

Pseudo-Canonical Formulae are Classical

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Summary. An original result about Hilbert Positive Propositional Calculus introduced in [11] is proven. That is, it is shown that the pseudo-canonical formulae of that calculus (and hence also the canonical ones, see [17]) are a subset of the classical tautologies.

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The notation and terminology used in this paper have been introduced in the following articles: [13], [1], [14], [10], [9], [15], [3], [4], [5], [6], [11], [16], [17], [2], [7], [18], [20], [22], [21], [12], [19], and [8].

1. PRELIMINARIES ABOUT INJECTIVITY, INVOLUTIVENESS, FIXED POINTS

From now on $a, b, c, x, y, z, A, B, C, X, Y$ denote sets, f, g denote functions, V denotes a SetValuation, P denotes a permutation of V , p, q, r, s denote elements of HP-WFF, and n denotes an element of \mathbb{N} .

Let us consider X and Y . Let f be a relation between X and Y . Note that $\text{id}_X \cdot f$ reduces to f and $f \cdot \text{id}_Y$ reduces to f .

Now we state the proposition:

- (1) Let us consider one-to-one functions f, g . If $f^{-1} = g^{-1}$, then $f = g$.

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One can verify that there exists a function which is involutive.

Let us consider A . Let us observe that there exists a permutation of A which is involutive.

Now we state the propositions:

- (2) Let us consider an involutive function f . Suppose $\text{rng } f \subseteq \text{dom } f$. Then $f \cdot f = \text{id}_{\text{dom } f}$.
- (3) Let us consider a function f . If $f \cdot f = \text{id}_{\text{dom } f}$, then f is involutive.
- (4) Let us consider an involutive function f from A into A . Then $f \cdot f = \text{id}_A$. The theorem is a consequence of (2).
- (5) Let us consider a function f from A into A . If $f \cdot f = \text{id}_A$, then f is involutive. The theorem is a consequence of (3).

Observe that every function which is involutive is also one-to-one.

Let us consider A . Let f be an involutive permutation of A . One can verify that f^{-1} is involutive.

Let n be a non zero natural number. Observe that $[0 \mapsto n, n \mapsto 0]$ is without fixpoints.

Let z be a zero natural number. Note that $\text{fixpoints}[z \mapsto n, n \mapsto z]$ is empty.

Let X be a non empty set. Observe that there exists a permutation of X which is non empty and involutive.

Let us consider A and B . Let f be an involutive function from A into A and g be an involutive function from B into B . Observe that $f \times g$ is involutive.

Let A, B be non empty sets, f be an involutive permutation of A , and g be an involutive permutation of B . Observe that $f \Rightarrow g$ is involutive.

2. FACTS ABOUT PERM'S FIXED POINTS

Now we state the propositions:

- (6) If x is a fixpoint of $\text{Perm}(P, q)$, then $\text{SetVal}(V, p) \mapsto x$ is a fixpoint of $\text{Perm}(P, p \Rightarrow q)$.
- (7) If $\text{Perm}(P, q)$ has fixpoints, then $\text{Perm}(P, p \Rightarrow q)$ has fixpoints. The theorem is a consequence of (6).
- (8) If $\text{Perm}(P, p)$ has fixpoints and $\text{Perm}(P, q)$ is without fixpoints, then $\text{Perm}(P, p \Rightarrow q)$ is without fixpoints.

3. AXIOM OF CHOICE IN FUNCTIONAL FORM VIA THE FRAENKEL OPERATOR

Let X be a set. The functor $\text{ChoiceOn } X$ yielding a set is defined by the term

(Def. 1) $\{\langle x, \text{the element of } x \rangle, \text{ where } x \text{ is an element of } X \setminus \{\emptyset\} : x \in X \setminus \{\emptyset\}\}.$

One can check that $\text{ChoiceOn } X$ is relation-like and function-like.

Let us consider f . The functor $\text{FieldCover } f$ yielding a set is defined by the term

(Def. 2) $\{\{x, f(x)\}, \text{ where } x \text{ is an element of } \text{dom } f : x \in \text{dom } f\}$.

The functor $\text{SomePoints } f$ yielding a set is defined by the term

(Def. 3) $\text{field } f \setminus \text{rng } \text{ChoiceOn } \text{FieldCover } f$.

The functor $\text{OtherPoints } f$ yielding a set is defined by the term

(Def. 4) $(\text{field } f \setminus \text{fixpoints } f) \setminus \text{SomePoints } f$.

Let us consider g . Let us observe that $\text{OtherPoints } g \cap \text{SomePoints } g$ is empty.

4. BUILDING A SUITABLE SET VALUATION AND A SUITABLE PERMUTATION OF IT

Let us consider x . The functor $\text{ToHilb}(x)$ yielding a set is defined by the term

(Def. 5) $(\text{id}_1 + \cdot (1 \times \emptyset^x) \cdot (\emptyset^x \times \{1\})) + \cdot (\{1\} \times \emptyset^x) \cdot (\emptyset^x \times \{0\})$.

Note that $\text{ToHilb}(x)$ is function-like and relation-like.

Now we state the propositions:

(9) If $x \neq \emptyset$, then $\text{ToHilb}(x) = \text{id}_1$.

(10) $\text{ToHilb}(\emptyset) = [0 \mapsto 1, 1 \mapsto 0]$.

Let v be a function. The functor $\text{ToHilbPerm}(v)$ yielding a set is defined by the term

(Def. 6) the set of all $\langle n, \text{ToHilb}(v(n)) \rangle$ where n is an element of \mathbb{N} .

The functor $\text{ToHilbVal}(v)$ yielding a set is defined by the term

(Def. 7) the set of all $\langle n, \text{dom } \text{ToHilb}(v(n)) \rangle$ where n is an element of \mathbb{N} .

One can check that $\text{ToHilbVal}(v)$ is function-like and relation-like and $\text{ToHilbPerm}(v)$ is function-like and relation-like and $\text{ToHilbVal}(v)$ is \mathbb{N} -defined and $\text{ToHilbVal}(v)$ is total and $\text{ToHilbPerm}(v)$ is \mathbb{N} -defined and $\text{ToHilbPerm}(v)$ is total.

One can verify that $\text{ToHilbVal}(v)$ is non-empty.

Let us consider x . Let us note that $\text{ToHilb}(x)$ is symmetric.

Let v be a function. Observe that the functor $\text{ToHilbPerm}(v)$ yields a permutation of $\text{ToHilbVal}(v)$.

A set valuation is a many sorted set indexed by \mathbb{N} . From now on v denotes a set valuation.

Let us consider p and v . Note that $\text{Perm}(\text{ToHilbPerm}(v), p)$ is involutive.

5. CLASSICAL SEMANTICS VIA SetVal_0 , AN EXTENSION OF SetVal

Let V be a set valuation. The functor $\text{SetVal}_0 V$ yielding a many sorted set indexed by HP-WFF is defined by

- (Def. 8) (i) $it(\text{VERUM}) = 1$, and
 (ii) for every n , $it(\text{prop } n) = V(n)$, and
 (iii) for every p and q , $it(p \wedge q) = it(p) \times it(q)$ and $it(p \Rightarrow q) = (it(q))^{it(p)}$.

Let us consider v and p . The functor $\text{SetVal}_0(v, p)$ yielding a set is defined by the term

- (Def. 9) $(\text{SetVal}_0 v)(p)$.

We say that p is classical if and only if

- (Def. 10) $\text{SetVal}_0(v, p) \neq \emptyset$.

One can check that every element of HP-WFF which is pseudo-canonical is also classical.

Let us consider v . Let p be a classical element of HP-WFF. Note that $\text{SetVal}_0(v, p)$ is non empty.

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