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Equivalent Expressions of Direct Sum Decomposition of Groups[1]

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Summary. In this article, the equivalent expressions of the direct sum decomposition of groups are mainly discussed. In the first section, we formalize the fact that the internal direct sum decomposition can be defined as normal subgroups and some of their properties. In the second section, we formalize an equivalent form of internal direct sum of commutative groups. In the last section, we formalize that the external direct sum leads an internal direct sum. We referred to [19], [18] [8] and [14] in the formalization.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [20], [6], [9], [10], [7], [22], [17], [16], [23], [24], [25], [26], [13], [3], [5], [11], [15], [28], [29], [27], and [12].

1. Internal Direct Sum Decomposition into Normal Subgroups

Let I be a set and G be a group.

A subgroup-family of I and G is a group-family of I and is defined by

(Def. 1) for every object i such that $i \in I$ holds $it(i)$ is a subgroup of G.

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Let F be a subgroup-family of I and G. We say that F is component commutative if and only if

(Def. 2) for every elements i, j of I and for every elements g_1, g_2 of G such that $i \neq j$ and $g_1 \in F(i)$ and $g_2 \in F(j)$ holds $g_1 \cdot g_2 = g_2 \cdot g_1$.

Let I be a non empty set. One can verify that there exists a subgroup-family of I and G which is component commutative.

Now we state the propositions:

(1) Let us consider a group G, a normal subgroup H of G, and elements x, y of G. Suppose $y \in H$. Then $x \cdot y \cdot x^{-1}$, $x \cdot (y \cdot x^{-1}) \in H$.

(2) Let us consider a non empty set I, a group G, a group-family F of I, and a function x from I into G. Suppose $x \in \prod F$. Then x is a function from I into $\bigcup$(the support of F).

Proof: For every object z such that $z \in \operatorname{rng} x$ holds $z \in \bigcup$(the support of F) by [10, (11)], [16, (5), (4)], [9, (3)]. □

(3) Let us consider a non empty set I, a group G, a subgroup H of G, a function x from I into G, and a function y from I into H. If $x = y$, then $\operatorname{support} x = \operatorname{support} y$.

Proof: For every object i, $i \in \operatorname{support} x$ iff $i \in \operatorname{support} y$ by [23, (44)]. □

(4) Let us consider a non empty set I, a group G, and a subgroup H of G. Then every finite-support function from I into H is a finite-support function from I into G. The theorem is a consequence of (3).

(5) Let us consider a non empty set I, a group G, a subgroup H of G, and a finite-support function x from I into G. Suppose $\operatorname{rng} x \subseteq \Omega_H$. Then x is a finite-support function from I into H. The theorem is a consequence of (3).

(6) Let us consider a non empty set I, a group G, a subgroup H of G, a finite-support function x from I into G, and a finite-support function y from I into H. If $x = y$, then $\prod x = \prod y$. The theorem is a consequence of (3).

(7) Let us consider a function f, and sets i, x. Then $f = (f +\cdot (i, x)) +\cdot (i, f(i))$.

(8) Let us consider a non empty set I, a group G, a component commutative subgroup-family F of I and G, finite-support functions x, y from I into G, and an element i of I. Suppose $y = x +\cdot (i, \mathbf{1}_{F(i)})$ and $x \in \prod F$. Then $\prod x = \prod y \cdot x(i) = x(i) \cdot \prod y$.

Proof: Reconsider $p_2 = y$ as an element of $\prod F$. Reconsider $s_1 = p_2$ as an element of sum F. Set $z = \mathbf{1}_{\prod F} +\cdot (i, x(i))$. Reconsider $s_2 = z$ as an element of sum F. $x = s_1 \cdot s_2$ by [16, (5), (24)], [23, (40)], [7, (31)].

$s_1 \cdot s_2 = s_2 \cdot s_1$ by [16, (27), (17)], [23, (43)], [16, (32)]. □

(9) Let us consider a non empty set I, a group G, a component commutative subgroup-family F of I and G, a subset U of G, an element i of I, and finite-support functions x, y from I into $\mathrm{gr}(U)$. Suppose $y = x +\cdot (i, \mathbf{1}_{F(i)})$ and $x \in \prod F$. Then $\prod x = \prod y \cdot x(i) = x(i) \cdot \prod y$. The theorem is a consequence of (4), (6), and (8).

(10) Let us consider a non empty set I, a group G, a component commutative subgroup-family F of I and G, a subset U of G, a finite-support function y from I into $\mathrm{gr}(U)$, an element i of I, and an element g of $\mathrm{gr}(U)$. Suppose $y \in \prod F$ and $y(i) = \mathbf{1}_{F(i)}$ and $g \in F(i)$. Then $\prod y \cdot g = g \cdot \prod y$. The theorem is a consequence of (7) and (9).

(11) Let us consider a non empty set I, a group G, a component commutative subgroup-family F of I and G, and a subset U of G. Suppose $U = \bigcup(\text{the support of } F)$. Let us consider an element g of G, a finite sequence H of elements of G, and a finite sequence K of elements of $\mathbb{Z}$. Suppose $\operatorname{len} H = \operatorname{len} K$ and $\operatorname{rng} H \subseteq U$ and $\prod H^K = g$. Then there exists a finite-support function f from I into G such that

   (i) $f \in \prod F$, and

   (ii) $g = \prod f$.

   PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every element g of G for every finite sequence H of elements of G for every finite sequence K of elements of $\mathbb{Z}$ such that $\operatorname{len} H = \$_1$ and $\operatorname{len} H = \operatorname{len} K$ and $\operatorname{rng} H \subseteq U$ and $\prod H^K = g$ there exists a finite-support function f from I into G such that $f \in \prod F$ and $g = \prod f$. $\mathcal{P}[0]$ by [25, (21)], [16, (12), (13), (16)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [28, (70)], [6, (4)], [21, (55)], [9, (3)]. For every natural number n, $\mathcal{P}[n]$ from [4, Sch. 2]. □

(12) Let us consider a non empty set I, a group G, a subgroup-family F of I and G, finite-support functions h, h_0 from I into G, an element i of I, and a subset U_1 of G. Suppose $U_1 = \bigcup((\text{the support of } F){\restriction}(I \setminus \{i\}))$ and $h_0 = h +\cdot (i, \mathbf{1}_{F(i)})$ and $h \in \prod F$. Then $\prod h_0 \in \mathrm{gr}(U_1)$.

   PROOF: For every object y such that $y \in \operatorname{rng} h_0$ holds $y \in \Omega_{\mathrm{gr}(U_1)}$ by [10, (113)], [7, (32)], [16, (5), (4)]. Reconsider $x_0 = h_0$ as a finite-support function from I into $\mathrm{gr}(U_1)$. $\prod x_0 = \prod h_0$. □

(13) Let us consider a non empty set I, a group G, a component commutative subgroup-family F of I and G, and a subset U of G. Suppose $U = \bigcup(\text{the support of } F)$. Let us consider an element g of G. Suppose $g \in \mathrm{gr}(U)$. Then there exists a finite-support function f from I into $\mathrm{gr}(U)$ such that

(i) $f \in \operatorname{sum} F$, and

(ii) $g = \prod f$.

The theorem is a consequence of (11), (2), (5), and (6).

(14) Let us consider a non empty set I, a group G, a component commutative subgroup-family F of I and G, and a subset U of G. Suppose $U = \bigcup$(the support of F). Let us consider an element i of I. Then $F(i)$ is a normal subgroup of $\operatorname{gr}(U)$.

PROOF: Reconsider $F_1 = F(i)$ as a subgroup of $\operatorname{gr}(U)$. For every element a of $\operatorname{gr}(U)$, $a \cdot F_1 \subseteq F_1 \cdot a$ by [23, (103), (42)], (13), [23, (40)]. $\square$

(15) Let us consider a non empty set I, a group G, and a component commutative subgroup-family F of I and G. Suppose for every element i of I, there exists a subset U_1 of G such that $U_1 = \bigcup(($the support of $F) \restriction (I \setminus \{i\}))$ and $\Omega_{\operatorname{gr}(U_1)} \cap \Omega_{F(i)} = \{\mathbf{1}_G\}$. Let us consider finite-support functions x_1, x_2 from I into G. If $x_1, x_2 \in \operatorname{sum} F$ and $\prod x_1 = \prod x_2$, then $x_1 = x_2$.

PROOF: Define $\mathcal{P}[$natural number$] \equiv$ for every finite-support functions x_1, x_2 from I into G such that $\overline{\overline{\text{support}\, x_1}} = \$_1$ and $x_1, x_2 \in \operatorname{sum} F$ and $\prod x_1 = \prod x_2$ holds $x_1 = x_2$. $\mathcal{P}[0]$ by [16, (15), (14)], [23, (42)], [16, (26)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [23, (42)], [16, (26)], [23, (44)], [16, (30), (25)]. For every natural number k, $\mathcal{P}[k]$ from [4, Sch. 2]. $\square$

(16) Let us consider a non empty set I, a strict group G, and a group-family F of I. Then F is an internal direct sum components of G and I if and only if for every element i of I, $F(i)$ is a normal subgroup of G and there exists a subset U of G such that $U = \bigcup$(the support of F) and $\operatorname{gr}(U) = G$ and for every element i of I, there exists a subset U_1 of G such that $U_1 = \bigcup(($the support of $F) \restriction (I \setminus \{i\}))$ and $\Omega_{\operatorname{gr}(U_1)} \cap \Omega_{F(i)} = \{\mathbf{1}_G\}$.

PROOF: Consider U being a subset of G such that $U = \bigcup$(the support of F) and $\operatorname{gr}(U) = G$. For every elements i, j of I such that $i \neq j$ holds $\Omega_{F(i)} \cap \Omega_{F(j)} = \{\mathbf{1}_G\}$ by [23, (46)], [12, (31)], [28, (62)], [9, (49)]. For every elements i, j of I and for every elements g_1, g_2 of G such that $i \neq j$ and $g_1 \in F(i)$ and $g_2 \in F(j)$ holds $g_1 \cdot g_2 = g_2 \cdot g_1$ by [23, (51)], (1), [22, (17)], [23, (50)]. For every element y of G, there exists a finite-support function x from I into G such that $x \in \operatorname{sum} F$ and $y = \prod x$. For every finite-support functions x_1, x_2 from I into G such that $x_1, x_2 \in \operatorname{sum} F$ and $\prod x_1 = \prod x_2$ holds $x_1 = x_2$. $\square$

2. Internal Direct Sum Decomposition for Commutative Group

Now we state the proposition:

(17) Let us consider a non empty set I, a commutative group G, and a group-family F of I. Then F is an internal direct sum components of G and I if and only if for every element i of I, $F(i)$ is a subgroup of G and for every elements i, j of I such that $i \neq j$ holds $\Omega_{F(i)} \cap \Omega_{F(j)} = \{\mathbf{1}_G\}$ and for every element y of G, there exists a finite-support function x from I into G such that $x \in \operatorname{sum} F$ and $y = \prod x$ and for every finite-support functions x_1, x_2 from I into G such that x_1, $x_2 \in \operatorname{sum} F$ and $\prod x_1 = \prod x_2$ holds $x_1 = x_2$.

3. Equivalence between Internal and External Direct Sum

Now we state the propositions:

(18) Let us consider a non empty set I, a group G, a subgroup-family F of I and G, a homomorphism h from $\operatorname{sum} F$ to G, and a finite-support function a from I into G. Suppose $a \in \operatorname{sum} F$ and for every element i of I and for every element x of $F(i)$, $h((\mathrm{1ProdHom}(F, i))(x)) = x$. Then $h(a) = \prod a$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite-support function b from I into G such that $b \in \operatorname{sum} F$ holds if $\overline{\overline{\operatorname{support} b}} = \$_1$, then $h(b) = \prod b$. $\mathcal{P}[0]$ by [16, (14)], [23, (44)], [26, (31)], [16, (15)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [16, (25)], [23, (44)], [16, (26)], [23, (40)]. For every natural number k, $\mathcal{P}[k]$ from [4, Sch. 2]. Consider k being a natural number such that $\overline{\overline{\operatorname{support} a}} = k$. $\square$

(19) Let us consider a non empty set I, a group G, and a direct sum components M of G and I. Then there exists a homomorphism f from $\operatorname{sum} M$ to G and there exists an internal direct sum components N of G and I such that f is bijective and for every element i of I, there exists a homomorphism q_1 from $M(i)$ to $N(i)$ such that $q_1 = f \cdot \mathrm{1ProdHom}(M, i)$ and q_1 is bijective.
Proof: Consider f being a homomorphism from $\operatorname{sum} M$ to G such that f is bijective. Define $\mathcal{D}(\text{element of } I) = f^{\circ}(\mathrm{ProjGroup}(M, \$_1))$. Consider N being a function such that $\operatorname{dom} N = I$ and for every element i of I such that $i \in I$ holds $N(i) = \mathcal{D}(i)$ from [2, Sch. 2]. For every object i such that $i \in I$ holds $N(i)$ is a strict subgroup of G. Define $\mathcal{E}(\text{element of } I) = f \cdot \mathrm{1ProdHom}(M, \$_1)$. Consider q being a function such that $\operatorname{dom} q = I$ and for every element i of I such that $i \in I$ holds $q(i) = \mathcal{E}(i)$ from [2, Sch. 2]. Reconsider $r = \mathrm{SumMap}(M, N, q)$ as a homomorphism from $\operatorname{sum} M$ to $\operatorname{sum} N$. Reconsider $s = r^{-1}$ as a homomorphism from $\operatorname{sum} N$ to $\operatorname{sum} M$.

Reconsider $g = f \cdot s$ as a function. For every element i of I and for every element n of $N(i)$, $g((1\mathrm{ProdHom}(N,i))(n)) = n$ by [16, (42)], [23, (40)], [9, (13), (34)]. For every finite-support function a from I into G such that $a \in \mathrm{sum}\, N$ holds $g(a) = \prod a$. For every element i of I, there exists a homomorphism q_1 from $M(i)$ to $N(i)$ such that $q_1 = f \cdot 1\mathrm{ProdHom}(M,i)$ and q_1 is bijective. $\square$

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