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The First Isomorphism Theorem and Other Properties of Rings

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Summary. Different properties of rings and fields are discussed [12], [41] and [17]. We introduce ring homomorphisms, their kernels and images, and prove the First Isomorphism Theorem, namely that for a homomorphism $f : R \longrightarrow S$ we have $R/_{\ker(f)} \cong \mathrm{Im}(f)$. Then we define prime and irreducible elements and show that every principal ideal domain is factorial. Finally we show that polynomial rings over fields are Euclidean and hence also factorial.

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The notation and terminology used in this paper have been introduced in the following articles: [22], [31], [2], [32], [24], [5], [11], [33], [7], [8], [26], [36], [37], [39], [30], [1], [35], [27], [34], [19], [3], [4], [9], [25], [18], [28], [29], [13], [6], [42], [43], [20], [14], [38], [23], [40], [15], [16], [21], and [10].

1. Preliminaries

Let R be a non empty set, f be a non empty finite sequence of elements of R, and x be an element of dom f. Note that the functor $f(x)$ yields an element of R. Let X be a set and F_1, F_2 be X-valued finite sequences. One can verify that $F_1 \frown F_2$ is X-valued.

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Now we state the propositions:

(1) Let us consider an add-associative, right zeroed, right complementable, distributive, well unital, non empty double loop structure R, and a finite sequence F of elements of R. Suppose there exists a natural number i such that $i \in \operatorname{dom} F$ and $F(i) = 0_R$. Then $\prod F = 0_R$.

(2) Let us consider an add-associative, right zeroed, right complementable, well unital, distributive, integral domain-like, non degenerated double loop structure R, and a finite sequence F of elements of R. Then $\prod F = 0_R$ if and only if there exists a natural number i such that $i \in \operatorname{dom} F$ and $F(i) = 0_R$. The theorem is a consequence of (1).

Let X be a set.

A chain of X is a sequence of X. Let X be a non empty set and C be a chain of X. We say that C is ascending if and only if

(Def. 1) for every natural number i, $C(i) \subseteq C(i+1)$.

We say that C is stagnating if and only if

(Def. 2) there exists a natural number i such that for every natural number j such that $j \geqslant i$ holds $C(j) = C(i)$.

Let x be an element of X. One can check that $\mathbb{N} \longmapsto x$ is ascending and stagnating as a chain of X and there exists a chain of X which is ascending and stagnating.

Now we state the proposition:

(3) Let us consider a non empty set X, an ascending chain C of X, and natural numbers i, j. If $i \leqslant j$, then $C(i) \subseteq C(j)$.

Let R be a ring. The functor $\operatorname{Ideals} R$ yielding a family of subsets of the carrier of R is defined by the term

(Def. 3) the set of all I where I is an ideal of R.

One can verify that $\operatorname{Ideals} R$ is non empty.

Now we state the propositions:

(4) Let us consider a commutative ring R, an ideal I of R, and an element a of R. If $a \in I$, then $\{a\}$–ideal $\subseteq I$.

(5) Let us consider a ring R, and an ascending chain C of $\operatorname{Ideals} R$. Then $\bigcup$ the set of all $C(i)$ where i is a natural number is an ideal of R.

Let R be a non empty double loop structure and S be a right zeroed, non empty double loop structure. Let us note that $R \longmapsto 0_S$ is additive.

Let S be an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure. Observe that $R \longmapsto 0_S$ is multiplicative.

Let R be a well unital, non empty double loop structure and S be a well unital, non degenerated double loop structure. Note that $R \longmapsto 0_S$ is non unity-preserving.

Let R be a non empty double loop structure. One can verify that id_R is additive, multiplicative, and unity-preserving and id_R is monomorphic and epimorphic.

Let S be a right zeroed, non empty double loop structure. Observe that there exists a function from R into S which is additive.

Let S be an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure. Let us observe that there exists a function from R into S which is multiplicative.

Let R, S be well unital, non empty double loop structures. One can verify that there exists a function from R into S which is unity-preserving.

Let R be a non empty double loop structure and S be an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure. One can verify that there exists a function from R into S which is additive and multiplicative.

2. Homomorphisms, Kernel and Image

Let R, S be rings. We say that S is R-homomorphic if and only if

(Def. 4) there exists a function f from R into S such that f inherits ring homomorphism.

Let R be a ring. One can verify that there exists a ring which is R-homomorphic.

Let R be a commutative ring. Let us observe that there exists a commutative ring which is R-homomorphic and there exists a ring which is R-homomorphic.

Let R be a field. Observe that there exists a field which is R-homomorphic and there exists a commutative ring which is R-homomorphic and there exists a ring which is R-homomorphic.

Let R be a ring and S be an R-homomorphic ring. Note that there exists a function from R into S which is additive, multiplicative, and unity-preserving.

A homomorphism from R to S is an additive, multiplicative, unity-preserving function from R into S. Let R, S, T be rings, f be a unity-preserving function from R into S, and g be a unity-preserving function from S into T. Observe that $g \cdot f$ is unity-preserving as a function from R into T.

Let R be a ring and S be an R-homomorphic ring. Note that every S-homomorphic ring is R-homomorphic.

Let R, S be non empty double loop structures. We introduce R and S are isomorphic as a synonym of R is ring isomorphic to S.

Now we state the propositions:

(6) Let us consider an add-associative, right zeroed, right complementable, non empty double loop structure R, an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure S, and an additive function f from R into S. Then $f(0_R) = 0_S$.

(7) Let us consider an add-associative, right zeroed, right complementable, non empty double loop structure R, an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure S, an additive function f from R into S, and an element x of R. Then $f(-x) = -f(x)$. The theorem is a consequence of (6).

(8) Let us consider an add-associative, right zeroed, right complementable, non empty double loop structure R, an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure S, an additive function f from R into S, and elements x, y of R. Then $f(x - y) = f(x) - f(y)$. The theorem is a consequence of (7).

(9) Let us consider a right unital, non empty double loop structure R, an add-associative, right zeroed, right complementable, right unital, Abelian, right distributive, integral domain-like, non empty double loop structure S, and a multiplicative function f from R into S. Then

   (i) $f(1_R) = 0_S$, or

   (ii) $f(1_R) = 1_S$.

Let us consider fields E, F and an additive, multiplicative function f from E into F. Now we state the propositions:

(10) $f(1_E) = 0_F$ if and only if $f = E \longmapsto 0_F$.

(11) $f(1_E) = 1_F$ if and only if f is monomorphic.

Let E, F be fields. One can check that every function from E into F which is additive, multiplicative, and unity-preserving is also monomorphic.

Let R be a ring and I be an ideal of R. The canonical homomorphism of I into quotient field yielding a function from R into $R/_I$ is defined by

(Def. 5) for every element a of R, $it(a) = [a]_{\mathrm{EqRel}(R,I)}$.

Let us note that the canonical homomorphism of I into quotient field is additive, multiplicative, and unity-preserving and the canonical homomorphism of I into quotient field is epimorphic and $R/_I$ is R-homomorphic.

Let R be an add-associative, right zeroed, right complementable, non empty double loop structure, S be an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure, and f be an additive function from R into S. One can check that $\ker f$ is non empty.

Let R be a non empty double loop structure and S be an add-associative, right zeroed, right complementable, non empty double loop structure. One can check that $\ker f$ is closed under addition.

Let S be an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure and f be a multiplicative function from R into S. Observe that $\ker f$ is left ideal.

Let S be an add-associative, right zeroed, right complementable, distributive, non empty double loop structure. Let us observe that $\ker f$ is right ideal.

Let R be a well unital, non empty double loop structure, S be a well unital, non degenerated double loop structure, and f be a unity-preserving function from R into S. Observe that $\ker f$ is proper.

Now we state the propositions:

(12) Let us consider a ring R, an R-homomorphic ring S, and a homomorphism f from R to S. Then f is monomorphic if and only if $\ker f = \{0_R\}$. The theorem is a consequence of (6) and (8).

(13) Let us consider a ring R, and an ideal I of R. Then ker the canonical homomorphism of I into quotient field $= I$.

(14) Let us consider a ring R, and a subset I of R. Then I is an ideal of R if and only if there exists an R-homomorphic ring S and there exists a homomorphism f from R to S such that $\ker f = I$. The theorem is a consequence of (13).

Let R be a ring, S be an R-homomorphic ring, and f be a homomorphism from R to S. The functor $\operatorname{Im} f$ yielding a strict double loop structure is defined by

(Def. 6) the carrier of $it = \operatorname{rng} f$ and the addition of $it =$ (the addition of S) $\restriction$ $\operatorname{rng} f$ and the multiplication of $it =$ (the multiplication of S) $\restriction \operatorname{rng} f$ and the one of $it = 1_S$ and the zero of $it = 0_S$.

Note that $\operatorname{Im} f$ is non empty and $\operatorname{Im} f$ is Abelian, add-associative, right zeroed, and right complementable and $\operatorname{Im} f$ is associative, well unital, and distributive.

Let R be a commutative ring and S be an R-homomorphic commutative ring. One can verify that $\operatorname{Im} f$ is commutative.

Let R be a ring and S be an R-homomorphic ring. Let us note that the functor $\operatorname{Im} f$ yields a strict subring of S. The canonical homomorphism of f into quotient field yielding a function from $R/_{\ker f}$ into $\operatorname{Im} f$ is defined by

(Def. 7) for every element a of R, $it([a]_{\mathrm{EqRel}(R,\ker f)}) = f(a)$.

One can check that the canonical homomorphism of f into quotient field is additive, multiplicative, and unity-preserving and the canonical homomorphism of f into quotient field is monomorphic and epimorphic.

Let us consider a ring R, an R-homomorphic ring S, and a homomorphism f from R to S. Now we state the propositions:

(15) $R/_{\ker f}$ and $\operatorname{Im} f$ are isomorphic.

(16) If f is onto, then $R/_{\ker f}$ and S are isomorphic.

Now we state the proposition:

(17) Let us consider a ring R. Then $R/_{\{0_R\}}$ and R are isomorphic. The theorem is a consequence of (12).

Let R be a ring. Let us note that $R/_{\Omega_R}$ is trivial.

3. Units and Non Units

Let L be a right unital, non empty multiplicative loop structure. Let us note that there exists an element of L which is unital.

A unit of L is a unital element of L. Let L be an add-associative, right zeroed, right complementable, left distributive, non degenerated double loop structure. One can check that there exists an element of L which is non unital.

A non-unit of L is a non unital element of L. Note that 0_L is non unital.

Let L be a right unital, non empty multiplicative loop structure. Let us note that 1_L is unital.

Let L be an add-associative, right zeroed, right complementable, left distributive, right unital, non degenerated double loop structure. One can verify that every unit of L is non zero.

Let F be a field. Note that every non zero element of F is unital.

Let R be an integral domain and u, v be unital elements of R. One can check that $u \cdot v$ is unital.

Let us consider a commutative ring R and elements a, b of R. Now we state the propositions:

(18) $a \mid b$ if and only if $b \in \{a\}$–ideal.

(19) $a \mid b$ if and only if $\{b\}$–ideal $\subseteq \{a\}$–ideal. The theorem is a consequence of (18).

Now we state the propositions:

(20) Let us consider a commutative ring R, and an element a of R. Then a is a unit of R if and only if $\{a\}$–ideal $= \Omega_R$. The theorem is a consequence of (18).

(21) Let us consider a commutative ring R, and elements a, b of R. Then a is associated to b if and only if $\{a\}$–ideal $= \{b\}$–ideal.

4. Prime and Irreducible Elements

Let R be a right unital, non empty double loop structure and x be an element of R. We say that x is prime if and only if

(Def. 8) $x \neq 0_R$ and x is not a unit of R and for every elements a, b of R such that $x \mid a \cdot b$ holds $x \mid a$ or $x \mid b$.

We say that x is irreducible if and only if

(Def. 9) $x \neq 0_R$ and x is not a unit of R and for every element a of R such that $a \mid x$ holds a is unit of R or associated to x.

We introduce x is reducible as an antonym for x is irreducible.

Note that there exists an element of R which is non prime and there exists an element of $\mathbb{Z}^{\mathrm{R}}$ which is prime.

Let R be a right unital, non empty double loop structure. Let us observe that every element of R which is prime is also non zero and non unital and every element of R which is irreducible is also non zero and non unital.

Let R be an integral domain. Observe that every element of R which is prime is also irreducible.

Let F be a field. Let us note that every element of F is reducible.

Let R be a right unital, non empty double loop structure. The functor $\mathrm{IRR}(R)$ yielding a subset of R is defined by the term

(Def. 10) $\{x$, where x is an element of $R : x$ is irreducible$\}$.

Let F be a field. One can check that $\mathrm{IRR}(F)$ is empty.

Now we state the propositions:

(22) Let us consider an integral domain R, a non zero element c of R, and elements b, a, d of R. Suppose $a \cdot b$ is associated to $c \cdot d$ and a is associated to c. Then b is associated to d.

(23) Let us consider an integral domain R, and elements a, b of R. Suppose a is irreducible and b is associated to a. Then b is irreducible.

Let us consider a non degenerated commutative ring R and a non zero element a of R. Now we state the propositions:

(24) a is prime if and only if $\{a\}$–ideal is prime. The theorem is a consequence of (18).

(25) If $\{a\}$–ideal is maximal, then a is irreducible. The theorem is a consequence of (19) and (18).

5. Principal Ideal Domains and Factorial Rings

Note that every field is PID and there exists a non empty double loop structure which is PID.

A principal ideal domain is a PID integral domain. Now we state the proposition:

(26) Let us consider a principal ideal domain R, and a non zero element a of R. Then $\{a\}$–ideal is maximal if and only if a is irreducible. The theorem is a consequence of (19), (20), (18), and (25).

Let R be a principal ideal domain. Observe that every element of R which is irreducible is also prime and every commutative ring which is Euclidean is also PID.

Let R be a principal ideal domain. One can verify that every chain of Ideals R which is ascending is also stagnating.

Let R be a right unital, non empty double loop structure, x be an element of R, and F be a non empty finite sequence of elements of R. We say that F is a factorization of x if and only if

(Def. 11) $x = \prod F$ and for every element i of $\operatorname{dom} F$, $F(i)$ is irreducible.

We say that x is factorizable if and only if

(Def. 12) there exists a non empty finite sequence F of elements of R such that F is a factorization of x.

Assume x is factorizable.

A factorization of x is a non empty finite sequence of elements of R and is defined by

(Def. 13) *it* is a factorization of x.

We say that x is uniquely factorizable if and only if

(Def. 14) x is factorizable and for every factorizations F, G of x, there exists a function B from $\operatorname{dom} F$ into $\operatorname{dom} G$ such that B is bijective and for every element i of $\operatorname{dom} F$, $G(B(i))$ is associated to $F(i)$.

One can verify that every element of R which is uniquely factorizable is also factorizable.

Let R be an integral domain. Let us observe that every element of R which is factorizable is also non zero and non unital.

Let R be a right unital, non empty double loop structure. Let us note that every element of R which is irreducible is also factorizable.

Now we state the propositions:

(27) Let us consider a right unital, non empty double loop structure R, and an element a of R. Then a is irreducible if and only if $\langle a \rangle$ is a factorization of a.

(28) Let us consider a well unital, associative, non empty double loop structure R, elements a, b of R, and non empty finite sequences F, G of elements of R. Suppose F is a factorization of a and G is a factorization of b. Then $F \frown G$ is a factorization of $a \cdot b$.

Let R be a principal ideal domain. Observe that every element of R which is factorizable is also uniquely factorizable.

Let R be a non degenerated ring. We say that R is factorial if and only if

(Def. 15) for every non zero element a of R such that a is a non-unit of R holds a is uniquely factorizable.

One can check that there exists a non degenerated ring which is factorial.

Let R be a factorial, non degenerated ring. Note that every element of R which is non zero and non unital is also factorizable.

A factorial ring is a factorial, non degenerated ring. One can check that every integral domain which is PID is also factorial.

6. Polynomial Rings over Fields

Let L be a field and p be a polynomial of L. The functor $\deg{*}\,p$ yielding a natural number is defined by the term

(Def. 16) $\begin{cases} \deg p, & \textbf{if } p \neq \mathbf{0}.\,L, \\ 0, & \textbf{otherwise}. \end{cases}$

The functor $\deg{*}\,L$ yielding a function from Polynom-Ring L into $\mathbb{N}$ is defined by

(Def. 17) for every polynomial p of L, $it(p) = \deg{*}\,p$.

Now we state the propositions:

(29) Let us consider a field L, a polynomial p of L, and a non zero polynomial q of L. Then $\deg(p \bmod q) < \deg q$.

(30) Let us consider a field L, an element p of Polynom-Ring L, and a non zero element q of Polynom-Ring L. Then there exist elements u, r of Polynom-Ring L such that

(i) $p = u \cdot q + r$, and

(ii) $r = 0_{\text{Polynom-Ring}\,L}$ or $(\deg{*}\,L)(r) < (\deg{*}\,L)(q)$.

The theorem is a consequence of (29).

Let L be a field. One can check that Polynom-Ring L is Euclidean.

Note that the functor $\deg{*}\,L$ yields a DegreeFunction of Polynom-Ring L.

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