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σ-ring and σ-algebra of Sets[1]

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Summary. In this article, semiring and semialgebra of sets are formalized so as to construct a measure of a given set in the next step. Although a semiring of sets has already been formalized in [13], that is, strictly speaking, a definition of a quasi semiring of sets suggested in the last few decades [15]. We adopt a classical definition of a semiring of sets here to avoid such a confusion. Ring of sets and algebra of sets have been formalized as non empty preboolean set [23] and field of subsets [18], respectively. In the second section, definitions of a ring and a σ-ring of sets, which are based on a semiring and a ring of sets respectively, are formalized and their related theorems are proved. In the third section, definitions of an algebra and a σ-algebra of sets, which are based on a semialgebra and an algebra of sets respectively, are formalized and their related theorems are proved. In the last section, mutual relationships between σ-ring and σ-algebra of sets are formalized and some related examples are given. The formalization is based on [15], and also referred to [9] and [16].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [3], [17], [21], [6], [14], [23], [10], [11], [7], [8], [22], [4], [5], [18], [19], [26], [27], [20], [13], [25], and [12].

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1. Preliminaries

Now we state the propositions:

(1) Let us consider finite sequences f_1, f_2, and a natural number k. Suppose $k \in \mathrm{Seg}(\mathrm{len}\, f_1 \cdot \mathrm{len}\, f_2)$. Then

(i) $(k -' 1 \bmod \mathrm{len}\, f_2) + 1 \in \mathrm{dom}\, f_2$, and

(ii) $(k -' 1 \,\mathrm{div}\, \mathrm{len}\, f_2) + 1 \in \mathrm{dom}\, f_1$.

(2) Let us consider a non empty, finite set S. Then $\bigcup \mathrm{CFS}(S) = \bigcup S$.

(3) Let us consider an object x. Then $\langle x \rangle$ is a disjoint valued finite sequence.

(4) Let us consider sets x, y, and a finite sequence F. If $F = \langle x, y \rangle$ and x misses y, then F is disjoint valued.

(5) Let us consider finite sequences f_1, f_2. Then there exists a finite sequence f such that

(i) $\bigcup f_1 \cap \bigcup f_2 = \bigcup f$, and

(ii) $\mathrm{dom}\, f = \mathrm{Seg}(\mathrm{len}\, f_1 \cdot \mathrm{len}\, f_2)$, and

(iii) for every natural number i such that $i \in \mathrm{dom}\, f$ holds $f(i) = f_1((i -' 1 \,\mathrm{div}\, \mathrm{len}\, f_2) + 1) \cap f_2((i -' 1 \bmod \mathrm{len}\, f_2) + 1)$.

Proof: For every natural number k such that $k \in \mathrm{Seg}(\mathrm{len}\, f_1 \cdot \mathrm{len}\, f_2)$ holds $(k -' 1 \bmod \mathrm{len}\, f_2) + 1 \in \mathrm{dom}\, f_2$ and $(k -' 1 \,\mathrm{div}\, \mathrm{len}\, f_2) + 1 \in \mathrm{dom}\, f_1$. Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv \$_2 = f_1((\$_1 -' 1 \,\mathrm{div}\, \mathrm{len}\, f_2) + 1) \cap f_2((\$_1 -' 1 \bmod \mathrm{len}\, f_2) + 1)$. Consider f being a finite sequence such that $\mathrm{dom}\, f = \mathrm{Seg}(\mathrm{len}\, f_1 \cdot \mathrm{len}\, f_2)$ and for every natural number k such that $k \in \mathrm{Seg}(\mathrm{len}\, f_1 \cdot \mathrm{len}\, f_2)$ holds $\mathcal{P}[k, f(k)]$ from [6, Sch. 1]. $\square$

(6) Let us consider disjoint valued finite sequences f_1, f_2. Then there exists a disjoint valued finite sequence f such that

(i) $\bigcup f_1 \cap \bigcup f_2 = \bigcup f$, and

(ii) $\mathrm{dom}\, f = \mathrm{Seg}(\mathrm{len}\, f_1 \cdot \mathrm{len}\, f_2)$, and

(iii) for every natural number i such that $i \in \mathrm{dom}\, f$ holds $f(i) = f_1((i -' 1 \,\mathrm{div}\, \mathrm{len}\, f_2) + 1) \cap f_2((i -' 1 \bmod \mathrm{len}\, f_2) + 1)$.

The theorem is a consequence of (5).

(7) Let us consider a set X, and a non empty, \-closed family S of subsets of X. Then $\emptyset \in S$.

Let X be a set. One can check that every family of subsets of X which is non empty and \-closed has also the empty element.

2. Classical Semiring, Ring and σ-ring of Sets

Let I_1 be a set. We say that I_1 is semi $\setminus$-closed if and only if

(Def. 1) for every sets X, Y such that $X, Y \in I_1$ there exists a disjoint valued finite sequence F of elements of I_1 such that $X \setminus Y = \bigcup F$.

Let X be a set. Let us note that 2^X is semi $\setminus$-closed and there exists a family of subsets of X which is non empty, semi $\setminus$-closed, and $\cap$-closed and there exists a family of subsets of X which is semi $\setminus$-closed and $\cap$-closed and has the empty element.

A semiring of X is a semi $\setminus$-closed, $\cap$-closed family of subsets of X with the empty element. Now we state the propositions:

(8) Let us consider a set X, a family S of subsets of X, and sets S_1, S_2. Suppose $S_1, S_2 \in S$ and S is semi $\setminus$-closed. Then there exists a finite subset x of S such that x is a partition of $S_1 \setminus S_2$.

(9) Let us consider a set X, and a non empty family S of subsets of X. Suppose S is semi $\setminus$-closed. Then S is $\setminus^{\subseteq}_{fp}$-closed. The theorem is a consequence of (8).

(10) Let us consider a set X, and a family S of subsets of X. Suppose S is $\cap_{fp}$-closed and $\setminus^{\subseteq}_{fp}$-closed and has the empty element. Then S is semi $\setminus$-closed. The theorem is a consequence of (2).

Note that every set which is $\setminus$-closed is also semi $\setminus$-closed and $\cap$-closed.

Let X be a set. Observe that there exists a family of subsets of X which is non empty and preboolean and every set which is non empty and preboolean has also the empty element.

Let X be a set and S be a semi $\setminus$-closed, $\cap$-closed family of subsets of X with the empty element. The ring generated by S yielding a non empty, preboolean family of subsets of X is defined by the term

(Def. 2) $\bigcap\{Z$, where Z is a non empty, preboolean family of subsets of $X : S \subseteq Z\}$.

Now we state the proposition:

(11) Let us consider a set X, and a semi $\setminus$-closed, $\cap$-closed family P of subsets of X with the empty element. Then $P \subseteq$ the ring generated by P.

Let X be a set and S be a semi $\setminus$-closed, $\cap$-closed family of subsets of X with the empty element. The functor DisUnion S yielding a non empty family of subsets of X is defined by the term

(Def. 3) $\{A$, where A is a subset of X : there exists a disjoint valued finite sequence F of elements of S such that $A = \bigcup F\}$.

Let us consider a set X and a semi \-closed, $\cap$-closed family S of subsets of X with the empty element. Now we state the propositions:

(12) $S \subseteq \mathrm{DisUnion}\, S$.

(13) $\mathrm{DisUnion}\, S$ is $\cap$-closed. The theorem is a consequence of (6) and (1).

Now we state the proposition:

(14) Let us consider a set X, a semi \-closed, $\cap$-closed family S of subsets of X with the empty element, and sets A, B, P. If $P = \mathrm{DisUnion}\, S$ and A, $B \in P$ and A misses B, then $A \cup B \in P$.

Let us consider a set X, a semi \-closed, $\cap$-closed family S of subsets of X with the empty element, and sets A, B. Now we state the propositions:

(15) If A, $B \in S$, then $B \setminus A \in \mathrm{DisUnion}\, S$.

(16) If $A \in S$ and $B \in \mathrm{DisUnion}\, S$, then $B \setminus A \in \mathrm{DisUnion}\, S$.
PROOF: Reconsider $A_1 = A$ as a subset of X. Consider B_1 being a subset of X such that $B = B_1$ and there exists a disjoint valued finite sequence F of elements of S such that $B_1 = \bigcup F$. Consider g_1 being a disjoint valued finite sequence of elements of S such that $B_1 = \bigcup g_1$. Reconsider $R_1 = \mathrm{DisUnion}\, S$ as a non empty set. Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv \$_2 = g_1(\$_1) \setminus A_1$. For every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, g_1$ there exists an element x of R_1 such that $\mathcal{P}[k, x]$ by [10, (3)], (15). Consider g_2 being a finite sequence of elements of R_1 such that $\mathrm{dom}\, g_2 = \mathrm{Seg}\,\mathrm{len}\, g_1$ and for every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, g_1$ holds $\mathcal{P}[k, g_2(k)]$ from [6, Sch. 5]. For every natural numbers n, m such that n, $m \in \mathrm{dom}\, g_2$ and $n \neq m$ holds $g_2(n)$ misses $g_2(m)$. Set $R = \mathrm{DisUnion}\, S$. Define $\mathcal{H}[\text{natural number}] \equiv \bigcup \mathrm{rng}(g_2 \restriction \$_1) \in R$. For every natural number k such that $\mathcal{H}[k]$ holds $\mathcal{H}[k+1]$ by [4, (13)], [6, (59), (82)], [24, (55)]. For every natural number k, $\mathcal{H}[k]$ from [4, Sch. 2]. $\square$

Now we state the propositions:

(17) Let us consider a set X, a semi \-closed, $\cap$-closed family S of subsets of X with the empty element, and sets A, B, R. Suppose $R = \mathrm{DisUnion}\, S$ and A, $B \in R$ and $A \neq \emptyset$. Then $B \setminus A \in R$.
PROOF: Consider A_1 being a subset of X such that $A = A_1$ and there exists a disjoint valued finite sequence F of elements of S such that $A_1 = \bigcup F$. Consider f_1 being a disjoint valued finite sequence of elements of S such that $A_1 = \bigcup f_1$. Consider B_1 being a subset of X such that $B = B_1$ and there exists a disjoint valued finite sequence F of elements of S such that $B_1 = \bigcup F$. Reconsider $R_1 = R$ as a non empty set. Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv \$_2 = B_1 \setminus f_1(\$_1)$. For every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, f_1$ there exists an element x of R_1 such that $\mathcal{P}[k, x]$ by [10, (3)], (16). Consider F being a finite sequence of elements of R_1 such

that $\operatorname{dom} F = \operatorname{Seg} \operatorname{len} f_1$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} f_1$ holds $\mathcal{P}[k, F(k)]$ from [6, Sch. 5]. Define $\mathcal{P}[\text{natural number}] \equiv \bigcap \operatorname{rng}(F \restriction \$_1) \in R$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [6, (82)], [4, (11)], [6, (59)], [24, (55)]. For every natural number k, $\mathcal{P}[k]$ from [4, Sch. 2]. □

(18) Let us consider a set X, and a semi $\setminus$-closed, $\cap$-closed family S of subsets of X with the empty element. Then the ring generated by $S = \operatorname{DisUnion} S$. The theorem is a consequence of (13), (17), and (14).

Let X be a set.

A σ-ring of subsets of X is a non empty, preboolean family of subsets of X and is defined by

(Def. 4) for every sequence F of subsets of X such that $\operatorname{rng} F \subseteq it$ holds $\bigcup F \in it$.

Let us observe that every σ-ring of subsets of X is σ-multiplicative.

Let S be a family of subsets of X. The functor σ-ring(S) yielding a σ-ring of subsets of X is defined by

(Def. 5) $S \subseteq it$ and for every set Z such that $S \subseteq Z$ and Z is a σ-ring of subsets of X holds $it \subseteq Z$.

Now we state the proposition:

(19) Let us consider a set X, and a semi $\setminus$-closed, $\cap$-closed family S of subsets of X with the empty element. Then σ-ring(the ring generated by S) $=$ σ-ring(S). The theorem is a consequence of (11).

3. Semialgebra, Algebra and σ-algebra of Sets

Let X be a set.

A semialgebra of sets of X is a semi $\setminus$-closed, $\cap$-closed family of subsets of X with the empty element and is defined by

(Def. 6) $X \in it$.

Now we state the proposition:

(20) Let us consider a set X. Then every field of subsets of X is a semialgebra of sets of X.

Let X be a set and S be a semialgebra of sets of X. The field generated by S yielding a non empty field of subsets of X is defined by the term

(Def. 7) $\bigcap \{Z$, where Z is a field of subsets of $X : S \subseteq Z\}$.

Now we state the propositions:

(21) Let us consider a set X, and a semialgebra P of sets of X. Then $P \subseteq$ the field generated by P.

(22) Let us consider a set X, and a semialgebra S of sets of X. Then the field generated by $S = \mathrm{DisUnion}\, S$. The theorem is a consequence of (13), (17), and (14).

(23) Let us consider a non empty set X, and a semialgebra S of sets of X. Then σ(the field generated by S) $= \sigma(S)$. The theorem is a consequence of (21).

4. Mutual Relationships between σ-ring and σ-algebra of Sets

Let us consider a set X and a set S. Now we state the propositions:

(24) If S is a σ-field of subsets of X, then S is a σ-ring of subsets of X.

(25) If S is a σ-ring of subsets of X and $X \in S$, then S is a σ-field of subsets of X.

Let us consider a family S of subsets of $\mathbb{R}$. Now we state the propositions:

(26) Suppose $S = \{I$, where I is a subset of $\mathbb{R}$: I is left open interval$\}$. Then S is semi $\setminus$-closed and $\cap$-closed and has the empty element. The theorem is a consequence of (10).

(27) Suppose $S = \{I$, where I is a subset of $\mathbb{R}$: I is right open interval$\}$. Then S is semi $\setminus$-closed and $\cap$-closed and has the empty element. The theorem is a consequence of (4) and (3).

Now we state the proposition:

(28) the set of all I where I is an interval is a semialgebra of sets of $\mathbb{R}$. The theorem is a consequence of (3) and (4).

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