$ and $5 = O_1(s_{13,1})$ and ... and $5 = O_1(s_{13,\operatorname{len}(s_1(3))})$ and for every i, $2 \cdot i - 1 = O_1(s_{1i,1})$ and ... and $2 \cdot i - 1 = O_1(s_{1i,\operatorname{len}(s_1(i))})$ holds $\overline{\overline{\operatorname{Coim}(e,1)}} = 2^{a_1}(s_{11,1}) + ((2^{a_1} \odot s_1)(1), 2) + \ldots$ and $\overline{\overline{\operatorname{Coim}(e,3)}} = 2^{a_1}(s_{12,1}) + ((2^{a_1} \odot s_1)(2), 2) + \ldots$ and $\overline{\overline{\operatorname{Coim}(e,5)}} = 2^{a_1}(s_{13,1}) + ((2^{a_1} \odot s_1)(3), 2) + \ldots$ and $\overline{\overline{\operatorname{Coim}(e, j \cdot 2 - 1)}} = 2^{a_1}(s_{1j,1}) + ((2^{a_1} \odot s_1)(j), 2) + \ldots$.

Proof: Consider O being an odd-valued finite sequence, a being a natural-valued finite sequence such that $\operatorname{len} O = \operatorname{len} d = \operatorname{len} a$ and $d = O \cdot 2^a$ and $d(1) = O(1) \cdot 2^{a(1)}$ and ... and $d(\operatorname{len} d) = O(\operatorname{len} d) \cdot 2^{a(\operatorname{len} d)}$. $n = d(1) + ((d, 2) + \ldots + (d, \operatorname{len} d))$ by [19, (22)]. $n = 2^{a(1)} \cdot O(1) + 2^{a(2)} \cdot O(2) + ((O \cdot 2^a, 3) + \ldots + (O \cdot 2^a, \operatorname{len} d))$ by [19, (20)], [25, (25)]. Reconsider $s_1 =$ the odd organization of O as a double reorganization of $\operatorname{dom} 2^a$. Consider μ being a $(2 \cdot \operatorname{len} s_1)$-element finite sequence of elements of $\mathbb{N}$ such that for every j, $\mu(2 \cdot j) = 0$ and $\mu(2 \cdot j - 1) = 2^a(s_{1j,1}) + ((2^a \odot s_1)(j), 2) + \ldots$. Set $\alpha = a \cdot s_1(1)$. Set $\beta = a \cdot s_1(2)$. Set $\gamma = a \cdot s_1(3)$. $n = (2^\alpha(1) + (2^\alpha, 2) + \ldots) \cdot 1 + (2^\beta(1) + (2^\beta, 2) + \ldots) \cdot 3 + (2^\gamma(1) + (2^\gamma, 2) + \ldots) \cdot 5 + ((\mathrm{id}_{\operatorname{dom} \mu} \cdot \mu), 7) + \ldots$ by [25, (29)], [19, (41)], [25, (25)], [9, (12)]. $n = \mu(1) \cdot 1 + \mu(3) \cdot 3 + \mu(5) \cdot 5 + ((\mathrm{id}_{\operatorname{dom} \mu} \cdot \mu), 7) + \ldots$ by [19, (42), (41), (25)]. Consider K being an odd-valued finite sequence such that K is non-decreasing and for every i, $\overline{\overline{\operatorname{Coim}(K,i)}} = \mu(i)$. $n = \overline{\overline{\operatorname{Coim}(K,1)}} \cdot 1 + \overline{\overline{\operatorname{Coim}(K,3)}} \cdot 3 + \overline{\overline{\operatorname{Coim}(K,5)}} \cdot 5 + ((\mathrm{id}_{\operatorname{dom} \mu} \cdot \mu), 7) + \ldots$. $n = \sum K$ by [19, (20)], (9). For every j such that $1 \leqslant j \leqslant \operatorname{len} d$ holds $O(j) = O_1(j)$ and $a(j) = a_1(j)$ by [25, (25)], [22, (9)], [4, (4)]. For every j, $\overline{\overline{\mathrm{Coim}(K, j \cdot 2 - 1)}} = 2^{a_1}(sort1_{j,1}) + ((2^{a_1} \odot sort1)(j), 2) + \ldots$ by [19, (42)], [25, (29)], [9, (72)], [19, (22)]. □

Let n be a natural number and p be a one-to-one partition of n. The Euler transformation p yielding an odd-valued partition of n is defined by

(Def. 5) for every odd-valued finite sequence O and for every natural-valued finite sequence a such that $\operatorname{len} O = \operatorname{len} p = \operatorname{len} a$ and $p = O \cdot 2^a$ for every double reorganization s_1 of $\operatorname{dom} p$ such that $1 = O(s_{1 1,1})$ and ... and $1 = O(s_{1 1,\operatorname{len}(s_1(1))})$ and $3 = O(s_{1 2,1})$ and ... and $3 = O(s_{1 2,\operatorname{len}(s_1(2))})$ and $5 = O(s_{1 3,1})$ and ... and $5 = O(s_{1 3,\operatorname{len}(s_1(3))})$ and for every i, $2 \cdot i - 1 = O(s_{1 i,1})$ and ... and $2 \cdot i - 1 = O(s_{1 i,\operatorname{len}(s_1(i))})$ holds $\overline{\overline{\mathrm{Coim}(it, 1)}} = 2^a(s_{1 1,1}) + ((2^a \odot s_1)(1), 2) + \ldots$ and $\overline{\overline{\mathrm{Coim}(it, 3)}} = 2^a(s_{1 2,1}) + ((2^a \odot s_1)(2), 2) + \ldots$ and $\overline{\overline{\mathrm{Coim}(it, 5)}} = 2^a(s_{1 3,1}) + ((2^a \odot s_1)(3), 2) + \ldots$ and $\overline{\overline{\mathrm{Coim}(it, j \cdot 2 - 1)}} = 2^a(s_{1 j,1}) + ((2^a \odot s_1)(j), 2) + \ldots$.

Now we state the proposition:

(12) Let us consider a natural number n, a one-to-one partition p of n, and an odd-valued partition e of n. Then $e =$ the Euler transformation p if and only if for every odd-valued finite sequence O and for every natural-valued finite sequence a and for every odd organization s_1 of O such that $\operatorname{len} O = \operatorname{len} p = \operatorname{len} a$ and $p = O \cdot 2^a$ for every j, $\overline{\overline{\mathrm{Coim}(e, j \cdot 2 - 1)}} = ((2^a \odot s_1)(j), 1) + \ldots$.

Proof: If $e =$ the Euler transformation p, then for every odd-valued finite sequence O and for every natural-valued finite sequence a and for every odd organization s_1 of O such that $\operatorname{len} O = \operatorname{len} p = \operatorname{len} a$ and $p = O \cdot 2^a$ for every j, $\overline{\overline{\mathrm{Coim}(e, j \cdot 2 - 1)}} = ((2^a \odot s_1)(j), 1) + \ldots$ by [25, (29)], [19, (42), (20)]. For every j and for every odd-valued finite sequence O and for every natural-valued finite sequence a such that $\operatorname{len} O = \operatorname{len} p = \operatorname{len} a$ and $p = O \cdot 2^a$ for every double reorganization s_1 of $\operatorname{dom} p$ such that $1 = O(s_{1 1,1})$ and ... and $1 = O(s_{1 1,\operatorname{len}(s_1(1))})$ and $3 = O(s_{1 2,1})$ and ... and $3 = O(s_{1 2,\operatorname{len}(s_1(2))})$ and $5 = O(s_{1 3,1})$ and ... and $5 = O(s_{1 3,\operatorname{len}(s_1(3))})$ and for every i, $2 \cdot i - 1 = O(s_{1 i,1})$ and ... and $2 \cdot i - 1 = O(s_{1 i,\operatorname{len}(s_1(i))})$ holds $\overline{\overline{\mathrm{Coim}(e, 1)}} = 2^a(s_{1 1,1}) + ((2^a \odot s_1)(1), 2) + \ldots$ and $\overline{\overline{\mathrm{Coim}(e, 3)}} = 2^a(s_{1 2,1}) + ((2^a \odot s_1)(2), 2) + \ldots$ and $\overline{\overline{\mathrm{Coim}(e, 5)}} = 2^a(s_{1 3,1}) + ((2^a \odot s_1)(3), 2) + \ldots$ and $\overline{\overline{\mathrm{Coim}(e, j \cdot 2 - 1)}} = 2^a(s_{1 j,1}) + ((2^a \odot s_1)(j), 2) + \ldots$ by [25, (29)], (4), [19, (42), (20)]. □

One can verify that every real-valued function which is one-to-one and non-decreasing is also increasing.

(13) Let us consider an odd-valued finite sequence O, a natural-valued finite sequence a, and an odd organization s of O. Suppose $\operatorname{len} O = \operatorname{len} a$ and $O \cdot 2^a$ is one-to-one. Then $(a \odot s)(i)$ is one-to-one.
Proof: $(a \odot s)(i)$ is one-to-one by [9, (11), (12)], [25, (25)]. □

(14) Let us consider one-to-one partitions p_1, p_2 of n. Suppose the Euler transformation $p_1 =$ the Euler transformation p_2. Then $p_1 = p_2$.

(15) Let us consider an odd-valued partition e of n. Then there exists a one-to-one partition p of n such that $\overline{\overline{e = \text{the}}}$ Euler transformation p.
Proof: Define $\mathcal{K}(\text{object}) = \overline{\overline{\operatorname{Coim}(e, \$_1)}}$. Consider H being a finite sequence such that $\operatorname{len} H = n$ and for every k such that $k \in \operatorname{dom} H$ holds $H(k) = \mathcal{K}(k)$ from [7, Sch. 2]. $\operatorname{rng} H \subseteq \mathbb{N}$. $\sum e = \sum(\operatorname{idseq}(n) \cdot H)$ by [25, (25)], [5, (14)], [9, (72)], [30, (5)]. Define $\mathcal{F}[\text{natural number}, \text{object}] \equiv$ there exists an increasing, natural-valued finite sequence f such that $H(\$_1) = 2^f(1) + (2^f, 2) + \ldots$ and $\$_2 = \$_1 \cdot 2^f$. There exists a finite sequence p of elements of $\mathbb{N}^*$ such that $\operatorname{dom} p = \operatorname{Seg} \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} H$ holds $\mathcal{F}[k, p(k)]$ by [19, (31)]. Consider p being a finite sequence of elements of $\mathbb{N}^*$ such that $\operatorname{dom} p = \operatorname{Seg} \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} H$ holds $\mathcal{F}[k, p(k)]$. For every k such that $p(k) \neq \emptyset$ holds k is odd by [18, (83)], [12, (85)], [19, (22)], [9, (72)]. Set $N =$ the concatenation of $\mathbb{N}$. Set $n_3 = N \odot p$. Set $s_2 = \operatorname{sort_a} n_3$. s_2 is a one-to-one partition of n by [19, (1)], [25, (25)], [12, (45)], [18, (83)]. For every odd-valued finite sequence O and for every natural-valued finite sequence a and for every odd organization s_1 of O such that $\operatorname{len} O = \operatorname{len} s_2 = \operatorname{len} a$ and $s_2 = O \cdot 2^a$ for every j, $\overline{\overline{\operatorname{Coim}(e, j \cdot 2 - 1)}} = ((2^a \odot s_1)(j), 1) + \ldots$ by [25, (29)], [5, (14)], [9, (72)], [25, (25)]. □

3. Main Theorem

Now we state the proposition:

(16) Euler's partition theorem:
$\overline{\overline{\text{the set of all } p \text{ where } p \text{ is an odd-valued partition of } n}} =$
$\overline{\overline{\text{the set of all } p \text{ where } p \text{ is a one-to-one partition of } n}}$. The theorem is a consequence of (15) and (14).

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