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Semiring of Sets: Examples

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Summary. This article proposes the formalization of some examples of semiring of sets proposed by Goguadze [8] and Schmets [13].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [14], [7], [17], [15], [5], [16], [9], [12], [19], [10], [18], and [6].

1. Preliminaries

From now on X denotes a set and S denotes a family of subsets of X.

Now we state the propositions:

(1) Let us consider sets X_1, X_2, a family S_1 of subsets of X_1, and a family S_2 of subsets of X_2. Then $\{a \times b$, where a is an element of S_1, b is an element of $S_2 : a \in S_1$ and $b \in S_2\} = \{s$, where s is a subset of $X_1 \times X_2 :$ there exist sets a, b such that $a \in S_1$ and $b \in S_2$ and $s = a \times b\}$. Proof: $\{a \times b$, where a is an element of S_1, b is an element of $S_2 : a \in S_1$ and $b \in S_2\} \subseteq \{s$, where s is a subset of $X_1 \times X_2 :$ there exist sets a, b such that $a \in S_1$ and $b \in S_2$ and $s = a \times b\}$ by [6, (96)]. $\square$

(2) Let us consider sets X_1, X_2, a non empty family S_1 of subsets of X_1, and a non empty family S_2 of subsets of X_2. Then $\{s$, where s is a subset of $X_1 \times X_2 :$ there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$ = the set of all $x_1 \times x_2$ where x_1 is an element of S_1, x_2 is an element of S_2.

(3) Let us consider sets X_1, X_2, a family S_1 of subsets of X_1, and a family S_2 of subsets of X_2. Suppose

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(i) S_1 is $\cap$-closed, and

(ii) S_2 is $\cap$-closed.

Then $\{s$, where s is a subset of $X_1 \times X_2$: there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$ is $\cap$-closed. Proof: Set $Y = \{s$, where s is a subset of $X_1 \times X_2$: there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$. Y is $\cap$-closed by [6, (100)]. $\square$

Let X be a set. Note that every σ-field of subsets of X is $\cap_{fp}$-closed and $\setminus_{fp}^{\subseteq}$-closed and has countable cover and empty element.

2. Ordinary Examples of Semirings of Sets

Now we state the proposition:

(4) Every σ-field of subsets of X is a semiring of sets of X.

Let X be a set. Note that 2^X is $\cap_{fp}$-closed and $\setminus_{fp}^{\subseteq}$-closed and has countable cover and empty element as a family of subsets of X.

Now we state the proposition:

(5) 2^X is a semiring of sets of X.

Let us consider X. Note that $\mathrm{Fin}\, X$ is $\cap_{fp}$-closed and $\setminus_{fp}^{\subseteq}$-closed and has empty element as a family of subsets of X.

Let D be a denumerable set. Observe that $\mathrm{Fin}\, D$ has countable cover as a family of subsets of D.

Now we state the propositions:

(6) $\mathrm{Fin}\, X$ is a semiring of sets of X.

(7) Let us consider sets X_1, X_2, a semiring S_1 of sets of X_1, and a semiring S_2 of sets of X_2. Then $\{s$, where s is a subset of $X_1 \times X_2$: there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$ is a semiring of sets of $X_1 \times X_2$. Proof: Set $Y = \{s$, where s is a subset of $X_1 \times X_2$: there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$. Y has empty element. Y is $\cap_{fp}$-closed by [6, (100)], [4, (8)], [1, (10)]. Y is $\setminus_{fp}$-closed by [1, (10)], [11, (39)], [4, (8)], [11, (45)]. $\square$

(8) Let us consider non empty sets X_1, X_2, a family S_1 of subsets of X_1 with countable cover, a family S_2 of subsets of X_2 with countable cover, and a family S of subsets of $X_1 \times X_2$. Suppose $S = \{s$, where s is a subset of $X_1 \times X_2$: there exist sets x_1, x_2 such that $x_1 \in S_1$ and $x_2 \in S_2$ and $s = x_1 \times x_2\}$. Then S has countable cover. Proof: There exists a countable subset U of S such that $\bigcup U = X_1 \times X_2$ and U is a subset of S by [6, (2), (77)], [2, (95)], [3, (7)]. $\square$

Let us consider a family S of subsets of $\mathbb{R}$. Now we state the propositions:

(9) Suppose $S = \{]a, b]$, where a, b are real numbers : $a \leqslant b\}$. Then

(i) S is $\cap$-closed, and

(ii) S is $\setminus_{fp}$-closed and has empty element, and

(iii) S has countable cover.

(10) Suppose $S = \{s$, where s is a subset of $\mathbb{R}$: s is left open interval$\}$. Then

(i) S is $\cap$-closed, and

(ii) S is $\setminus_{fp}$-closed and has empty element, and

(iii) S has countable cover.

PROOF: S is $\cap$-closed. S has empty element. S is $\setminus_{fp}$-closed by [11, (39)], [6, (75)]. $\square$

3. Numerical Example

The functor sring_8^4 yielding a family of subsets of $\{1, 2, 3, 4\}$ is defined by the term

(Def. 1) $\{\{1, 2, 3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1\}, (\{2\}), (\{3\}), (\{4\}), (\emptyset)\}$.

One can verify that sring_8^4 has empty element and sring_8^4 is $\cap_{fp}$-closed and non $\cap$-closed and sring_8^4 is $\setminus_{fp}$-closed.

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