FORMALIZED MATHEMATICS
Vol. 21, No. 4, Pages 249–260, 2013
DOI: 10.2478/forma-2013-0027

DE GRUYTER OPEN
degruyter.com/view/j/forma

Isometric Differentiable Functions on Real Normed Space[1]

Yuichi Futa
Japan Advanced Institute of Science and Technology
Ishikawa, Japan

Noboru Endou
Gifu National College of Technology
Gifu, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we formalize isometric differentiable functions on real normed space [17], and their properties.

MSC: 58C20 46G05 03B35

Keywords: isometric differentiable function

MML identifier: NDIFF_7, version: 8.1.02 5.22.1194

The notation and terminology used in this paper have been introduced in the following articles: [3], [2], [8], [4], [5], [18], [10], [11], [19], [14], [16], [1], [6], [9], [15], [23], [24], [21], [22], [13], [25], and [7].

1. Preliminaries

From now on S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, Z denotes a subset of S, and i, n denote natural numbers.

Now we state the propositions:

(1) Let us consider a set X and functions I, f. Then $(f \restriction X) \cdot I = (f \cdot I) \restriction I^{-1}(X)$.

(2) Let us consider real normed spaces S, T, a linear operator L from S into T, and points x, y of S. Then $L(x) - L(y) = L(x - y)$.

[1]This work was supported by JSPS KAKENHI 23500029 and 22300285.

ISSN 1426–2630(Print), 1898-9934(Online)

(3) Let us consider real normed spaces X, Y, W, a function I from X into Y, and partial functions f_1, f_2 from Y to W. Then

(i) $(f_1 + f_2) \cdot I = f_1 \cdot I + f_2 \cdot I$, and

(ii) $(f_1 - f_2) \cdot I = f_1 \cdot I - f_2 \cdot I$.

Proof: Set D_1 = the carrier of X. For every element s of D_1, $s \in \mathrm{dom}((f_1+f_2)\cdot I)$ iff $s \in \mathrm{dom}(f_1\cdot I+f_2\cdot I)$ by [4, (11)]. For every element z of D_1 such that $z \in \mathrm{dom}((f_1+f_2)\cdot I)$ holds $((f_1+f_2)\cdot I)(z) = (f_1\cdot I+f_2\cdot I)(z)$ by [4, (11), (12)]. For every element s of D_1, $s \in \mathrm{dom}((f_1 - f_2) \cdot I)$ iff $s \in \mathrm{dom}(f_1 \cdot I - f_2 \cdot I)$ by [4, (11)]. For every element z of D_1 such that $z \in \mathrm{dom}((f_1 - f_2) \cdot I)$ holds $((f_1 - f_2) \cdot I)(z) = (f_1 \cdot I - f_2 \cdot I)(z)$ by [4, (11), (12)]. □

(4) Let us consider real normed spaces X, Y, W, a function I from X into Y, a partial function f from Y to W, and a real number r. Then $r \cdot (f \cdot I) = (r \cdot f) \cdot I$. Proof: Set D_1 = the carrier of X. For every element s of D_1, $s \in \mathrm{dom}((r \cdot f) \cdot I)$ iff $s \in \mathrm{dom}(f \cdot I)$ by [4, (11)]. For every element s of D_1, $s \in \mathrm{dom}((r \cdot f) \cdot I)$ iff $I(s) \in \mathrm{dom}(r \cdot f)$ by [4, (11)]. For every element z of D_1 such that $z \in \mathrm{dom}(r \cdot (f \cdot I))$ holds $(r \cdot (f \cdot I))(z) = ((r \cdot f) \cdot I)(z)$ by [4, (12)]. □

(5) Let us consider a partial function f from T to W, a function g from S into T, and a point x of S. Suppose

(i) $x \in \mathrm{dom}\, g$, and

(ii) $g_x \in \mathrm{dom}\, f$, and

(iii) g is continuous in x, and

(iv) f is continuous in g_x.

Then $f \cdot g$ is continuous in x. Proof: Set $h = f \cdot g$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in \mathrm{dom}\, h$ and $\|x_1 - x\| < s$ holds $\|h_{x_1} - h_x\| < r$ by [14, (7)], [12, (3), (4)]. □

Let X, Y be real normed spaces and x be an element of $X \times Y$. The functor reproj1(x) yielding a function from X into $X \times Y$ is defined by

(Def. 1) Let us consider an element r of X. Then $it(r) = \langle r, \mathbf{x_2} \rangle$.

The functor reproj2(x) yielding a function from Y into $X \times Y$ is defined by

(Def. 2) Let us consider an element r of Y. Then $it(r) = \langle \mathbf{x_1}, r \rangle$.

2. Isometries

Now we state the propositions:

(6) Let us consider a linear operator I from S into T and a point x of S. If I is isometric, then I is continuous in x.

(7) Let us consider real normed spaces S, T and a linear operator f from S into T. Then f is isometric if and only if for every element x of S, $\|f(x)\| = \|x\|$. The theorem is a consequence of (2).

(8) Let us consider a linear operator I from S into T and a subset Z of S. If I is isometric, then I is continuous on Z. The theorem is a consequence of (6).

(9) Let us consider a linear operator I from S into T. Suppose I is one-to-one, onto, and isometric. Then there exists a linear operator J from T into S such that

(i) $J = I^{-1}$, and

(ii) J is one-to-one, onto, and isometric.

The theorem is a consequence of (7). PROOF: Reconsider $J = I^{-1}$ as a function from T into S. For every points v, w of T, $J(v+w) = J(v)+J(w)$ by [5, (113)], [4, (34)]. For every point v of T and for every real number r, $J(r \cdot v) = r \cdot J(v)$ by [5, (113)], [4, (34)]. For every point v of T, $\|J(v)\| = \|v\|$ by [5, (113)], [4, (34)]. □

Let us consider a linear operator I from S into T and a sequence s_1 of S. Now we state the propositions:

(10) If I is isometric and s_1 is convergent, then $I \cdot s_1$ is convergent and $\lim(I \cdot s_1) = I(\lim s_1)$.

(11) If I is one-to-one, onto, and isometric, then s_1 is convergent iff $I \cdot s_1$ is convergent.

Let us consider a linear operator I from S into T and a subset Z of S. Now we state the propositions:

(12) If I is one-to-one, onto, and isometric, then Z is closed iff $I^\circ Z$ is closed.

(13) If I is one-to-one, onto, and isometric, then Z is open iff $I^\circ Z$ is open.

(14) If I is one-to-one, onto, and isometric, then Z is compact iff $I^\circ Z$ is compact.

Now we state the propositions:

(15) Let us consider a partial function f from T to W and a linear operator I from S into T. Suppose I is one-to-one, onto, and isometric. Let us consider a point x of S. Suppose $I(x) \in \operatorname{dom} f$. Then $f \cdot I$ is continuous in x if and only if f is continuous in $I(x)$. The theorem is a consequence of (9), (6), and (5).

(16) Let us consider a partial function f from T to W, a linear operator I from S into T, and a set X. Suppose

(i) $X \subseteq$ the carrier of T, and

(ii) I is one-to-one, onto, and isometric.

Then f is continuous on X if and only if $f \cdot I$ is continuous on $I^{-1}(X)$. The theorem is a consequence of (15) and (1). PROOF: For every point y of T such that $y \in X$ holds $f \restriction X$ is continuous in y by [5, (113)], [23, (57)]. □

Let X, Y be real normed spaces. The functor IsoCPNrSP(X, Y) yielding a linear operator from $X \times Y$ into $\prod\langle X, Y\rangle$ is defined by

(Def. 3) Let us consider a point x of X and a point y of Y. Then $it(x, y) = \langle x,\ y\rangle$.

Now we state the proposition:

(17) Let us consider real normed spaces X, Y. Then $0_{\prod\langle X,Y\rangle} =$ (IsoCPNrSP$(X, Y))(0_{X \times Y})$.

Let X, Y be real normed spaces. Observe that IsoCPNrSP(X, Y) is one-to-one onto and isometric.

Let us note that there exists a linear operator from $X \times Y$ into $\prod\langle X, Y\rangle$ which is one-to-one, onto, and isometric.

Let f be a one-to-one onto isometric linear operator from $X \times Y$ into $\prod\langle X, Y\rangle$. Let us note that the functor f^{-1} yields a linear operator from $\prod\langle X, Y\rangle$ into $X \times Y$. One can verify that f^{-1} is one-to-one onto and isometric as a linear operator from $\prod\langle X, Y\rangle$ into $X \times Y$.

Observe that there exists a linear operator from $\prod\langle X, Y\rangle$ into $X \times Y$ which is one-to-one, onto, and isometric.

Now we state the propositions:

(18) Let us consider real normed spaces X, Y, a point x of X, and a point y of Y. Then (IsoCPNrSP$(X, Y))^{-1}(\langle x, y\rangle) = \langle\!\langle x,\ y\rangle\!\rangle$. PROOF: Set $I =$ IsoCPNrSP(X, Y). Set $J = I^{-1}$. For every point x of X and for every point y of Y, $J(\langle x, y\rangle) = \langle\!\langle x,\ y\rangle\!\rangle$ by [4, (34)]. □

(19) Let us consider real normed spaces X, Y. Then (IsoCPNrSP$(X, Y))^{-1}(0_{\prod\langle X,Y\rangle}) = 0_{X \times Y}$. The theorem is a consequence of (17).

(20) Let us consider real normed spaces X, Y and a subset Z of $X \times Y$. Then IsoCPNrSP(X, Y) is continuous on Z.

(21) Let us consider real normed spaces X, Y and a subset Z of $\prod\langle X, Y\rangle$. Then (IsoCPNrSP$(X, Y))^{-1}$ is continuous on Z.

(22) Let us consider real normed spaces S, T, W, a point f of the real norm space of bounded linear operators from S into W, a point g of the real norm space of bounded linear operators from T into W, and a linear operator I from S into T. Suppose
   (i) I is one-to-one, onto, and isometric, and
   (ii) $f = g \cdot I$.

   Then $\|f\| = \|g\|$. The theorem is a consequence of (9) and (7). PROOF: Consider J being a linear operator from T into S such that $J = I^{-1}$ and

J is one-to-one, onto, and isometric. Reconsider $g_0 = g$ as a Lipschitzian linear operator from T into W. Reconsider $g_3 = g \cdot I$ as a Lipschitzian linear operator from S into W. For every element x, $x \in \{\|g_0(t)\|$, where t is a vector of $T : \|t\| \leqslant 1\}$ iff $x \in \{\|g_3(w)\|$, where w is a vector of $S : \|w\| \leqslant 1\}$ by [4, (13), (35)]. □

Let us consider S and T. One can verify that every linear operator from S into T which is isometric is also Lipschitzian.

3. Isometric Differentiable Functions on Real Normed Space

Let us consider a real norm space sequence G, a real normed space F, a set i, partial functions f, g from $\prod G$ to F, and a subset X of $\prod G$. Now we state the propositions:

(23) Suppose X is open and $i \in \operatorname{dom} G$ and f is partially differentiable on X w.r.t. i and g is partially differentiable on X w.r.t. i. Then

(i) $f + g$ is partially differentiable on X w.r.t. i, and

(ii) $(f + g){\restriction^i}X = (f{\restriction^i}X) + (g{\restriction^i}X)$.

(24) Suppose X is open and $i \in \operatorname{dom} G$ and f is partially differentiable on X w.r.t. i and g is partially differentiable on X w.r.t. i. Then

(i) $f - g$ is partially differentiable on X w.r.t. i, and

(ii) $(f - g){\restriction^i}X = (f{\restriction^i}X) - (g{\restriction^i}X)$.

Now we state the propositions:

(25) Let us consider a real norm space sequence G, a real normed space F, a set i, a partial function f from $\prod G$ to F, a real number r, and a subset X of $\prod G$. Suppose

(i) X is open, and

(ii) $i \in \operatorname{dom} G$, and

(iii) f is partially differentiable on X w.r.t. i.

Then

(iv) $r \cdot f$ is partially differentiable on X w.r.t. i, and

(v) $r \cdot f{\restriction^i}X = r \cdot (f{\restriction^i}X)$.

Proof: Set $h = r \cdot f$. For every point x of $\prod G$ such that $x \in X$ holds h is partially differentiable in x w.r.t. i and $\operatorname{partdiff}(h, x, i) = r \cdot \operatorname{partdiff}(f, x, i)$ by [19, (24), (30)]. Set $f_3 = f{\restriction^i}X$. For every point x of $\prod G$ such that $x \in X$ holds $(r \cdot f_3)_x = \operatorname{partdiff}(h, x, i)$. □

(26) Let us consider real normed spaces S, T, a Lipschitzian linear operator L from S into T, and a point x_0 of S. Then

(i) L is differentiable in x_0, and

(ii) $L'(x_0) = L$.

The theorem is a consequence of (2). PROOF: Reconsider $L_0 = L$ as a point of the real norm space of bounded linear operators from S into T. Reconsider $R =$ (the carrier of S) $\longmapsto 0_T$ as a partial function from S to T. Set $N =$ the neighbourhood of x_0. For every point x of S such that $x \in N$ holds $L_{0x} - L_{0x_0} = L(x - x_0) + R_{x-x_0}$ by [20, (7)], [21, (4)]. □

(27) Let us consider a partial function f from T to W, a Lipschitzian linear operator I from S into T, and a point I_0 of the real norm space of bounded linear operators from S into T. Suppose $I_0 = I$. Let us consider a point x of S. Suppose f is differentiable in $I(x)$. Then

(i) $f \cdot I$ is differentiable in x, and

(ii) $(f \cdot I)'(x) = f'(I(x)) \cdot I_0$.

The theorem is a consequence of (26).

(28) Let us consider a partial function f from T to W and a linear operator I from S into T. Suppose

(i) I is one-to-one and onto, and

(ii) I is isometric.

Let us consider a point x of S. Then $f \cdot I$ is differentiable in x if and only if f is differentiable in $I(x)$. The theorem is a consequence of (9), (26), and (27).

(29) Let us consider a partial function f from T to W, a linear operator I from S into T, and a set X. Suppose

(i) $X \subseteq$ the carrier of T, and

(ii) I is one-to-one and onto, and

(iii) I is isometric.

Then f is differentiable on X if and only if $f \cdot I$ is differentiable on $I^{-1}(X)$. The theorem is a consequence of (28) and (1). PROOF: For every point y of T such that $y \in X$ holds $f \restriction X$ is differentiable in y by [5, (113)]. □

(30) Let us consider real normed spaces X, Y, a partial function f from $\prod\langle X, Y\rangle$ to W, and a subset D of $\prod\langle X, Y\rangle$. Suppose f is differentiable on D. Let us consider a point z of $\prod\langle X, Y\rangle$. Suppose $z \in \operatorname{dom} f'_{\restriction D}$. Then $f'_{\restriction D}(z) = ((f \cdot \mathrm{IsoCPNrSP}(X, Y))'_{\restriction (\mathrm{IsoCPNrSP}(X,Y))^{-1}(D)})_{(\mathrm{IsoCPNrSP}(X,Y))^{-1}(z)} \cdot (\mathrm{IsoCPNrSP}(X, Y))^{-1}$. The theorem is a consequence of (17), (29), and (27). PROOF: Set $I = \mathrm{IsoCPNrSP}(X, Y)$. Set $J = (\mathrm{IsoCPNrSP}(X, Y))^{-1}$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point z of $\prod\langle X, Y\rangle$ such that $z \in \operatorname{dom} f'_{\restriction D}$ holds $f'_{\restriction D}(z) = (g'_{\restriction E})_{J(z)} \cdot I^{-1}$ by [10, (31)], [5, (113)], [23, (36)]. □

(31) Let us consider real normed spaces X, Y, a partial function f from $X \times Y$ to W, and a subset D of $X \times Y$. Suppose f is differentiable on D. Let us consider a point z of $X \times Y$. Suppose $z \in \operatorname{dom} f'_{\restriction D}$. Then $f'_{\restriction D}(z) = ((f \cdot (\text{IsoCPNrSP}(X,Y))^{-1})'_{\restriction((\text{IsoCPNrSP}(X,Y))^{-1})^{-1}(D)})_{(\text{IsoCPNrSP}(X,Y))(z)} \cdot ((\text{IsoCPNrSP}(X,Y))^{-1})^{-1}$. The theorem is a consequence of (18), (19), (17), (29), and (27). Proof: Set $I = (\text{IsoCPNrSP}(X,Y))^{-1}$. Set $J = \text{IsoCPNrSP}(X,Y)$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point z of $X \times Y$ such that $z \in \operatorname{dom} f'_{\restriction D}$ holds $f'_{\restriction D}(z) = (g'_{\restriction E})_{J(z)} \cdot I^{-1}$ by [10, (31)], [5, (113)], [23, (36)]. □

(32) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then

(i) $\text{reproj1}(z) = (\text{IsoCPNrSP}(X,Y))^{-1} \cdot \text{reproj}(1(\in \operatorname{dom}\langle X,Y\rangle), (\text{IsoCPNrSP}(X,Y))(z))$, and

(ii) $\text{reproj2}(z) = (\text{IsoCPNrSP}(X,Y))^{-1} \cdot \text{reproj}(2(\in \operatorname{dom}\langle X,Y\rangle), (\text{IsoCPNrSP}(X,Y))(z))$.

The theorem is a consequence of (18).

Let X, Y be real normed spaces and z be a point of $X \times Y$. Let us note that the functor $z_{\mathbf{1}}$ yields a point of X. One can verify that the functor $z_{\mathbf{2}}$ yields a point of Y. Let X, Y, W be real normed spaces. Let f be a partial function from $X \times Y$ to W. We say that f is partially differentiable in z w.r.t. 1 if and only if

(Def. 4) $f \cdot \text{reproj1}(z)$ is differentiable in $z_{\mathbf{1}}$.

We say that f is partially differentiable in z w.r.t. 2 if and only if

(Def. 5) $f \cdot \text{reproj2}(z)$ is differentiable in $z_{\mathbf{2}}$.

Now we state the propositions:

(33) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then

(i) $z_{\mathbf{1}} =$ the projection onto $1(\in \operatorname{dom}\langle X,Y\rangle)((\text{IsoCPNrSP}(X,Y))(z))$, and

(ii) $z_{\mathbf{2}} =$ the projection onto $2(\in \operatorname{dom}\langle X,Y\rangle)((\text{IsoCPNrSP}(X,Y))(z))$.

(34) Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, and a partial function f from $X \times Y$ to W. Then

(i) f is partially differentiable in z w.r.t. 1 iff $f \cdot (\text{IsoCPNrSP}(X,Y))^{-1}$ is partially differentiable in $(\text{IsoCPNrSP}(X,Y))(z)$ w.r.t. 1, and

(ii) f is partially differentiable in z w.r.t. 2 iff $f \cdot (\text{IsoCPNrSP}(X,Y))^{-1}$ is partially differentiable in $(\text{IsoCPNrSP}(X,Y))(z)$ w.r.t. 2.

The theorem is a consequence of (32) and (33).

Let X, Y, W be real normed spaces, z be a point of $X \times Y$, and f be a partial function from $X \times Y$ to W. The functor $\text{partdiff}(f,z)$ w.r.t. 1 yielding a point of the real norm space of bounded linear operators from X into W is defined by the term

(Def. 6) $(f \cdot \mathrm{reproj1}(z))'(z_{\mathbf{1}})$.

The functor partdiff(f, z) w.r.t. 2 yielding a point of the real norm space of bounded linear operators from Y into W is defined by the term

(Def. 7) $(f \cdot \mathrm{reproj2}(z))'(z_{\mathbf{2}})$.

Now we state the proposition:

(35) Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, and a partial function f from $X \times Y$ to W. Then

(i) partdiff(f, z) w.r.t. $1 =$ partdiff$(f \cdot (\mathrm{IsoCPNrSP}(X, Y))^{-1}, (\mathrm{IsoCPNrSP}(X, Y))(z), 1)$, and

(ii) partdiff(f, z) w.r.t. $2 =$ partdiff$(f \cdot (\mathrm{IsoCPNrSP}(X, Y))^{-1}, (\mathrm{IsoCPNrSP}(X, Y))(z), 2)$.

The theorem is a consequence of (32) and (33).

Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, and partial functions f_1, f_2 from $X \times Y$ to W. Now we state the propositions:

(36) Suppose f_1 is partially differentiable in z w.r.t. 1 and f_2 is partially differentiable in z w.r.t. 1. Then

(i) $f_1 + f_2$ is partially differentiable in z w.r.t. 1, and

(ii) partdiff$(f_1 + f_2, z)$ w.r.t. $1 =$ partdiff(f_1, z) w.r.t. $1 +$ partdiff(f_2, z) w.r.t. 1, and

(iii) $f_1 - f_2$ is partially differentiable in z w.r.t. 1, and

(iv) partdiff$(f_1 - f_2, z)$ w.r.t. $1 =$ partdiff(f_1, z) w.r.t. $1 -$ partdiff(f_2, z) w.r.t. 1.

(37) Suppose f_1 is partially differentiable in z w.r.t. 2 and f_2 is partially differentiable in z w.r.t. 2. Then

(i) $f_1 + f_2$ is partially differentiable in z w.r.t. 2, and

(ii) partdiff$(f_1 + f_2, z)$ w.r.t. $2 =$ partdiff(f_1, z) w.r.t. $2 +$ partdiff(f_2, z) w.r.t. 2, and

(iii) $f_1 - f_2$ is partially differentiable in z w.r.t. 2, and

(iv) partdiff$(f_1 - f_2, z)$ w.r.t. $2 =$ partdiff(f_1, z) w.r.t. $2 -$ partdiff(f_2, z) w.r.t. 2.

Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, a real number r, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(38) Suppose f is partially differentiable in z w.r.t. 1. Then

(i) $r \cdot f$ is partially differentiable in z w.r.t. 1, and

(ii) partdiff$(r \cdot f, z)$ w.r.t. $1 = r \cdot$ partdiff(f, z) w.r.t. 1.

(39) Suppose f is partially differentiable in z w.r.t. 2. Then

(i) $r \cdot f$ is partially differentiable in z w.r.t. 2, and

(ii) $\mathrm{partdiff}(r \cdot f, z)$ w.r.t. $2 = r \cdot \mathrm{partdiff}(f, z)$ w.r.t. 2.

Let X, Y, W be real normed spaces, Z be a set, and f be a partial function from $X \times Y$ to W. We say that f is partially differentiable on Z w.r.t. 1 if and only if

(Def. 8) (i) $Z \subseteq \mathrm{dom}\, f$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $f{\restriction}Z$ is partially differentiable in z w.r.t. 1.

We say that f is partially differentiable on Z w.r.t. 2 if and only if

(Def. 9) (i) $Z \subseteq \mathrm{dom}\, f$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $f{\restriction}Z$ is partially differentiable in z w.r.t. 2.

Now we state the proposition:

(40) Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W. Then

(i) f is partially differentiable on Z w.r.t. 1 iff $f \cdot (\mathrm{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable on $((\mathrm{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$ w.r.t. 1, and

(ii) f is partially differentiable on Z w.r.t. 2 iff $f \cdot (\mathrm{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable on $((\mathrm{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$ w.r.t. 2.

The theorem is a consequence of (18), (19), (17), (34), and (1). Proof: Set $I = (\mathrm{IsoCPNrSP}(X, Y))^{-1}$. Set $g = f \cdot I$. Set $E = I^{-1}(Z)$. f is partially differentiable on Z w.r.t. 1 iff g is partially differentiable on E w.r.t. 1 by [5, (113)], [4, (34)], [5, (38)]. f is partially differentiable on Z w.r.t. 2 iff g is partially differentiable on E w.r.t. 2 by [5, (113)], [4, (34)], [5, (38)]. □

Let X, Y, W be real normed spaces, Z be a set, and f be a partial function from $X \times Y$ to W. Assume f is partially differentiable on Z w.r.t. 1. The functor $f \restriction^1 Z$ yielding a partial function from $X \times Y$ to the real norm space of bounded linear operators from X into W is defined by

(Def. 10) (i) $\mathrm{dom}\, it = Z$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $it_z = \mathrm{partdiff}(f, z)$ w.r.t. 1.

Assume f is partially differentiable on Z w.r.t. 2. The functor $f \restriction^2 Z$ yielding a partial function from $X \times Y$ to the real norm space of bounded linear operators from Y into W is defined by

(Def. 11) (i) $\mathrm{dom}\, it = Z$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $it_z = \mathrm{partdiff}(f, z)$ w.r.t. 2.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(41) Suppose f is partially differentiable on Z w.r.t. 1. Then $f \restriction^1 Z = (f \cdot (\mathrm{IsoCPNrSP}(X,Y))^{-1} \restriction^1 ((\mathrm{IsoCPNrSP}(X,Y))^{-1})^{-1}(Z)) \cdot \mathrm{IsoCPNrSP}(X,Y)$.

(42) Suppose f is partially differentiable on Z w.r.t. 2. Then $f \restriction^2 Z = (f \cdot (\mathrm{IsoCPNrSP}(X,Y))^{-1} \restriction^2 ((\mathrm{IsoCPNrSP}(X,Y))^{-1})^{-1}(Z)) \cdot \mathrm{IsoCPNrSP}(X,Y)$.

(43) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 1 if and only if $Z \subseteq \mathrm{dom}\, f$ and for every point x of $X \times Y$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. 1.

(44) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 2 if and only if $Z \subseteq \mathrm{dom}\, f$ and for every point x of $X \times Y$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. 2.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and partial functions f, g from $X \times Y$ to W. Now we state the propositions:

(45) Suppose Z is open and f is partially differentiable on Z w.r.t. 1 and g is partially differentiable on Z w.r.t. 1. Then

(i) $f + g$ is partially differentiable on Z w.r.t. 1, and

(ii) $(f + g) \restriction^1 Z = (f \restriction^1 Z) + (g \restriction^1 Z)$.

(46) Suppose Z is open and f is partially differentiable on Z w.r.t. 1 and g is partially differentiable on Z w.r.t. 1. Then

(i) $f - g$ is partially differentiable on Z w.r.t. 1, and

(ii) $(f - g) \restriction^1 Z = (f \restriction^1 Z) - (g \restriction^1 Z)$.

(47) Suppose Z is open and f is partially differentiable on Z w.r.t. 2 and g is partially differentiable on Z w.r.t. 2. Then

(i) $f + g$ is partially differentiable on Z w.r.t. 2, and

(ii) $(f + g) \restriction^2 Z = (f \restriction^2 Z) + (g \restriction^2 Z)$.

(48) Suppose Z is open and f is partially differentiable on Z w.r.t. 2 and g is partially differentiable on Z w.r.t. 2. Then

(i) $f - g$ is partially differentiable on Z w.r.t. 2, and

(ii) $(f - g) \restriction^2 Z = (f \restriction^2 Z) - (g \restriction^2 Z)$.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, a real number r, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(49) Suppose Z is open and f is partially differentiable on Z w.r.t. 1. Then

(i) $r \cdot f$ is partially differentiable on Z w.r.t. 1, and

(ii) $r \cdot f \restriction^1 Z = r \cdot (f \restriction^1 Z)$.

(50) Suppose Z is open and f is partially differentiable on Z w.r.t. 2. Then

(i) $r \cdot f$ is partially differentiable on Z w.r.t. 2, and

(ii) $r \cdot f \restriction^2 Z = r \cdot (f \restriction^2 Z)$.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(51) Suppose f is differentiable on Z. Then $f'_{\restriction Z}$ is continuous on Z if and only if $(f \cdot (\mathrm{IsoCPNrSP}(X,Y))^{-1})'_{\restriction((\mathrm{IsoCPNrSP}(X,Y))^{-1})^{-1}(Z)}$ is continuous on $((\mathrm{IsoCPNrSP}(X,Y))^{-1})^{-1}(Z)$.

(52) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 1 and f is partially differentiable on Z w.r.t. 2 and $f \restriction^1 Z$ is continuous on Z and $f \restriction^2 Z$ is continuous on Z if and only if f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[9] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. The product space of real normed spaces and its properties. *Formalized Mathematics*, 15(**3**):81–85, 2007. doi:10.2478/v10037-007-0010-y.

[10] Hiroshi Imura, Morishige Kimura, and Yasunari Shidama. The differentiable functions on normed linear spaces. *Formalized Mathematics*, 12(**3**):321–327, 2004.

[11] Hiroshi Imura, Yuji Sakai, and Yasunari Shidama. Differentiable functions on normed linear spaces. Part II. *Formalized Mathematics*, 12(**3**):371–374, 2004.

[12] Jarosław Kotowicz. Partial functions from a domain to a domain. *Formalized Mathematics*, 1(**4**):697–702, 1990.

[13] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[14] Takaya Nishiyama, Keiji Ohkubo, and Yasunari Shidama. The continuous functions on normed linear spaces. *Formalized Mathematics*, 12(**3**):269–275, 2004.

[15] Hiroyuki Okazaki, Noboru Endou, and Yasunari Shidama. Cartesian products of family of real linear spaces. *Formalized Mathematics*, 19(**1**):51–59, 2011. doi:10.2478/v10037-011-0009-2.

[16] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.

[17] Laurent Schwartz. *Cours d'analyse*. Hermann, 1981.

[18] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(**1**):39–48, 2004.

[19] Yasunari Shidama. Differentiable functions on normed linear spaces. *Formalized Mathematics*, 20(**1**):31–40, 2012. doi:10.2478/v10037-012-0005-1.

[20] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1 (**2**):329–334, 1990.

[21] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.
[24] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[25] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(**2**):171–175, 1992.

Received December 31, 2013