Differential Equations on Functions from $\mathbb{R}$ into Real Banach Space

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Summary. In this article, we describe the differential equations on functions from $\mathbb{R}$ into real Banach space. The descriptions are based on the article [20]. As preliminary to the proof of these theorems, we proved some properties of differentiable functions on real normed space. For the proof we referred to descriptions and theorems in the article [21] and the article [32]. And applying the theorems of Riemann integral introduced in the article [22], we proved the ordinary differential equations on real Banach space. We referred to the methods of proof in [30].

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The notation and terminology used in this paper have been introduced in the following articles: [29], [5], [11], [3], [6], [7], [19], [13], [34], [31], [33], [1], [15], [25], [32], [18], [24], [23], [26], [27], [20], [2], [8], [14], [16], [28], [12], [37], [38], [9], [35], [36], [17], and [10].

1. Some Properties of Differentiable Functions on Real Normed Space

From now on $Y$ denotes a real normed space.

Now we state the propositions:

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Let us consider a real normed space $Y$, a function $J$ from $\langle E^1, \| \cdot \| \rangle$ into $\mathbb{R}$, a point $x_0$ of $\langle E^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle E^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $J = \text{proj}(1, 1)$, and

(ii) $x_0 \in \text{dom} f$, and

(iii) $y_0 \in \text{dom} g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f = g \cdot J$.

Then $f$ is continuous in $x_0$ if and only if $g$ is continuous in $y_0$. Proof: If $f$ is continuous in $x_0$, then $g$ is continuous in $y_0$ by [14, (2)], [6, (39)], [37, (36)]. □

Let us consider a real normed space $Y$, a function $I$ from $\mathbb{R}$ into $\langle E^1, \| \cdot \| \rangle$, a point $x_0$ of $\langle E^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle E^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$, and

(ii) $x_0 \in \text{dom} f$, and

(iii) $y_0 \in \text{dom} g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f \cdot I = g$.

Then $f$ is continuous in $x_0$ if and only if $g$ is continuous in $y_0$. Proof: If $f$ is continuous in $x_0$, then $g$ is continuous in $y_0$ by [14, (1)], [21, (33)], [26, (15)]. □

Let us consider a function $I$ from $\mathbb{R}$ into $\langle E^1, \| \cdot \| \rangle$. Suppose $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$. Then

(i) for every rest $R$ of $\langle E^1, \| \cdot \| \rangle$, $Y, R \cdot I$ is a rest of $Y$, and

(ii) for every linear operator $L$ from $\langle E^1, \| \cdot \| \rangle$ into $Y$, $L \cdot I$ is a linear of $Y$.

Proof: For every rest $R$ of $\langle E^1, \| \cdot \| \rangle$, $Y, R \cdot I$ is a rest of $Y$ by [15, (23)], [5, (47)], [14, (3)]. Reconsider $L_0 = L$ as a function from $\mathbb{R}^1$ into $Y$. Reconsider $L_1 = L_0 \cdot I$ as a partial function from $\mathbb{R}$ to $Y$. Reconsider $r = L_1(jj)$ as a point of $Y$. For every real number $p$, $L_{1p} = p \cdot r$ by [6, (13)], [14, (3)], [6, (12)]. □

Let us consider a function $J$ from $\langle E^1, \| \cdot \| \rangle$ into $\mathbb{R}$. Suppose $J = \text{proj}(1, 1)$. Then

(i) for every rest $R$ of $Y$, $R \cdot J$ is a rest of $\langle E^1, \| \cdot \| \rangle$, $Y$, and

(ii) for every linear $L$ of $Y$, $L \cdot J$ is a Lipschitzian linear operator from $\langle E^1, \| \cdot \| \rangle$ into $Y$. 


Proof: For every rest $R$ of $Y$, $R \cdot J$ is a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, $Y$ by \cite{14} (4), \cite{15} (6), \cite{13} (47). Consider $r$ being a point of $Y$ such that for every real number $p$, $L_p = p \cdot r$. □

(5) Let us consider a function $I$ from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point $x_0$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $I = (\text{proj}(1,1) \text{ qua function})^{-1}$, and

(ii) $x_0 \in \text{dom } f$, and

(iii) $y_0 \in \text{dom } g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f \cdot I = g$, and

(vi) $f$ is differentiable in $x_0$.

Then

(vii) $g$ is differentiable in $y_0$, and

(viii) $g'(y_0) = f'(x_0)(\langle 1 \rangle)$, and

(ix) for every element $r$ of $\mathbb{R}$, $f'(x_0)(\langle r \rangle) = r \cdot g'(y_0)$.

The theorem is a consequence of (3). Proof: Consider $N_1$ being a neighbourhood of $x_0$ such that $N_1 \subseteq \text{dom } f$ and there exists a point $L$ of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $Y$ and there exists a rest $R$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, $Y$ such that for every point $x$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $x \in N_1$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x - x_0}$. Consider $e$ being a real number such that $0 < e$ and $\{ z, \text{ where } z \text{ is a point of } \langle \mathcal{E}^1, \| \cdot \| \rangle : \| z - x_0 \| < e \} \subseteq N_1$. Consider $L$ being a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $Y, R$ being a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle, Y$ such that for every point $x_3$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $x_3 \in N_1$ holds $f_{x_3} - f_{x_0} = L(x_3 - x_0) + R_{x_3 - x_0}$. Reconsider $R_0 = R \cdot I$ as a rest of $Y$. Reconsider $L_0 = L \cdot I$ as a linear of $Y$. Set $N = \{ z, \text{ where } z \text{ is a point of } \langle \mathcal{E}^1, \| \cdot \| \rangle : \| z - x_0 \| < e \}$. $N \subseteq \text{the carrier of } \langle \mathcal{E}^1, \| \cdot \| \rangle$. Set $N_0 = \{ z, \text{ where } z \text{ is an element of } \mathbb{R} : \| z - y_0 \| < e \}$. $y_0 - e, y_0 + e \subseteq N_0$ by \cite{28} (1). $N_0 \subseteq \| y_0 - e, y_0 + e \|$ by \cite{28} (1). For every real number $y_1$ such that $y_1 \in N_0$ holds $(f \cdot I)_{y_1} - (f \cdot I)_{y_0} = L_{y_1 - y_0} + R_{y_1 - y_0}$ by \cite{26} (12), \cite{17} (35), \cite{14} (3). □

(6) Let us consider a function $I$ from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point $x_0$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a real number $y_0$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $I = (\text{proj}(1,1) \text{ qua function})^{-1}$, and

(ii) $x_0 \in \text{dom } f$, and

(iii) $y_0 \in \text{dom } g$, and
(iv) \( x_0 = \langle y_0 \rangle \), and
(v) \( f \cdot I = g \).

Then \( f \) is differentiable in \( x_0 \) if and only if \( g \) is differentiable in \( y_0 \). The theorem is a consequence of (5) and (4). **Proof:** Reconsider \( J = \text{proj}(1, 1) \) as a function from \( \langle \mathcal{E}, \| \cdot \| \rangle \) into \( \mathbb{R} \). Consider \( N_0 \) being a neighbourhood of \( y_0 \) such that \( N_0 \subseteq \text{dom}(f \cdot I) \) and there exists a linear \( L \) of \( Y \) and there exists a rest \( R \) of \( Y \) such that for every real number \( y \) such that \( y \in N_0 \) holds \( (f \cdot I)y - (f \cdot I)y_0 = L_{y-y_0} + R_{y-y_0} \). Consider \( e_0 \) being a real number such that \( 0 < e_0 \) and \( N_0 = \{ y_0 - e_0, y_0 + e_0 \} \). Reconsider \( e = e_0 \) as an element of \( \mathbb{R} \). Set \( N = \{ z, \text{ where } z \text{ is a point of } \langle \mathcal{E}, \| \cdot \| \rangle : \| z - x_0 \| < e \} \). Consider \( L \) being a linear of \( Y \), \( R \) being a rest of \( Y \) such that for every real number \( y_1 \) such that \( y_1 \in N_0 \) holds \( (f \cdot I)y_1 - (f \cdot I)y_0 = L_{y_1-y_0} + R_{y_1-y_0} \). Reconsider \( R_0 = R \cdot J \) as a rest of \( \langle \mathcal{E}, \| \cdot \| \rangle \), \( Y \). Reconsider \( L_0 = L \cdot J \) as a Lipschitzian linear operator from \( \langle \mathcal{E}, \| \cdot \| \rangle \) into \( Y \). \( N \subseteq \) the carrier of \( \langle \mathcal{E}, \| \cdot \| \rangle \). For every point \( y \) of \( \langle \mathcal{E}, \| \cdot \| \rangle \) such that \( y \in N \) holds \( f_y - f_{x_0} = L_0(y-y_0) + R_{0y-x_0} \) by \([6] \ (13)), \ [7] \ (35)), \ [14] \ (4)) \). \( \square \)

(7) Let us consider a function \( J \) from \( \langle \mathcal{E}, \| \cdot \| \rangle \) into \( \mathbb{R} \), a point \( x_0 \) of \( \langle \mathcal{E}, \| \cdot \| \rangle \), an element \( y_0 \) of \( \mathbb{R} \), a partial function \( g \) from \( \mathbb{R} \) to \( Y \), and a partial function \( f \) from \( \langle \mathcal{E}, \| \cdot \| \rangle \) to \( Y \). Suppose
(i) \( J = \text{proj}(1, 1) \), and
(ii) \( x_0 \in \text{dom} f \), and
(iii) \( y_0 \in \text{dom} g \), and
(iv) \( x_0 = \langle y_0 \rangle \), and
(v) \( f = g \cdot J \).

Then \( f \) is differentiable in \( x_0 \) if and only if \( g \) is differentiable in \( y_0 \). The theorem is a consequence of (6).

(8) Let us consider a function \( I \) from \( \mathbb{R} \) into \( \langle \mathcal{E}, \| \cdot \| \rangle \), a point \( x_0 \) of \( \langle \mathcal{E}, \| \cdot \| \rangle \), an element \( y_0 \) of \( \mathbb{R} \), a partial function \( g \) from \( \mathbb{R} \) to \( Y \), and a partial function \( f \) from \( \langle \mathcal{E}, \| \cdot \| \rangle \) to \( Y \). Suppose
(i) \( I = \text{proj}(1, 1) \text{ qua function}^{-1} \), and
(ii) \( x_0 \in \text{dom} f \), and
(iii) \( y_0 \in \text{dom} g \), and
(iv) \( x_0 = \langle y_0 \rangle \), and
(v) \( f \cdot I = g \), and
(vi) \( f \) is differentiable in \( x_0 \).

Then \( \| g'(y_0) \| = \| f'(x_0) \| \). The theorem is a consequence of (5). **Proof:** Reconsider \( d_1 = f'(x_0) \) as a Lipschitzian linear operator from \( \langle \mathcal{E}, \| \cdot \| \rangle \) into \( Y \). Set \( A = \text{PreNorms}(d_1) \). For every real number \( r \) such that \( r \in A \) holds \( r \leq \| g'(y_0) \| \) by \([14] \ (1), \ (4)) \). \( \square \)
Let us consider real numbers $a, b, z$ and points $p, q, x$ of $\langle E^1, \| \cdot \| \rangle$. Now we state the propositions:

(9) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then
   (i) if $z \in [a, b]$, then $x \in ]p, q[$, and
   (ii) if $x \in ]p, q[$, then $a \neq b$ and if $a < b$, then $z \in ]a, b[$ and if $a > b$, then $z \in ]b, a[.$

(10) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then
   (i) if $z \in [a, b]$, then $x \in ]p, q[$, and
   (ii) if $x \in ]p, q[$, then if $a \leq b$, then $z \in [a, b]$ and if $a > b$, then $z \in [b, a]$. 

Now we state the propositions:

(11) Let us consider real numbers $a, b$, points $p, q$ of $\langle E^1, \| \cdot \| \rangle$, and a function $I$ from $\mathbb{R}$ into $\langle E^1, \| \cdot \| \rangle$. Suppose
   (i) $p = \langle a \rangle$, and
   (ii) $q = \langle b \rangle$, and
   (iii) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$.

Then
   (iv) if $a \leq b$, then $I^0[a, b] = ]p, q[,$ and
   (v) if $a < b$, then $I^0[a, b[ = ]p, q[.$

The theorem is a consequence of (10) and (9).

(12) Let us consider a real normed space $Y$, a partial function $g$ from $\mathbb{R}$ to the carrier of $Y$, and real numbers $a, b, M$. Suppose
   (i) $a \leq b$, and
   (ii) $[a, b] \subseteq \text{dom } g$, and
   (iii) for every real number $x$ such that $x \in [a, b]$ holds $g$ is continuous in $x$, and
   (iv) for every real number $x$ such that $x \in ]a, b[$ holds $g$ is differentiable in $x$, and
   (v) for every real number $x$ such that $x \in ]a, b[$ holds $\|g'(x)\| \leq M$.

Then $\|g_b - g_a\| \leq M \cdot |b - a|$. The theorem is a consequence of (11), (10), (1), (9), (7), and (8).
2. Differential Equations

In the sequel $X$, $Y$ denote real Banach spaces, $Z$ denotes an open subset of $\mathbb{R}$, $a$, $b$, $c$, $d$, $e$, $r$, $x_0$ denote real numbers, $y_0$ denotes a vector of $X$, and $G$ denotes a function from $X$ into $X$.

Now we state the propositions:

(13) Let us consider a real Banach space $X$, a partial function $F$ from $\mathbb{R}$ to the carrier of $X$, and a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $[a, b] \subseteq \text{dom } f$, and
(ii) $]a, b[ \subseteq \text{dom } F$, and

(iii) for every real number $x$ such that $x \in ]a, b[ \$ holds $F_x = \int_a^x f(x)dx$, and

(iv) $x_0 \in ]a, b[$, and
(v) $f$ is continuous in $x_0$.

Then

(ivi) $F$ is differentiable in $x_0$, and
(vii) $F'(x_0) = f_{x_0}$.

(14) Let us consider a partial function $F$ from $\mathbb{R}$ to the carrier of $X$ and a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $\text{dom } f = [a, b]$, and

(ii) $\text{dom } F = [a, b]$, and

(iii) for every real number $t$ such that $t \in [a, b]$ holds $F_t = \int_a^t f(x)dx$.

Let us consider a real number $x$. If $x \in [a, b]$, then $F$ is continuous in $x$.

(15) Let us consider a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$. If $a \in \text{dom } f$, then $\int_a^a f(x)dx = 0_X$.

Let us consider a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$ and a partial function $g$ from $\mathbb{R}$ to the carrier of $X$. Now we state the propositions:

(16) Suppose $a \leq b$ and $\text{dom } f = [a, b]$ and for every real number $t$ such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t f(x)dx$. Then $g_a = y_0$. 
(17) Suppose \( \text{dom } f = [a, b] \) and \( \text{dom } g = [a, b] \) and \( Z = ]a, b[ \) and for every real number \( t \) such that \( t \in [a, b] \) holds \( g_t = y_0 + \int_a^t f(x)dx \). Then

(i) \( g \) is continuous and differentiable on \( Z \), and

(ii) for every real number \( t \) such that \( t \in Z \) holds \( g'(t) = f_t \).

Let us consider a partial function \( f \) from \( \mathbb{R} \) to the carrier of \( X \). Now we state the propositions:

(18) Suppose \( a \leq b \) and \( [a, b] \subseteq \text{dom } f \) and for every real number \( x \) such that \( x \in [a, b] \) holds \( f \) is continuous in \( x \) and \( f \) is differentiable on \( ]a, b[ \) and for every real number \( x \) such that \( x \in ]a, b[ \) holds \( f'(x) = 0_X \). Then \( f_b = f_a \).

(19) Suppose \( [a, b] \subseteq \text{dom } f \) and for every real number \( x \) such that \( x \in [a, b] \) holds \( f \) is continuous in \( x \) and \( f \) is differentiable on \( ]a, b[ \) and for every real number \( x \) such that \( x \in ]a, b[ \) holds \( f'(x) = 0_X \). Then \( f|]a, b[ \) is constant.

Now we state the propositions:

(20) Let us consider a continuous partial function \( f \) from \( \mathbb{R} \) to the carrier of \( X \). Suppose

(i) \( [a, b] = \text{dom } f \), and

(ii) \( f|]a, b[ \) is constant.

Let us consider a real number \( x \). If \( x \in [a, b] \), then \( f_x = f_a \).

(21) Let us consider continuous partial functions \( y, G_1 \) from \( \mathbb{R} \) to the carrier of \( X \) and a partial function \( g \) from \( \mathbb{R} \) to the carrier of \( X \). Suppose

(i) \( a \leq b \), and

(ii) \( Z = ]a, b[ \), and

(iii) \( \text{dom } y = [a, b] \), and

(iv) \( \text{dom } g = [a, b] \), and

(v) \( \text{dom } G_1 = [a, b] \), and

(vi) \( y \) is differentiable on \( Z \), and

(vii) \( y_a = y_0 \), and

(viii) for every real number \( t \) such that \( t \in Z \) holds \( y'(t) = G_1(t) \), and

(ix) for every real number \( t \) such that \( t \in [a, b] \) holds \( g_t = y_0 + \int_a^t G_1(x)dx \).

Then \( y = g \). The theorem is a consequence of (17), (16), (19), and (20).

**Proof:** Recomconsider \( h = y - g \) as a continuous partial function from \( \mathbb{R} \) to the carrier of \( X \). For every real number \( x \) such that \( x \in \text{dom } h \) holds \( h_x = 0_X \) by [35] (15)]. For every element \( x \) of \( \mathbb{R} \) such that \( x \in \text{dom } y \) holds \( y(x) = g(x) \) by [35] (21)]. □
Let $X$ be a real Banach space, $y_0$ be a vector of $X$, $G$ be a function from $X$ into $X$, and $a, b$ be real numbers. Assume $a \leq b$ and $G$ is continuous on $\text{dom} G$. The functor $\text{Fredholm}(G, a, b, y_0)$ yielding a function from the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$ into the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$ is defined by

(Def. 1) Let us consider a vector $x$ of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$. Then there exist continuous partial functions $f, g, G_1$ from $\mathbb{R}$ to the carrier of $X$ such that

(i) $x = f$, and
(ii) $it(x) = g$, and
(iii) $\text{dom} f = [a, b]$, and
(iv) $\text{dom} g = [a, b]$, and
(v) $G_1 = G \cdot f$, and
(vi) for every real number $t$ such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t G_1(x)dx$.

Now we state the propositions:

(22) Suppose $a \leq b$ and $0 < r$ and for every vectors $y_1, y_2$ of $X$, $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$. Let us consider vectors $u, v$ of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$ and continuous partial functions $g, h$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $g = (\text{Fredholm}(G, a, b, y_0))(u)$, and
(ii) $h = (\text{Fredholm}(G, a, b, y_0))(v)$.

Let us consider a real number $t$. Suppose $t \in [a, b]$. Then $\|g_t - h_t\| \leq (r \cdot (t - a)) \cdot \|u - v\|$. PROOF: Set $F = \text{Fredholm}(G, a, b, y_0)$. Consider $f_1, g_1, G_3$ being continuous partial functions from $\mathbb{R}$ to the carrier of $X$ such that $u = f_1$ and $F(u) = g_1$ and $\text{dom} f_1 = [a, b]$ and $\text{dom} g_1 = [a, b]$ and $G_3 = G \cdot f_1$ and for every real number $t$ such that $t \in [a, b]$ holds $g_{1t} = y_0 + \int_a^t G_3(x)dx$. Consider $f_2, g_2, G_5$ being continuous partial functions from $\mathbb{R}$ to the carrier of $X$ such that $v = f_2$ and $F(v) = g_2$ and $\text{dom} f_2 = [a, b]$ and $\text{dom} g_2 = [a, b]$ and $G_5 = G \cdot f_2$ and for every real number $t$ such that $t \in [a, b]$ holds $g_{2t} = y_0 + \int_a^t G_5(x)dx$. Set $G_4 = G_3 - G_5$.

For every real number $x$ such that $x \in [a, t]$ holds $\|G_{4x}\| \leq r \cdot \|u - v\|$ by (20) (26), [5] (12)]. □

(23) Suppose $a \leq b$ and $0 < r$ and for every vectors $y_1, y_2$ of $X$, $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$. Let us consider vectors $u, v$ of the $\mathbb{R}$-norm space of
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continuous functions of \([a, b]\) and \(X\), an element \(m\) of \(\mathbb{N}\), and continuous partial functions \(g, h\) from \(\mathbb{R}\) to the carrier of \(X\). Suppose

(i) \(g = (\text{Fredholm}(G, a, b, y_0))^{m+1}(u)\), and

(ii) \(h = (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)\).

Let us consider a real number \(t\). Suppose \(t \in [a, b]\). Then \(\|g_t - h_t\| \leq \frac{(r \cdot |t-a|)^{m+1}}{(m+1)!} \cdot \|u - v\|\). The theorem is a consequence of (22). Proof: Set \(F = \text{Fredholm}(G, a, b, y_0)\). Define \(P[\text{natural number}] \equiv \) for every continuous partial functions \(g, h\) from \(\mathbb{R}\) to the carrier of \(X\) such that \(g = F^{m+1}(u_1)\) and \(h = F^{m+1}(v_1)\) for every real number \(t\) such that \(t \in [a, b]\) holds \(\|g_t - h_t\| \leq \frac{(r \cdot |t-a|)^{m+1}}{(m+1)!} \cdot \|u_1 - v_1\|\). \(P[0]\) by \([4, (70)]\), \([18, (5), (13)]\). For every natural number \(k\) such that \(P[k]\) holds \(P[k+1]\) by \([4, (71)]\), \([6, (13)]\), \([37, (27)]\). For every natural number \(k\), \(P[k]\) from \([1, \text{Sch. 2}]\). \(\square\)

(24) Let us consider a natural number \(m\). Suppose

(i) \(a \leq b\), and

(ii) \(0 < r\), and

(iii) for every vectors \(y_1, y_2\) of \(X\), \(\|Gy_1 - Gy_2\| \leq r \cdot \|y_1 - y_2\|\).

Let us consider vectors \(u, v\) of the \(\mathbb{R}\)-norm space of continuous functions of \([a, b]\) and \(X\).

Then \(\|(\text{Fredholm}(G, a, b, y_0))^{m+1}(u) - (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)\| \leq \frac{(r \cdot |b-a|)^{m+1}}{(m+1)!} \cdot \|u - v\|\). The theorem is a consequence of (23).

(25) If \(a < b\) and \(G\) is Lipschitzian on the carrier of \(X\), then there exists a natural number \(m\) such that \((\text{Fredholm}(G, a, b, y_0))^{m+1}\) is contraction. The theorem is a consequence of (24).

(26) If \(a < b\) and \(G\) is Lipschitzian on the carrier of \(X\), then Fredholm\((G, a, b, y_0)\) has unique fixpoint. The theorem is a consequence of (25).

(27) Let us consider continuous partial functions \(f, g\) from \(\mathbb{R}\) to the carrier of \(X\). Suppose

(i) \(\text{dom } f = [a, b]\), and

(ii) \(\text{dom } g = [a, b]\), and

(iii) \(Z = ]a, b[\), and

(iv) \(a < b\), and

(v) \(G\) is Lipschitzian on the carrier of \(X\), and

(vi) \(g = (\text{Fredholm}(G, a, b, y_0))(f)\).

Then

(vii) \(g_a = y_0\), and

(viii) \(g\) is differentiable on \(Z\), and

\[\ldots\]
(ix) for every real number $t$ such that $t \in Z$ holds $g'(t) = (G \cdot f)_t$.

The theorem is a consequence of (17) and (16).

(28) Let us consider a continuous partial function $y$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $a < b$, and
(ii) $Z = ]a, b[$, and
(iii) $G$ is Lipschitzian on the carrier of $X$, and
(iv) $\text{dom } y = [a, b]$, and
(v) $y$ is differentiable on $Z$, and
(vi) $y_a = y_0$, and
(vii) for every real number $t$ such that $t \in Z$ holds $y'(t) = G(y_t)$.

Then $y$ is a fixpoint of $\text{Fredholm}(G, a, b, y, y_0)$. The theorem is a consequence of (21).

Proof: Consider $f$, $g$, $G_1$ being continuous partial functions from $\mathbb{R}$ to the carrier of $X$ such that $y = f$ and $(\text{Fredholm}(G, a, b, y_0))(y) = g$ and $\text{dom } f = [a, b]$ and $\text{dom } g = [a, b]$ and $G_1 = G \cdot f$ and for every real number $t$ such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t G_1(x)dx$. For every real number $t$ such that $t \in Z$ holds $y'(t) = G_1(t)$ by [6, (13)]. □

(29) Let us consider continuous partial functions $y_1$, $y_2$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $a < b$, and
(ii) $Z = ]a, b[$, and
(iii) $G$ is Lipschitzian on the carrier of $X$, and
(iv) $\text{dom } y_1 = [a, b]$, and
(v) $y_1$ is differentiable on $Z$, and
(vi) $y_{1a} = y_0$, and
(vii) for every real number $t$ such that $t \in Z$ holds $y'_1(t) = G(y_{1t})$, and
(viii) $\text{dom } y_2 = [a, b]$, and
(ix) $y_2$ is differentiable on $Z$, and
(x) $y_{2a} = y_0$, and
(xi) for every real number $t$ such that $t \in Z$ holds $y'_2(t) = G(y_{2t})$.

Then $y_1 = y_2$. The theorem is a consequence of (26) and (28).

(30) Suppose $a < b$ and $Z = ]a, b[$ and $G$ is Lipschitzian on the carrier of $X$. Then there exists a continuous partial function $y$ from $\mathbb{R}$ to the carrier of $X$ such that
(i) \( \text{dom } y = [a, b] \), and 
(ii) \( y \) is differentiable on \( Z \), and 
(iii) \( y_a = y_0 \), and 
(iv) for every real number \( t \) such that \( t \in Z \) holds \( y'(t) = G(y_t) \).

The theorem is a consequence of (26) and (27).

**References**


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