Tietze Extension Theorem for \( n \)-dimensional Spaces\(^1\)

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Summary. In this article we prove the Tietze extension theorem for an arbitrary convex compact subset of \( \mathcal{E}^n \) with a non-empty interior. This theorem states that, if \( T \) is a normal topological space, \( X \) is a closed subset of \( T \), and \( A \) is a convex compact subset of \( \mathcal{E}^n \) with a non-empty interior, then a continuous function \( f : X \to A \) can be extended to a continuous function \( g : T \to \mathcal{E}^n \). Additionally we show that a subset \( A \) is replaceable by an arbitrary subset of a topological space that is homeomorphic with a convex compact subset of \( \mathcal{E}^n \) with a non-empty interior. This article is based on [20]; [23] and [22] can also serve as reference books.

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The notation and terminology used in this paper have been introduced in the following articles: [8], [36], [24], [30], [1], [15], [21], [16], [25], [6], [9], [17], [37], [10], [11], [3], [34], [5], [12], [26], [33], [35], [41], [42], [13], [40], [19], [31], [28], [43], [18], [44], [29], and [14].

1. Closed Hypercube

From now on \( n, m, i \) denote natural numbers, \( p, q \) denote points of \( \mathcal{E}_T^n \), \( r, s \) denote real numbers, and \( R \) denotes a real-valued finite sequence.

Note that every finite sequence which is empty is also non-negative yielding.

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Let $n$ be a non zero natural number, $X$ be a set, and $F$ be an element of $(\text{the carrier of } \mathbb{R}^1)^X$. Let us note that the functor $\Pi^* F$ yields a function from $X$ into $\mathcal{E}_n$. Now we state the proposition:

(1) Let us consider sets $X$, $Y$, a function yielding function $F$, and objects $x$, $y$. Suppose

(i) $F$ is $(Y^X)$-valued, or

(ii) $y \in \text{dom } \Pi^* F$.

Then $F(x)(y) = (\Pi^* F)(y)(x)$.

Let us consider $n$, $p$, and $r$. The functor $\text{OpenHypercube}(p, r)$ yielding an open subset of $\mathcal{E}_n$ is defined by

(Def. 1) There exists a point $e$ of $\mathcal{E}_n$ such that

(i) $p = e$, and

(ii) it $= \text{OpenHypercube}(e, r)$.

Now we state the propositions:

(2) If $q \in \text{OpenHypercube}(p, r)$ and $s \in [p(i) - r, p(i) + r]$, then $q + (i, s) \in \text{OpenHypercube}(p, r)$. PROOF: Consider $e$ being a point of $\mathcal{E}_n$ such that $p = e$ and $\text{OpenHypercube}(p, r) = \text{OpenHypercube}(e, r)$. Set $I = \text{Intervals}(e, r)$. Set $q_3 = q + (i, s)$. For every object $x$ such that $x \in \text{dom } I$ holds $q_3(x) \in I(x)$ by [2, (9)], [7, (31), (32)]. □

(3) If $i \in \text{Seg } n$, then $(\text{PROJ}(n, i))^c(\text{OpenHypercube}(p, r)) = [p(i) - r, p(i) + r]$. The theorem is a consequence of (2).

(4) $q \in \text{OpenHypercube}(p, r)$ if and only if for every $i$ such that $i \in \text{Seg } n$ holds $q(i) \in [p(i) - r, p(i) + r]$. The theorem is a consequence of (3).

Let us consider $n$, $p$, and $R$. The functor $\text{ClosedHypercube}(p, R)$ yielding a subset of $\mathcal{E}_n$ is defined by

(Def. 2) $q \in \text{it}$ if and only if for every $i$ such that $i \in \text{Seg } n$ holds $q(i) \in [p(i) - R(i), p(i) + R(i)]$.

Now we state the propositions:

(5) If there exists $i$ such that $i \in \text{Seg } n \cap \text{dom } R$ and $R(i) < 0$, then $\text{ClosedHypercube}(p, R)$ is empty.

(6) If for every $i$ such that $i \in \text{Seg } n \cap \text{dom } R$ holds $R(i) \geq 0$, then $p \in \text{ClosedHypercube}(p, R)$.

Let us consider $n$ and $p$. Let $R$ be a non-negative yielding real-valued finite sequence. One can check that $\text{ClosedHypercube}(p, R)$ is non empty.

Let us consider $R$. Let us observe that $\text{ClosedHypercube}(p, R)$ is convex and compact.

Now we state the propositions:
(7) If \( i \in \text{Seg } n \) and \( q \in \text{ClosedHypercube}(p, R) \) and \( r \in [p(i) - R(i), p(i) + R(i)] \), then \( q + (i, r) \in \text{ClosedHypercube}(p, R) \). \textbf{Proof:} Let \( p_4 = q + (i, r) \).

For every natural number \( j \) such that \( j \in \text{Seg } n \) holds \( p_4(j) \in [p(j) - R(j), p(j) + R(j)] \) by \([21, (32), (31)]\). \( \square \)

(8) Suppose \( i \in \text{Seg } n \) and \( \text{ClosedHypercube}(p, R) \) is not empty.

Then \((\text{PROJ}(n, i))^c(\text{ClosedHypercube}(p, R)) = [p(i) - R(i), p(i) + R(i)]\). The theorem is a consequence of (5), (7), and (6).

(9) If \( n \leq \inf \text{rng } R \),

then \( \text{OpenHypercube}(p, r) \subseteq \text{ClosedHypercube}(p, R) \).

(10) \( q \in \text{Fr } \text{ClosedHypercube}(p, R) \) if and only if \( q \in \text{ClosedHypercube}(p, R) \) and there exists \( i \) such that \( i \in \text{Seg } n \) and \( q(i) = p(i) - R(i) \) or \( q(i) = p(i) + R(i) \). \textbf{Proof:} Set \( T_4 = \mathcal{E}_n^p \). If \( q \in \text{Fr } \text{ClosedHypercube}(p, R) \), then \( q \in \text{ClosedHypercube}(p, R) \) and there exists \( i \) such that \( i \in \text{Seg } n \) and \( q(i) = p(i) - R(i) \) or \( q(i) = p(i) + R(i) \) by \([16, (22)], [32, (105)], [14, (33)], [6, (3)]\). For every subset \( S \) of \( T_4 \) such that \( S \) is open and \( q \in S \) holds \( \text{ClosedHypercube}(p, R) \) meets \( S \) and \((\text{ClosedHypercube}(p, R))^c \) meets \( S \) by \([16, (67)], [43, (23)], [38, (5)], [31, (13)]\). \( \square \)

(11) If \( r > 0 \), then \( p \in \text{ClosedHypercube}(p, n \mapsto r) \).

(12) If \( r > 0 \), then \( \text{Int } \text{ClosedHypercube}(p, n \mapsto r) = \text{OpenHypercube}(p, r) \).

\textbf{Proof:} Set \( O = \text{OpenHypercube}(p, r) \). Set \( C = \text{ClosedHypercube}(p, n \mapsto r) \). Set \( T_4 = \mathcal{E}_n^p \). Set \( R = n \mapsto r \). Consider \( e \) being a point of \( \mathcal{E}^n \) such that \( p = e \) and \( \text{OpenHypercube}(p, r) = \text{OpenHypercube}(e, r) \).

\( \text{Int } C \subseteq O \) by \([33, (39)], [9, (57)], (10), [39, (29)]\). Reconsider \( q = x \) as a point of \( T_4 \).

For every \( i \) such that \( i \in \text{Seg } n \) holds \( q(i) \in [p(i) - R(i), p(i) + R(i)] \) by \([9, (57)], (3)\). Consider \( i \) such that \( i \in \text{Seg } n \) and \( q(i) = p(i) - R(i) \) or \( q(i) = p(i) + R(i) \). \((\text{PROJ}(n, i))^cO = \{e(i) - r, e(i) + r\} \). \( \square \)

(13) \( \text{ClosedHypercube}(p, r) \subseteq \text{ClosedHypercube}(p, n \mapsto r) \).

(14) If \( r < s \), then \( \text{ClosedHypercube}(p, n \mapsto r) \subseteq \text{ClosedHypercube}(p, s) \).

The theorem is a consequence of (4).

Let us consider \( n \) and \( p \). Let \( r \) be a positive real number. Let us note that \( \text{ClosedHypercube}(p, n \mapsto r) \) is non boundary.

\[ 2. \text{Properties of the Product of Closed Hypercube} \]

From now on \( T_1, T_2, S_1, S_2 \) denote non empty topological spaces, \( t_1 \) denotes a point of \( T_1 \), \( t_2 \) denotes a point of \( T_2 \), \( p_2, q_2 \) denote points of \( \mathcal{E}_T^p \), and \( p_1, q_1 \) denote points of \( \mathcal{E}_T^q \).

Now we state the propositions:

(15) Let us consider a function \( f \) from \( T_1 \) into \( T_2 \) and a function \( g \) from \( S_1 \) into \( S_2 \). Suppose
(14) Suppose $r > 0$ and $s > 0$. Then there exists a function $h$ from
$(E^n_m \upharpoonright \text{ClosedHypercube}(p_2, n \mapsto r)) \times (E^m_m \upharpoonright \text{ClosedHypercube}(p_1, m \mapsto s))$
into $E^{n+m}_m \upharpoonright \text{ClosedHypercube}(0_{E^{n+m}_m}, (n + m) \mapsto 1)$ such that
(i) $h$ is a homeomorphism, and
(ii) $h^o(\text{OpenHypercube}(p_2, r) \times \text{OpenHypercube}(p_1, s)) = \text{OpenHypercube}(0_{E^{n+m}_m}, 1)$.

Proof: Set $T_6 = E^n_n$. Set $T_5 = E^m_m$. Set $n_1 = n + m$. Set $T_7 = E^{n_1}_1$. Set $R_2 =\text{ClosedHypercube}(0_{T_6}, n \mapsto 1)$. Set $R_4 = \text{ClosedHypercube}(p_2, n \mapsto r)$. Set $R_5 = \text{ClosedHypercube}(p_1, m \mapsto s)$. Set $R_1 = \text{ClosedHypercube}(0_{T_6}, m \mapsto 1)$. Set $R_3 = \text{ClosedHypercube}(0_{T_7}, n_1 \mapsto 1)$. Reconsider $R_{10} = R_5$, $R_6 = R_1$ as a non empty subset of $T_5$. Consider $h_3$ being a function from $T_5 \upharpoonright R_{10}$ into $T_6 \upharpoonright R_6$ such that $h_3$ is a homeomorphism and $h_3^o(Fr R_6) = Fr R_6$. Reconsider $R_9 = R_4$, $R_7 = R_2$ as a non empty subset of $T_6$. Consider $h_4$ being a function from $T_6 \upharpoonright R_9$ into $T_6 \upharpoonright R_7$ such that $h_4$ is a homeomorphism and $h_4^o(Fr R_6) = Fr R_7$. Set $O_8 = \text{OpenHypercube}(p_2, r)$. Set $O_9 = \text{OpenHypercube}(p_1, s)$. Set $O_6 = \text{OpenHypercube}(0_{T_7}, 1)$. Int $R_{10} = O_9$. Set $O_5 = \text{OpenHypercube}(0_{T_5}, 1)$. Reconsider $R_8 = R_3$ as a non empty subset of $T_7$. Consider $f$ being a function from $T_6 \times T_3$ into $T_7$ such that $f$ is a homeomorphism and for every element $f_5$ of $T_6$ and for every element $f_6$ of $T_5$, $f(f_5, f_6) = f_5 \circ f_6$. $f^o(R_7 \times R_6) \subseteq R_8$ by [14, (87)], [9, (57)], [6, (25)]. $R_8 \subseteq f^o(R_7 \times R_6)$ by [9, (23)], [27, (17)], [4, (11)], [6, (5)]. Set $h_5 = h_4 \times h_3$. $h_5$ is a homeomorphism. Int $R_7 = O_5$. Reconsider $f_1 = f \upharpoonright (R_7 \times R_6)$ as a function from $(T_5 \upharpoonright R_7) \times (T_3 \upharpoonright R_6)$ into $T_7 \upharpoonright R_8$. Reconsider $h = f_1 \cdot h_5$ as a function from $(T_5 \upharpoonright R_4) \times (T_3 \upharpoonright R_5)$ into $T_7 \upharpoonright R_8$. Int $R_6 = O_7$. Int $R_9 = O_8$. $h^o(O_8 \times O_9) \subseteq O_8$ by [14, (87)], [10, (12)], [43, (40)], [10, (49)]. Reconsider $p_3 = y$ as a point of $T_7$. Consider $p$, $q$ being finite sequences of elements of $R$ such that $len p = n$ and $len q = m$ and $p_3 = p \circ q$. $q \in O_7$. $q \in R_6$. Consider $x_2$ being an object such that $x_2 \in dom h_3$ and $h_3(x_2) = q \in O_5$. $p \in R_7$. Consider $x_1$ being an object such that $x_1 \in dom h_4$ and $h_4(x_1) = p$. □

(15) Suppose $r > 0$ and $s > 0$. Let us consider a function $f$ from $T_1$ into
$E^n_n \upharpoonright \text{ClosedHypercube}(p_2, n \mapsto r)$ and a function $g$ from $T_2$ into
$E^m_m \upharpoonright \text{ClosedHypercube}(p_1, m \mapsto s)$. Suppose
(i) $f$ is a homeomorphism, and
(ii) $g$ is a homeomorphism.
Then there exists a function $h$ from $T_1 \times T_2$ into
Let us consider a non boundary convex compact subset \( A \). Note that \( A \) is a non empty subset of \( \mathbb{R}^n \) such that \( h \) is a homeomorphism.

(iii) for every \( t_1 \) and \( t_2 \), \( f(t_1) \in \text{OpenHypercube}(p_2, r) \) and \( g(t_2) \in \text{OpenHypercube}(p_1, s) \) iff \( h(t_1, t_2) \in \text{OpenHypercube}(0_{\mathbb{E}^{n+m}}, 1) \).

**Proof:** Set \( n_1 = n + m \). Set \( T_6 = \mathbb{E}^n_1 \). Set \( T_5 = \mathbb{E}^m_1 \). Set \( T_7 = \mathbb{E}^{n_1}_1 \). Set \( R_7 = n \mapsto r \). Set \( R_6 = m \mapsto s \). Set \( R_8 = n_1 \mapsto 1 \). Set \( R_4 = \text{ClosedHypercube}(p_2, R_7) \). Set \( R_5 = \text{ClosedHypercube}(p_1, R_6) \). Set \( C_2 = \text{ClosedHypercube}(0_{T_5}, R_8) \). Consider \( R_{10} = R_5 \) as a non empty subset of \( T_5 \). Reconsider \( R_9 = R_4 \) as a non empty subset of \( T_6 \). Set \( O_9 = \text{OpenHypercube}(p_2, r) \). Set \( O_9 = \text{OpenHypercube}(p_1, s) \). Set \( O = \text{OpenHypercube}(0_{T_7}, 1) \). Consider \( h \) being a function from \( (T_6 \times T_9) \times (T_5 \times R_{10}) \) into \( T_7 \times C_2 \) such that \( h \) is a homeomorphism and \( h^\circ(O_9 \times O_9) = O \). Reconsider \( G = g \) as a function from \( T_2 \) into \( T_5 \times R_{10} \). Reconsider \( F = f \) as a function from \( T_1 \) into \( T_6 \times R_9 \). Consider \( f_4 = h \cdot (F \times G) \) as a function from \( T_1 \times T_2 \) into \( T_7 \times C_2 \). \( F \times G \) is a homeomorphism. \( O_9 \subseteq R_{10} \). \( O_8 \subseteq R_9 \). If \( f(t_1) \in O_8 \) and \( g(t_2) \in O_9 \), then \( f_4(t_1, t_2) \in O \) by [14, (87)], [15, (12)].

Consider \( x_3 \) being an object such that \( x_3 \in \text{dom} h \) and \( x_3 \in O_8 \). Then \( h(x_3) = h((f(t_1), g(t_2))) \). \( \square \)

Let us consider \( n \). One can check that there exists a subset of \( \mathbb{E}^n_1 \), which is non boundary, convex, and compact.

Now we state the propositions:

(18) Let us consider a non boundary convex compact subset \( A \) of \( \mathbb{E}^n_1 \), a non boundary convex compact subset \( B \) of \( \mathbb{E}^m_1 \), a non boundary convex compact subset \( C \) of \( \mathbb{E}^{n+m}_1 \), a function \( f \) from \( T_1 \) into \( \mathbb{E}^n_1 \), and a function \( g \) from \( T_2 \) into \( \mathbb{E}^m_1 \). Suppose

(i) \( f \) is a homeomorphism, and

(ii) \( g \) is a homeomorphism.

Then there exists a function \( h \) from \( T_1 \times T_2 \) into \( \mathbb{E}^{n+m}_1 \times C \) such that

(iii) \( h \) is a homeomorphism, and

(iv) for every \( t_1 \) and \( t_2 \), \( f(t_1) \in \text{Int} A \) and \( g(t_2) \in \text{Int} B \) iff \( h(t_1, t_2) \in \text{Int} C \).

**Proof:** Set \( T_6 = \mathbb{E}^n_1 \). Set \( T_5 = \mathbb{E}^m_1 \). Set \( n_1 = n + m \). Set \( T_7 = \mathbb{E}^{n_1}_1 \). Set \( R_7 = \text{ClosedHypercube}(0_{T_6}, n \mapsto 1) \). Set \( R_6 = \text{ClosedHypercube}(0_{T_5}, m \mapsto 1) \). Set \( R_8 = \text{ClosedHypercube}(0_{T_5}, n_1 \mapsto 1) \). Consider \( g_1 \) being a function from \( T_5 \times B \) into \( T_5 \times R_6 \) such that \( g_1 \) is a homeomorphism and \( g_1^\circ(\text{Fr} B) = \text{Fr} R_6 \). Reconsider \( g_2 = g_1 \cdot g \) as a function from \( T_2 \) into \( T_5 \times R_6 \). Consider \( f_7 \) being a function from \( T_6 \times A \) into \( T_6 \times R_7 \) such that \( f_7 \) is a homeomorphism and \( f_7^\circ(\text{Fr} A) = \text{Fr} R_7 \). Reconsider \( f_8 = f_7 \cdot f \) as a function from \( T_1 \) into \( T_6 \times R_7 \). Set \( O_3 = \text{OpenHypercube}(0_{T_6}, 1) \). Set \( O_2 = \text{OpenHypercube}(0_{T_5}, 1) \). Set \( O_4 = \text{OpenHypercube}(0_{T_5}, 1) \). Consider \( H \)
being a function from $T_7|R_8$ into $T_7|C$ such that $H$ is a homeomorphism and $H^o(Fr R_8) = Fr C$. Int $R_6 = O_2$. Consider $P$ being a function from $T_1 \times T_2$ into $T_7|R_8$ such that $P$ is a homeomorphism and for every $t_1$ and $t_2$, $f_8(t_1) \in O_3$ and $g_2(t_2) \in O_2$ iff $P(t_1, t_2) \in O_4$. Reconsider $H_1 = H \cdot P$ as a function from $T_1 \times T_2$ into $T_7|C$. Int $R_8 = O_4$. If $f(t_1) \in Int A$ and $g(t_2) \in Int B$, then $H_1(t_1, t_2) \in Int C$ by \[10] (11), (12)], (12). $P(t_1, t_2) \in R_8$. $P(t_1, t_2) \in O_4$. Int $R_7 = O_3$. $f(t_1) \in Int A$ by [43, (40)]. \]

(19) Let us consider a point $p_2$ of $E^n_T$, a point $p_1$ of $E^n_{T^r}$, $r$, and $s$. Suppose

(i) $r > 0$, and

(ii) $s > 0$.

Then there exists a function $h$ from $Tdisk(p_2, r) \times Tdisk(p_1, s)$ into $Tdisk(0_{e^{n+m}}, 1)$ such that

(iii) $h$ is a homeomorphism, and

(iv) $h^o(Ball(p_2, r) \times Ball(p_1, s)) = Ball(0_{e^{n+m}}, 1)$.

**Proof:** Set $T_6 = E^n_T$. Set $T_5 = E^n_{T^r}$. Set $n_1 = n + m$. Set $T_7 = E^{n_1}_T$. Reconsider $C_4 = Ball(p_2, r)$ as a non empty subset of $T_6$. Reconsider $C_3 = Ball(p_1, s)$ as a non empty subset of $T_5$. Reconsider $C_5 = Ball(0_{T^r}, 1)$ as a non empty subset of $T_7$. Set $R_7 = ClosedHypercube(0_{T^r}, 1 \rightarrow 1)$. Set $R_6 = ClosedHypercube(0_{T^r}, m \rightarrow 1)$. Consider $f_7$ being a function from $T_5|C_4$ into $T_5|R_7$ such that $f_7$ is a homeomorphism and $f_7^o(Fr C_4) = Fr R_6$. Consider $g_1$ being a function from $T_5|C_3$ into $T_5|R_6$ such that $g_1$ is a homeomorphism and $g_1^o(Fr C_3) = Fr R_6$. Consider $P$ being a function from $Tdisk(p_2, r) \times Tdisk(p_1, s)$ into $Tdisk(0_{T^r}, 1)$ such that $P$ is a homeomorphism and for every point $t_1$ of $T_6|C_4$ and for every point $t_2$ of $T_5|C_3$, $f_2(t_1) \in Int R_7$ and $g_1(t_2) \in Int R_6$ iff $P(t_1, t_2) \in Int C_5$. $P^o(Ball(p_2, r) \times Ball(p_1, s)) \subseteq Ball(0_{T^r}, 1)$ by [30], (3)], [43, (40)]. Consider $x$ being an object such that $x \in dom P$ and $P(x) = y$. Consider $y_1$, $y_2$ being objects such that $y_1 \in C_4$ and $y_2 \in C_3$ and $x = (y_1, y_2)$. \]

(20) Suppose $r > 0$ and $s > 0$ and $T_1$ and $E^n_T|Ball(p_2, r)$ are homeomorphic and $T_2$ and $E^n_{T^r}|Ball(p_1, s)$ are homeomorphic. Then $T_1 \times T_2$ and $E^{n+m}_T|Ball(0_{e^{n+m}}, 1)$ are homeomorphic.

3. Tietze Extension Theorem

In the sequel $T$, $S$ denote topological spaces, $A$ denotes a closed subset of $T$, and $B$ denotes a subset of $S$.

Now we state the propositions:

(21) Let us consider a non zero natural number $n$ and an element $F$ of $((\text{the carrier of } \mathbb{R}^1)^n)^n$. Suppose If $i \in dom F$, then for every function
h from $T$ into $\mathbb{R}^1$ such that $h = F(i)$ holds $h$ is continuous. Then $\prod^* F$ is continuous, where $\alpha$ is the carrier of $T$. \textbf{Proof}: Set $T_4 = E^m_T$. Set $F_1 = \prod^* F$. For every subset $Y$ of $T_4$ such that $Y$ is open holds $F_1^{-1}(Y)$ is open by [16 (67)], [11 (2)], (1), [19 (17)]. $\square$

(22) Suppose $T$ is normal. Let us consider a function $f$ from $T|A$ into $E^n_T|\text{ClosedHypercube}(0_{E^n_T}, n \mapsto 1)$. Suppose $f$ is continuous. Then there exists a function $g$ from $T$ into $E^n_T|\text{ClosedHypercube}(0_{E^n_T}, n \mapsto 1)$ such that

(i) $g$ is continuous, and

(ii) $g|A = f$.

The theorem is a consequence of (8), (1), and (21).

(23) Suppose $T$ is normal. Let us consider a subset $X$ of $E^n_T$. Suppose $X$ is compact, non boundary, and convex. Let us consider a function $f$ from $T|A$ into $E^n_T|X$. Suppose $f$ is continuous. Then there exists a function $g$ from $T$ into $E^n_T|X$ such that

(i) $g$ is continuous, and

(ii) $g|A = f$.

The theorem is a consequence of (22).

Now we state the proposition:

(24) \textbf{The First Implication of Tietze Extension Theorem for n-dimensional Spaces}: Suppose $T$ is normal. Let us consider a subset $X$ of $E^n_T$. Suppose

(i) $X$ is compact, non boundary, and convex, and

(ii) $B$ and $X$ are homeomorphic.

Let us consider a function $f$ from $T|A$ into $S|B$. Suppose $f$ is continuous. Then there exists a function $g$ from $T$ into $S|B$ such that

(iii) $g$ is continuous, and

(iv) $g|A = f$.

The theorem is a consequence of (23).

Now we state the proposition:

(25) \textbf{The Second Implication of Tietze Extension Theorem for n-dimensional Spaces}: Let us consider a non empty topological space $T$ and $n$. Suppose

(i) $n \geq 1$, and

(ii) for every topological space $S$ and for every non empty closed subset $A$ of $T$ and for every subset $B$ of $S$ such that there exists a subset $X$ of $E^n_T$ such that $X$ is compact, non boundary, and convex and $B$ and
$X$ are homeomorphic for every function $f$ from $T|A$ into $S|B$ such that $f$ is continuous there exists a function $g$ from $T$ into $S|B$ such that $g$ is continuous and $g|A = f$.

Then $T$ is normal. **Proof:** Set $C_1 = [-1, 1]_T$. For every non-empty closed subset $A$ of $T$ and for every continuous function $f$ from $T|A$ into $C_1$, there exists a continuous function $g$ from $T$ into $[-1, 1]_T$ such that $g|A = f$ by [19 (18), (17)], [11 (2)], [33 (26)]. \[\square\]

**References**

Tietze extension theorem for $n$-dimensional spaces


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