Semiring of Sets: Examples

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Summary. This article proposes the formalization of some examples of semiring of sets proposed by Goguadze [8] and Schmets [13].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [14], [7], [17], [15], [5], [16], [9], [12], [19], [10], [18], and [6].

1. Preliminaries

From now on $X$ denotes a set and $S$ denotes a family of subsets of $X$. Now we state the propositions:

(1) Let us consider sets $X_1$, $X_2$, a family $S_1$ of subsets of $X_1$, and a family $S_2$ of subsets of $X_2$. Then \{ $a \times b$, where $a$ is an element of $S_1$, $b$ is an element of $S_2$ : $a \in S_1$ and $b \in S_2$ \} = \{ $s$, where $s$ is a subset of $X_1 \times X_2$ : there exist sets $a$, $b$ such that $a \in S_1$ and $b \in S_2$ and $s = a \times b$ \}. PROOF: \{ $a \times b$, where $a$ is an element of $S_1$, $b$ is an element of $S_2$ : $a \in S_1$ and $b \in S_2$ \} $\subseteq$ \{ $s$, where $s$ is a subset of $X_1 \times X_2$ : there exist sets $a$, $b$ such that $a \in S_1$ and $b \in S_2$ and $s = a \times b$ \} by [6] (96). ☐

(2) Let us consider sets $X_1$, $X_2$, a non empty family $S_1$ of subsets of $X_1$, and a non empty family $S_2$ of subsets of $X_2$. Then \{ $s$, where $s$ is a subset of $X_1 \times X_2$ : there exist sets $x_1$, $x_2$ such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2$ \} = the set of all $x_1 \times x_2$ where $x_1$ is an element of $S_1$, $x_2$ is an element of $S_2$.

(3) Let us consider sets $X_1$, $X_2$, a family $S_1$ of subsets of $X_1$, and a family $S_2$ of subsets of $X_2$. Suppose
(i) \( S_1 \) is \( \cap \)-closed, and
(ii) \( S_2 \) is \( \cap \)-closed.

Then \( \{ s \text{, where } s \text{ is a subset of } X \times X : \text{there exist sets } x_1, x_2 \text{ such that } x_1 \in S_1 \text{ and } x_2 \in S_2 \text{ and } s = x_1 \times x_2 \} \) is \( \cap \)-closed. \( \text{PROOF: Set } \) \( Y = \{ s \text{, where } s \text{ is a subset of } X \times X : \text{there exist sets } x_1, x_2 \text{ such that } x_1 \in S_1 \text{ and } x_2 \in S_2 \text{ and } s = x_1 \times x_2 \} \). \( Y \) is \( \cap \)-closed by [6] (100). \( \square \)

Let \( X \) be a set. Note that every \( \sigma \)-field of subsets of \( X \) is \( \cap_{fp} \)-closed and \( \setminus_{fp} \)-closed and has countable cover and empty element.

2. Ordinary Examples of Semirings of Sets

Now we state the proposition:

(4) Every \( \sigma \)-field of subsets of \( X \) is a semiring of sets of \( X \).

Let \( X \) be a set. Note that \( 2^X \) is \( \cap_{fp} \)-closed and \( \setminus_{fp} \)-closed and has countable cover and empty element as a family of subsets of \( X \).

Now we state the proposition:

(5) \( 2^X \) is a semiring of sets of \( X \).

Let us consider \( X \). Note that \( \text{Fin} \ X \) is \( \cap_{fp} \)-closed and \( \setminus_{fp} \)-closed and has empty element as a family of subsets of \( X \).

Let \( D \) be a denumerable set. Observe that \( \text{Fin} \ D \) has countable cover as a family of subsets of \( D \).

Now we state the propositions:

(6) \( \text{Fin} \ X \) is a semiring of sets of \( X \).

(7) Let us consider sets \( X_1, X_2 \), a semiring \( S_1 \) of sets of \( X_1 \), and a semiring \( S_2 \) of sets of \( X_2 \). Then \( \{ s \text{, where } s \text{ is a subset of } X_1 \times X_2 : \text{there exist sets } x_1, x_2 \text{ such that } x_1 \in S_1 \text{ and } x_2 \in S_2 \text{ and } s = x_1 \times x_2 \} \) is a semiring of sets of \( X_1 \times X_2 \). \( \text{PROOF: Set } Y = \{ s \text{, where } s \text{ is a subset of } X_1 \times X_2 : \text{there exist sets } x_1, x_2 \text{ such that } x_1 \in S_1 \text{ and } x_2 \in S_2 \text{ and } s = x_1 \times x_2 \} \). \( Y \) has empty element. \( Y \) is \( \cap_{fp} \)-closed by [6] (100), [4] (8), [11] (10). \( Y \) is \( \setminus_{fp} \)-closed by [11] (10), [11] (39), [4] (8), [11] (45). \( \square \)

(8) Let us consider non empty sets \( X_1, X_2 \), a family \( S_1 \) of subsets of \( X_1 \) with countable cover, a family \( S_2 \) of subsets of \( X_2 \) with countable cover, and a family \( S \) of subsets of \( X_1 \times X_2 \). Suppose \( S = \{ s \text{, where } s \text{ is a subset of } X_1 \times X_2 : \text{there exist sets } x_1, x_2 \text{ such that } x_1 \in S_1 \text{ and } x_2 \in S_2 \text{ and } s = x_1 \times x_2 \} \). Then \( S \) has countable cover. \( \text{PROOF: There exists a countable subset } U \text{ of } S \text{ such that } \bigcup U = X_1 \times X_2 \text{ and } U \text{ is a subset of } S \text{ by [6] (2), (77), [2] (95), [3] (7)). } \) \( \square \)

Let us consider a family \( S \) of subsets of \( \mathbb{R} \). Now we state the propositions:

(9) Suppose \( S = \{ [a, b], \text{ where } a, b \text{ are real numbers } : a \leq b \} \). Then
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(i) $S$ is $\cap$-closed, and
(ii) $S$ is $\setminus_{FP}$-closed and has empty element, and
(iii) $S$ has countable cover.

(10) Suppose $S = \{ s, \text{ where } s \text{ is a subset of } \mathbb{R} : s \text{ is left open interval} \}$. Then
(i) $S$ is $\cap$-closed, and
(ii) $S$ is $\setminus_{FP}$-closed and has empty element, and
(iii) $S$ has countable cover.

**Proof:** $S$ is $\cap$-closed. $S$ has empty element. $S$ is $\setminus_{FP}$-closed by [11, (39)], [6] (75). □

3. Numerical Example

The functor $\text{sring}_{\mathbb{Z}}$ yielding a family of subsets of $\{1, 2, 3, 4\}$ is defined by the term

(Def. 1) $\{ \{1, 2, 3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1\}, \{\{2\}\}, \{\{3\}\}, \{\{4\}\}, \emptyset \}$.

One can verify that $\text{sring}_{\mathbb{Z}}$ has empty element and $\text{sring}_{\mathbb{Z}}$ is $\setminus_{FP}$-closed and non $\cap$-closed and $\text{sring}_{\mathbb{Z}}$ is $\setminus_{FP}$-closed.

References

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