The First Isomorphism Theorem and Other Properties of Rings

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Summary. Different properties of rings and fields are discussed [12], [41] and [17]. We introduce ring homomorphisms, their kernels and images, and prove the First Isomorphism Theorem, namely that for a homomorphism \( f : R \rightarrow S \) we have \( R/\ker(f) \cong \text{Im}(f) \). Then we define prime and irreducible elements and show that every principal ideal domain is factorial. Finally we show that polynomial rings over fields are Euclidean and hence also factorial.

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The notation and terminology used in this paper have been introduced in the following articles: [22], [31], [2], [32], [24], [5], [11], [33], [7], [8], [26], [36], [37], [39], [30], [1], [35], [27], [34], [19], [3], [4], [9], [25], [18], [28], [29], [13], [6], [42], [43], [20], [14], [38], [23], [40], [15], [16], [24], and [10].

1. Preliminaries

Let \( R \) be a non empty set, \( f \) be a non empty finite sequence of elements of \( R \), and \( x \) be an element of \( \text{dom} f \). Note that the functor \( f(x) \) yields an element of \( R \). Let \( X \) be a set and \( F_1 \), \( F_2 \) be \( X \)-valued finite sequences. One can verify that \( F_1 \circ F_2 \) is \( X \)-valued.
Now we state the propositions:

(1) Let us consider an add-associative, right zeroed, right complementable, distributive, well unital, non empty double loop structure $R$, and a finite sequence $F$ of elements of $R$. Suppose there exists a natural number $i$ such that $i \in \text{dom } F$ and $F(i) = 0_R$. Then $\prod F = 0_R$.

(2) Let us consider an add-associative, right zeroed, right complementable, well unital, distributive, integral domain-like, non degenerated double loop structure $R$, and a finite sequence $F$ of elements of $R$. Then $\prod F = 0_R$ if and only if there exists a natural number $i$ such that $i \in \text{dom } F$ and $F(i) = 0_R$. The theorem is a consequence of (1).

Let $X$ be a set.

A chain of $X$ is a sequence of $X$. Let $X$ be a non empty set and $C$ be a chain of $X$. We say that $C$ is ascending if and only if

(Def. 1) for every natural number $i$, $C(i) \subseteq C(i + 1)$.

We say that $C$ is stagnating if and only if

(Def. 2) there exists a natural number $i$ such that for every natural number $j$ such that $j \geq i$ holds $C(j) = C(i)$.

Let $x$ be an element of $X$. One can check that $\mathbb{N} \rightarrow x$ is ascending and stagnating as a chain of $X$ and there exists a chain of $X$ which is ascending and stagnating.

Now we state the proposition:

(3) Let us consider a non empty set $X$, an ascending chain $C$ of $X$, and natural numbers $i, j$. If $i \leq j$, then $C(i) \subseteq C(j)$.

Let $R$ be a ring. The functor $\text{Ideals } R$ yielding a family of subsets of the carrier of $R$ is defined by the term

(Def. 3) the set of all $I$ where $I$ is an ideal of $R$.

One can verify that $\text{Ideals } R$ is non empty.

Now we state the propositions:

(4) Let us consider a commutative ring $R$, an ideal $I$ of $R$, and an element $a$ of $R$. If $a \in I$, then $\{a\}$–ideal $\subseteq I$.

(5) Let us consider a ring $R$, and an ascending chain $C$ of $\text{Ideals } R$. Then $\bigcup$ the set of all $C(i)$ where $i$ is a natural number is an ideal of $R$.

Let $R$ be a non empty double loop structure and $S$ be a right zeroed, non empty double loop structure. Let us note that $R \rightarrow 0_S$ is additive.

Let $S$ be an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure. Observe that $R \rightarrow 0_S$ is multiplicative.
Let $R$ be a well unital, non empty double loop structure and $S$ be a well unital, non degenerated double loop structure. Note that $R \rightarrow 0_S$ is non unity-preserving.

Let $R$ be a non empty double loop structure. One can verify that $\text{id}_R$ is additive, multiplicative, and unity-preserving and $\text{id}_R$ is monomorphic and epimorphic.

Let $S$ be a right zeroed, non empty double loop structure. Observe that there exists a function from $R$ into $S$ which is additive.

Let $S$ be an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure. Let us observe that there exists a function from $R$ into $S$ which is multiplicative.

Let $R, S$ be well unital, non empty double loop structures. One can verify that there exists a function from $R$ into $S$ which is unity-preserving.

Let $R$ be a non empty double loop structure and $S$ be an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure. One can verify that there exists a function from $R$ into $S$ which is additive and multiplicative.

### 2. Homomorphisms, Kernel and Image

Let $R, S$ be rings. We say that $S$ is $R$-homomorphic if and only if

$$(\text{Def. 4}) \quad \text{there exists a function } f \text{ from } R \text{ into } S \text{ such that } f \text{ inherits ring homomorphism.}$$

Let $R$ be a ring. One can verify that there exists a ring which is $R$-homomorphic.

Let $R$ be a commutative ring. Let us observe that there exists a commutative ring which is $R$-homomorphic and there exists a ring which is $R$-homomorphic.

Let $R$ be a field. Observe that there exists a field which is $R$-homomorphic and there exists a commutative ring which is $R$-homomorphic and there exists a ring which is $R$-homomorphic.

Let $R$ be a ring and $S$ be an $R$-homomorphic ring. Note that there exists a function from $R$ into $S$ which is additive, multiplicative, and unity-preserving.

A homomorphism from $R$ to $S$ is an additive, multiplicative, unity-preserving function from $R$ into $S$. Let $R, S, T$ be rings, $f$ be a unity-preserving function from $R$ into $S$, and $g$ be a unity-preserving function from $S$ into $T$. Observe that $g \cdot f$ is unity-preserving as a function from $R$ into $T$.

Let $R$ be a ring and $S$ be an $R$-homomorphic ring. Note that every $S$-homomorphic ring is $R$-homomorphic.

Let $R, S$ be non empty double loop structures. We introduce $R$ and $S$ are isomorphic as a synonym of $R$ is ring isomorphic to $S$.

Now we state the propositions:
Let us consider an add-associative, right zeroed, right complementable, non empty double loop structure $R$, an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure $S$, and an additive function $f$ from $R$ into $S$. Then $f(0_R) = 0_S$.

Let us consider an add-associative, right zeroed, right complementable, non empty double loop structure $R$, an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure $S$, an additive function $f$ from $R$ into $S$, and an element $x$ of $R$. Then $f(-x) = -f(x)$. The theorem is a consequence of (6).

Let us consider an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure $S$, an additive function $f$ from $R$ into $S$, and elements $x, y$ of $R$. Then $f(x - y) = f(x) - f(y)$. The theorem is a consequence of (7).

Let us consider a right unital, non empty double loop structure $R$, an add-associative, right zeroed, right complementable, right unital, Abelian, right distributive, integral domain-like, non empty double loop structure $S$, and a multiplicative function $f$ from $R$ into $S$. Then

(i) $f(1_R) = 0_S$, or

(ii) $f(1_R) = 1_S$.

Let us consider fields $E, F$ and an additive, multiplicative function $f$ from $E$ into $F$. Now we state the propositions:

(10) $f(1_E) = 0_F$ if and only if $f = E \mapsto 0_F$.

(11) $f(1_E) = 1_F$ if and only if $f$ is monomorphic.

Let $E, F$ be fields. One can check that every function from $E$ into $F$ which is additive, multiplicative, and unity-preserving is also monomorphic.

Let $R$ be a ring and $I$ be an ideal of $R$. The canonical homomorphism of $I$ into quotient field yielding a function from $R$ into $R/I$ is defined by

(Def. 5) for every element $a$ of $R$, it(a) = $[a]_{\text{EqRel}(R, I)}$.

Let us note that the canonical homomorphism of $I$ into quotient field is additive, multiplicative, and unity-preserving and the canonical homomorphism of $I$ into quotient field is epimorphic and $R/I$ is $R$-homomorphic.

Let $R$ be an add-associative, right zeroed, right complementable, non empty double loop structure, $S$ be an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure, and $f$ be an additive function from $R$ into $S$. One can check that ker $f$ is non empty.

Let $R$ be a non empty double loop structure and $S$ be an add-associative, right zeroed, right complementable, non empty double loop structure. One can
check that $\ker f$ is closed under addition.

Let $S$ be an add-associative, right zeroed, right complementable, right distributive, non empty double loop structure and $f$ be a multiplicative function from $R$ into $S$. Observe that $\ker f$ is left ideal.

Let $S$ be an add-associative, right zeroed, right complementable, distributive, non empty double loop structure. Let us observe that $\ker f$ is right ideal.

Let $R$ be a well unital, non empty double loop structure, $S$ be a well unital, non degenerated double loop structure, and $f$ be a unity-preserving function from $R$ into $S$. Observe that $\ker f$ is proper.

Now we state the propositions:

(12) Let us consider a ring $R$, an $R$-homomorphic ring $S$, and a homomorphism $f$ from $R$ to $S$. Then $f$ is monomorphic if and only if $\ker f = \{0_R\}$. The theorem is a consequence of (6) and (8).

(13) Let us consider a ring $R$, and an ideal $I$ of $R$. Then $\ker$ the canonical homomorphism of $I$ into quotient field $= I$.

(14) Let us consider a ring $R$, and a subset $I$ of $R$. Then $I$ is an ideal of $R$ if and only if there exists an $R$-homomorphic ring $S$ and there exists a homomorphism $f$ from $R$ to $S$ such that $\ker f = I$. The theorem is a consequence of (13).

Let $R$ be a ring, $S$ be an $R$-homomorphic ring, and $f$ be a homomorphism from $R$ to $S$. The functor $\text{Im } f$ yielding a strict double loop structure is defined by

\begin{align*}
(\text{Def. 6}) & \quad \text{the carrier of } it = \text{rng } f \text{ and the addition of } it = (\text{the addition of } S) \upharpoonright \text{rng } f \text{ and the multiplication of } it = (\text{the multiplication of } S) \upharpoonright \text{rng } f \text{ and the one of } it = 1_S \text{ and the zero of } it = 0_S.
\end{align*}

Note that $\text{Im } f$ is non empty and $\text{Im } f$ is Abelian, add-associative, right zeroed, and right complementable and $\text{Im } f$ is associative, well unital, and distributive.

Let $R$ be a commutative ring and $S$ be an $R$-homomorphic commutative ring. One can verify that $\text{Im } f$ is commutative.

Let $R$ be a ring and $S$ be an $R$-homomorphic ring. Let us note that the functor $\text{Im } f$ yields a strict subring of $S$. The canonical homomorphism of $f$ into quotient field yielding a function from $R/\ker f$ into $\text{Im } f$ is defined by

\begin{align*}
(\text{Def. 7}) & \quad \text{for every element } a \text{ of } R, \quad it([a]_{\text{EqRel}(R, \ker f)}) = f(a).
\end{align*}

One can check that the canonical homomorphism of $f$ into quotient field is additive, multiplicative, and unity-preserving and the canonical homomorphism of $f$ into quotient field is monomorphic and epimorphic.
Let us consider a ring $R$, an $R$-homomorphic ring $S$, and a homomorphism $f$ from $R$ to $S$. Now we state the propositions:

(15) $R/\ker f$ and $\text{Im } f$ are isomorphic.

(16) If $f$ is onto, then $R/\ker f$ and $S$ are isomorphic.

Now we state the proposition:

(17) Let us consider a ring $R$. Then $R/(\{0_R\})$ and $R$ are isomorphic. The theorem is a consequence of (12).

Let $R$ be a ring. Let us note that $R/\Omega_R$ is trivial.

3. Units and Non Units

Let $L$ be a right unital, non empty multiplicative loop structure. Let us note that there exists an element of $L$ which is unital.

A unit of $L$ is a unital element of $L$. Let $L$ be an add-associative, right zeroed, right complementable, left distributive, non degenerated double loop structure. One can check that there exists an element of $L$ which is non unital.

A non-unit of $L$ is a non unital element of $L$. Note that $0_L$ is non unital.

Let $L$ be a right unital, non empty multiplicative loop structure. Let us note that $1_L$ is unital.

Let $L$ be an add-associative, right zeroed, right complementable, left distributive, right unital, non degenerated double loop structure. One can verify that every unit of $L$ is non zero.

Let $F$ be a field. Note that every non zero element of $F$ is unital.

Let $R$ be an integral domain and $u, v$ be unital elements of $R$. One can check that $u \cdot v$ is unital.

Let us consider a commutative ring $R$ and elements $a, b$ of $R$. Now we state the propositions:

(18) $a \mid b$ if and only if $b \in \{a\}$–ideal.

(19) $a \mid b$ if and only if $\{b\}$–ideal $\subseteq \{a\}$–ideal. The theorem is a consequence of (18).

Now we state the propositions:

(20) Let us consider a commutative ring $R$, and an element $a$ of $R$. Then $a$ is a unit of $R$ if and only if $\{a\}$–ideal $= \Omega_R$. The theorem is a consequence of (18).

(21) Let us consider a commutative ring $R$, and elements $a, b$ of $R$. Then $a$ is associated to $b$ if and only if $\{a\}$–ideal $= \{b\}$–ideal.
Let $R$ be a right unital, non empty double loop structure and $x$ be an element of $R$. We say that $x$ is prime if and only if

(Def. 8) $x \neq 0_R$ and $x$ is not a unit of $R$ and for every elements $a$, $b$ of $R$ such that $x \mid a \cdot b$ holds $x \mid a$ or $x \mid b$.

We say that $x$ is irreducible if and only if

(Def. 9) $x \neq 0_R$ and $x$ is not a unit of $R$ and for every element $a$ of $R$ such that $a \mid x$ holds $a$ is unit of $R$ or associated to $x$.

We introduce $x$ is reducible as an antonym for $x$ is irreducible.

Note that there exists an element of $R$ which is non prime and there exists an element of $\mathbb{Z}^R$ which is prime.

Let $R$ be a right unital, non empty double loop structure. Let us observe that every element of $R$ which is prime is also non zero and non unital and every element of $R$ which is irreducible is also non zero and non unital.

Let $R$ be an integral domain. Observe that every element of $R$ which is prime is also irreducible.

Let $F$ be a field. Let us note that every element of $F$ is reducible.

Let $R$ be a right unital, non empty double loop structure. The functor $\text{IRR}(R)$ yielding a subset of $R$ is defined by the term

(Def. 10) $\{x, \text{ where } x \text{ is an element of } R : x \text{ is irreducible}\}$.

Let $F$ be a field. One can check that $\text{IRR}(F)$ is empty.

Now we state the propositions:

(22) Let us consider an integral domain $R$, a non zero element $c$ of $R$, and elements $b$, $a$, $d$ of $R$. Suppose $a \cdot b$ is associated to $c \cdot d$ and $a$ is associated to $c$. Then $b$ is associated to $d$.

(23) Let us consider an integral domain $R$, and elements $a$, $b$ of $R$. Suppose $a$ is irreducible and $b$ is associated to $a$. Then $b$ is irreducible.

Let us consider a non degenerated commutative ring $R$ and a non zero element $a$ of $R$. Now we state the propositions:

(24) $a$ is prime if and only if $\{a\}$–ideal is prime. The theorem is a consequence of (18).

(25) If $\{a\}$–ideal is maximal, then $a$ is irreducible. The theorem is a consequence of (19) and (18).
5. Principal Ideal Domains and Factorial Rings

Note that every field is PID and there exists a non empty double loop structure which is PID.

A principal ideal domain is a PID integral domain. Now we state the proposition:

(26) Let us consider a principal ideal domain \( R \), and a non zero element \( a \) of \( R \). Then \( \{a\} \)–ideal is maximal if and only if \( a \) is irreducible. The theorem is a consequence of (19), (20), (18), and (25).

Let \( R \) be a principal ideal domain. Observe that every element of \( R \) which is irreducible is also prime and every commutative ring which is Euclidean is also PID.

Let \( R \) be a principal ideal domain. One can verify that every chain of Ideals \( R \) which is ascending is also stagnating.

Let \( R \) be a right unital, non empty double loop structure, \( x \) be an element of \( R \), and \( F \) be a non empty finite sequence of elements of \( R \). We say that \( F \) is a factorization of \( x \) if and only if

\[
\text{(Def. 11)} \quad x = \prod F \quad \text{and for every element } i \text{ of } \text{dom} \, F, \, F(i) \text{ is irreducible.}
\]

We say that \( x \) is factorizable if and only if

\[
\text{(Def. 12)} \quad \text{there exists a non empty finite sequence } F \text{ of elements of } R \text{ such that } F \text{ is a factorization of } x.
\]

Assume \( x \) is factorizable.

A factorization of \( x \) is a non empty finite sequence of elements of \( R \) and is defined by

\[
\text{(Def. 13)} \quad \text{it is a factorization of } x.
\]

We say that \( x \) is uniquely factorizable if and only if

\[
\text{(Def. 14)} \quad x \text{ is factorizable and for every factorizations } F, \, G \text{ of } x, \text{ there exists a function } B \text{ from } \text{dom} \, F \text{ into } \text{dom} \, G \text{ such that } B \text{ is bijective and for every element } i \text{ of } \text{dom} \, F, \, G(B(i)) \text{ is associated to } F(i).
\]

One can verify that every element of \( R \) which is uniquely factorizable is also factorizable.

Let \( R \) be an integral domain. Let us observe that every element of \( R \) which is factorizable is also non zero and non unital.

Let \( R \) be a right unital, non empty double loop structure. Let us note that every element of \( R \) which is irreducible is also factorizable.

Now we state the propositions:

(27) Let us consider a right unital, non empty double loop structure \( R \), and an element \( a \) of \( R \). Then \( a \) is irreducible if and only if \( \langle a \rangle \) is a factorization of \( a \).
Let us consider a well unital, associative, non empty double loop structure \( R \), elements \( a, b \) of \( R \), and non empty finite sequences \( F, G \) of elements of \( R \). Suppose \( F \) is a factorization of \( a \) and \( G \) is a factorization of \( b \). Then \( F \bowtie G \) is a factorization of \( a \cdot b \).

Let \( R \) be a principal ideal domain. Observe that every element of \( R \) which is factorizable is also uniquely factorizable.

Let \( R \) be a non degenerated ring. We say that \( R \) is factorial if and only if (Def. 15) for every non zero element \( a \) of \( R \) such that \( a \) is a non-unit of \( R \) holds \( a \) is uniquely factorizable.

One can check that there exists a non degenerated ring which is factorial.

Let \( R \) be a factorial, non degenerated ring. Note that every element of \( R \) which is non zero and non unital is also factorizable.

A factorial ring is a factorial, non degenerated ring. One can check that every integral domain which is PID is also factorial.

6. Polynomial Rings over Fields

Let \( L \) be a field and \( p \) be a polynomial of \( L \). The functor \( \deg^* p \) yielding a natural number is defined by the term

\[
\text{(Def. 16)} \quad \left\{ \begin{array}{ll}
\deg p, & \text{if } p \neq 0, L, \\
0, & \text{otherwise.}
\end{array} \right.
\]

The functor \( \deg^* L \) yielding a function from Polynom-Ring \( L \) into \( \mathbb{N} \) is defined by

\[
\text{(Def. 17)} \quad \text{for every polynomial } p \text{ of } L, \quad \text{it}(p) = \deg^* p.
\]

Now we state the propositions:

(29) Let us consider a field \( L \), a polynomial \( p \) of \( L \), and a non zero polynomial \( q \) of \( L \). Then \( \deg(p \mod q) < \deg q \).

(30) Let us consider a field \( L \), an element \( p \) of Polynom-Ring \( L \), and a non zero element \( q \) of Polynom-Ring \( L \). Then there exist elements \( u, r \) of Polynom-Ring \( L \) such that

(i) \( p = u \cdot q + r \), and

(ii) \( r = 0_{\text{Polynom-Ring } L} \) or \( (\deg^* L)(r) < (\deg^* L)(q) \).

The theorem is a consequence of (29).

Let \( L \) be a field. One can check that Polynom-Ring \( L \) is Euclidean.

Note that the functor \( \deg^* L \) yields a DegreeFunction of Polynom-Ring \( L \).
References


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