FORMALIZED MATHEMATICS
Vol. 21, No. 2, Pages 95–102, 2013
DOI: 10.2478/forma-2013-0011

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degruyter.com/view/j/forma

Differentiation in Normed Spaces[1]

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Summary. In this article we formalized the Fréchet differentiation. It is defined as a generalization of the differentiation of a real-valued function of a single real variable to more general functions whose domain and range are subsets of normed spaces [14].

MSC: 58C20 46G05 03B35

Keywords: formalization of Fréchet derivative; Fréchet differentiability

MML identifier: `NDIFF_6`, version: `8.1.02 5.17.1179`

The notation and terminology used in this paper have been introduced in the following articles: [5], [1], [4], [10], [6], [7], [16], [15], [11], [12], [13], [3], [8], [19], [20], [17], [18], [21], and [9].

Let us consider non empty sets D, E, F. Now we state the propositions:

(1) There exists a function I from $(F^E)^D$ into $F^{D\times E}$ such that

(i) I is bijective, and

(ii) for every function f from D into F^E and for every elements d, e such that $d \in D$ and $e \in E$ holds $I(f)(d,e) = f(d)(e)$.

(2) There exists a function I from $(F^E)^D$ into $F^{E\times D}$ such that

(i) I is bijective, and

(ii) for every function f from D into F^E and for every elements e, d such that $e \in E$ and $d \in D$ holds $I(f)(e,d) = f(d)(e)$.

Now we state the propositions:

(3) Let us consider non-empty non empty finite sequences D, E and a non empty set F. Then there exists a function L from $(F^{\prod E})^{\prod D}$ into $F^{\prod (E^\frown D)}$ such that

(i) L is bijective, and

[1] This work was supported by JSPS KAKENHI 23500029 and 22300285.

ISSN 1426–2630(Print), 1898-9934(Online)

(ii) for every function f from $\prod D$ into $F^{\prod E}$ and for every finite sequences e, d such that $e \in \prod E$ and $d \in \prod D$ holds $L(f)(e \frown d) = f(d)(e)$.

The theorem is a consequence of (2). PROOF: Consider I being a function from $(F^{\prod E})^{\prod D}$ into $F^{\prod E \times \prod D}$ such that I is bijective and for every function f from $\prod D$ into $F^{\prod E}$ and for every elements e, d such that $e \in \prod E$ and $d \in \prod D$ holds $I(f)(e,d) = f(d)(e)$. Consider J being a function from $\prod E \times \prod D$ into $\prod(E \frown D)$ such that J is one-to-one and onto and for every finite sequences x, y such that $x \in \prod E$ and $y \in \prod D$ holds $J(x,y) = x^\frown y$. Reconsider $K = J^{-1}$ as a function from $\prod(E^\frown D)$ into $\prod E \times \prod D$. Define $\mathcal{G}(\text{element}) = I(\$_1) \cdot K$. For every element x such that $x \in (F^{\prod E})^{\prod D}$ holds $\mathcal{G}(x) \in F^{\prod(E^\frown D)}$ by [7, (5), (8), (128)]. Consider L being a function from $(F^{\prod E})^{\prod D}$ into $F^{\prod(E^\frown D)}$ such that for every element e such that $e \in (F^{\prod E})^{\prod D}$ holds $L(e) = \mathcal{G}(e)$ from [7, Sch. 2]. For every function f from $\prod D$ into $F^{\prod E}$ and for every finite sequences e, d such that $e \in \prod E$ and $d \in \prod D$ holds $L(f)(e \frown d) = f(d)(e)$ by [9, (87)], [7, (26), (8), (5)]. □

(4) Let us consider non empty sets X, Y. Then there exists a function I from $X \times Y$ into $X \times \prod\langle Y\rangle$ such that

(i) I is bijective, and

(ii) for every elements x, y such that $x \in X$ and $y \in Y$ holds $I(x,y) = \langle\!\langle x, \langle y\rangle\rangle\!\rangle$.

PROOF: Consider J being a function from Y into $\prod\langle Y\rangle$ such that J is one-to-one and onto and for every element y such that $y \in Y$ holds $J(y) = \langle y\rangle$. Define $\mathcal{P}[\text{element}, \text{element}, \text{element}] \equiv \$_3 = \langle\!\langle \$_1, \langle \$_2\rangle\rangle\!\rangle$. For every elements x, y such that $x \in X$ and $y \in Y$ there exists an element z such that $z \in X \times \prod\langle Y\rangle$ and $\mathcal{P}[x,y,z]$ by [7, (5)], [9, (87)]. Consider I being a function from $X \times Y$ into $X \times \prod\langle Y\rangle$ such that for every elements x, y such that $x \in X$ and $y \in Y$ holds $\mathcal{P}[x,y,I(x,y)]$ from [5, Sch. 1]. □

(5) Let us consider a non-empty non empty finite sequence X and a non empty set Y. Then there exists a function K from $\prod X \times Y$ into $\prod(X ^\frown \langle Y\rangle)$ such that

(i) K is bijective, and

(ii) for every finite sequence x and for every element y such that $x \in \prod X$ and $y \in Y$ holds $K(x,y) = x \frown \langle y\rangle$.

The theorem is a consequence of (4). PROOF: Consider I being a function from $\prod X \times Y$ into $\prod X \times \prod\langle Y\rangle$ such that I is bijective and for every element x and for every element y such that $x \in \prod X$ and $y \in Y$ holds $I(x,y) = \langle\!\langle x, \langle y\rangle\rangle\!\rangle$. Consider J being a function from $\prod X \times \prod\langle Y\rangle$ into $\prod(X \frown \langle Y\rangle)$ such that J is one-to-one and onto and for every finite sequences x, y such that $x \in \prod X$ and $y \in \prod\langle Y\rangle$ holds $J(x,y) = x \frown y$. Set

$K = J \cdot I$. For every finite sequence x and for every element y such that $x \in \prod X$ and $y \in Y$ holds $K(x, y) = x \frown \langle y \rangle$ by [9, (87)], [7, (5), (15)]. □

(6) Let us consider a non empty set D, a non-empty non empty finite sequence E, and a non empty set F. Then there exists a function L from $(F^{\prod E})^D$ into $F^{\prod (E \frown \langle D \rangle)}$ such that

(i) L is bijective, and

(ii) for every function f from D into $F^{\prod E}$ and for every finite sequence e and for every element d such that $e \in \prod E$ and $d \in D$ holds $L(f)(e \frown \langle d \rangle) = f(d)(e)$.

The theorem is a consequence of (2) and (5). PROOF: Consider I being a function from $(F^{\prod E})^D$ into $F^{\prod E \times D}$ such that I is bijective and for every function f from D into $F^{\prod E}$ and for every elements e, d such that $e \in \prod E$ and $d \in D$ holds $I(f)(e, d) = f(d)(e)$. Consider J being a function from $\prod E \times D$ into $\prod (E \frown \langle D \rangle)$ such that J is bijective and for every finite sequence x and for every element y such that $x \in \prod E$ and $y \in D$ holds $J(x, y) = x \frown \langle y \rangle$. Reconsider $K = J^{-1}$ as a function from $\prod (E \frown \langle D \rangle)$ into $\prod E \times D$. Define $\mathcal{G}(\text{element}) = I(\$_1) \cdot K$. For every element x such that $x \in (F^{\prod E})^D$ holds $\mathcal{G}(x) \in F^{\prod (E \frown \langle D \rangle)}$ by [7, (5), (8), (128)]. Consider L being a function from $(F^{\prod E})^D$ into $F^{\prod (E \frown \langle D \rangle)}$ such that for every element e such that $e \in (F^{\prod E})^D$ holds $L(e) = \mathcal{G}(e)$ from [7, Sch. 2]. For every function f from D into $F^{\prod E}$ and for every finite sequence e and for every element d such that $e \in \prod E$ and $d \in D$ holds $L(f)(e \frown \langle d \rangle) = f(d)(e)$ by [7, (5), (26), (8)]. □

In this paper S, T denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, Z denotes a subset of S, and i, n denote natural numbers.

Let S be a set. Assume S is a real normed space. The functor $\mathrm{NormSp}_{\mathbb{R}}(S)$ yielding a real normed space is defined by the term

(Def. 1) S.

Let S, T be real normed spaces. The functor $\mathrm{diff}_{\mathrm{SP}}(S, T)$ yielding a function is defined by

(Def. 2) (i) $\mathrm{dom}\, it = \mathbb{N}$, and

(ii) $it(0) = T$, and

(iii) for every natural number i, $it(i{+}1) =$ the real norm space of bounded linear operators from S into $\mathrm{NormSp}_{\mathbb{R}}(it(i))$.

Now we state the proposition:

(7) (i) $(\mathrm{diff}_{\mathrm{SP}}(S, T))(0) = T$, and

(ii) $(\mathrm{diff}_{\mathrm{SP}}(S, T))(1) =$ the real norm space of bounded linear operators from S into T, and

(iii) $(\mathrm{diff}_{\mathrm{SP}}(S,T))(2) =$ the real norm space of bounded linear operators from S into the real norm space of bounded linear operators from S into T.

Let us consider a natural number i. Now we state the propositions:

(8) $(\mathrm{diff}_{\mathrm{SP}}(S,T))(i)$ is a real normed space.

(9) There exists a real normed space H such that

(i) $H = (\mathrm{diff}_{\mathrm{SP}}(S,T))(i)$, and

(ii) $(\mathrm{diff}_{\mathrm{SP}}(S,T))(i+1) =$ the real norm space of bounded linear operators from S into H.

Let S, T be real normed spaces and i be a natural number. The functor $\mathrm{diff}_{\mathrm{SP}}(S^i,T)$ yielding a real normed space is defined by the term

(Def. 3) $(\mathrm{diff}_{\mathrm{SP}}(S,T))(i)$.

Now we state the proposition:

(10) Let us consider a natural number i. Then $\mathrm{diff}_{\mathrm{SP}}(S^{(i+1)},T) =$ the real norm space of bounded linear operators from S into $\mathrm{diff}_{\mathrm{SP}}(S^i,T)$. The theorem is a consequence of (9).

Let S, T be real normed spaces and f be a set. Assume f is a partial function from S to T. The functor $\mathrm{PartFuncs}(f,S,T)$ yielding a partial function from S to T is defined by the term

(Def. 4) f.

Let f be a partial function from S to T and Z be a subset of S. The functor $f'(Z)$ yielding a function is defined by

(Def. 5) (i) $\mathrm{dom}\, it = \mathbb{N}$, and

(ii) $it(0) = f{\restriction}Z$, and

(iii) for every natural number i, $it(i+1) = (\mathrm{PartFuncs}(it(i), S, \mathrm{diff}_{\mathrm{SP}}(S^i,T)))'_{\restriction Z}$.

Now we state the propositions:

(11) (i) $f'(Z)(0) = f{\restriction}Z$, and

(ii) $f'(Z)(1) = (f{\restriction}Z)'_{\restriction Z}$, and

(iii) $f'(Z)(2) = ((f{\restriction}Z)'_{\restriction Z})'_{\restriction Z}$.

The theorem is a consequence of (7).

(12) Let us consider a natural number i. Then $f'(Z)(i)$ is a partial function from S to $\mathrm{diff}_{\mathrm{SP}}(S^i,T)$. The theorem is a consequence of (7). Proof: Define $\mathcal{P}[\text{natural number}] \equiv f'(Z)(\$_1)$ is a partial function from S to $\mathrm{diff}_{\mathrm{SP}}(S^{\$_1},T)$. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

Let S, T be real normed spaces, f be a partial function from S to T, Z be a subset of S, and i be a natural number. The functor $\mathrm{diff}_Z(f,i)$ yielding a partial function from S to $\mathrm{diff}_{\mathrm{SP}}(S^i,T)$ is defined by the term

(Def. 6) $f'(Z)(i)$.

Now we state the proposition:

(13) $\operatorname{diff}_Z(f, i+1) = \operatorname{diff}_Z(f, i)'_{\restriction Z}$. The theorem is a consequence of (12) and (8).

Let S, T be real normed spaces, f be a partial function from S to T, Z be a subset of S, and n be a natural number. We say that f is differentiable n times on Z if and only if

(Def. 7) (i) $Z \subseteq \operatorname{dom} f$, and

(ii) for every natural number i such that $i \leqslant n-1$ holds $\operatorname{PartFuncs}(f'(Z)(i), S, \operatorname{diff}_{\mathrm{SP}}(S^i, T))$ is differentiable on Z.

Now we state the propositions:

(14) f is differentiable n times on Z if and only if $Z \subseteq \operatorname{dom} f$ and for every natural number i such that $i \leqslant n-1$ holds $\operatorname{diff}_Z(f, i)$ is differentiable on Z.

(15) f is differentiable 1 times on Z if and only if $Z \subseteq \operatorname{dom} f$ and $f{\restriction}Z$ is differentiable on Z. The theorem is a consequence of (14) and (7). PROOF: For every natural number i such that $i \leqslant 1-1$ holds $\operatorname{diff}_Z(f, i)$ is differentiable on Z. $\square$

(16) f is differentiable 2 times on Z if and only if $Z \subseteq \operatorname{dom} f$ and $f{\restriction}Z$ is differentiable on Z and $(f{\restriction}Z)'_{\restriction Z}$ is differentiable on Z. The theorem is a consequence of (14), (7), and (11). PROOF: For every natural number i such that $i \leqslant 2-1$ holds $\operatorname{diff}_Z(f, i)$ is differentiable on Z by [2, (14)]. $\square$

(17) Let us consider real normed spaces S, T, a partial function f from S to T, a subset Z of S, and a natural number n. Suppose f is differentiable n times on Z. Let us consider a natural number m. If $m \leqslant n$, then f is differentiable m times on Z.

(18) Let us consider a natural number n and a partial function f from S to T. If $1 \leqslant n$ and f is differentiable n times on Z, then Z is open. The theorem is a consequence of (17) and (15).

(19) Let us consider a natural number n and a partial function f from S to T. Suppose

(i) $1 \leqslant n$, and

(ii) f is differentiable n times on Z.

Let us consider a natural number i. Suppose $i \leqslant n$. Then

(iii) $(\operatorname{diff}_{\mathrm{SP}}(S, T))(i)$ is a real normed space, and

(iv) $f'(Z)(i)$ is a partial function from S to $\operatorname{diff}_{\mathrm{SP}}(S^i, T)$, and

(v) $\operatorname{dom} \operatorname{diff}_Z(f, i) = Z$.

The theorem is a consequence of (13) and (14).

(20) Let us consider a natural number n and partial functions f, g from S to T. Suppose

(i) $1 \leqslant n$, and

(ii) f is differentiable n times on Z, and

(iii) g is differentiable n times on Z.

Let us consider a natural number i. Suppose $i \leqslant n$. Then $\mathrm{diff}_Z(f+g,i) = \mathrm{diff}_Z(f,i) + \mathrm{diff}_Z(g,i)$. The theorem is a consequence of (18), (14), (19), (13), and (10). PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then $\mathrm{diff}_Z(f+g,\$_1) = \mathrm{diff}_Z(f,\$_1) + \mathrm{diff}_Z(g,\$_1)$. $\mathcal{P}[0]$ by [21, (27)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [2, (11)], [11, (39)], [8, (5)]. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. $\square$

(21) Let us consider a natural number n and partial functions f, g from S to T. Suppose

(i) $1 \leqslant n$, and

(ii) f is differentiable n times on Z, and

(iii) g is differentiable n times on Z.

Then $f+g$ is differentiable n times on Z. The theorem is a consequence of (18), (14), (19), and (20). PROOF: For every natural number i such that $i \leqslant n-1$ holds $\mathrm{diff}_Z(f+g,i)$ is differentiable on Z by [11, (39)]. $\square$

(22) Let us consider a natural number n and partial functions f, g from S to T. Suppose

(i) $1 \leqslant n$, and

(ii) f is differentiable n times on Z, and

(iii) g is differentiable n times on Z.

Let us consider a natural number i. Suppose $i \leqslant n$. Then $\mathrm{diff}_Z(f-g,i) = \mathrm{diff}_Z(f,i) - \mathrm{diff}_Z(g,i)$. The theorem is a consequence of (18), (14), (19), (13), and (10). PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then $\mathrm{diff}_Z(f-g,\$_1) = \mathrm{diff}_Z(f,\$_1) - \mathrm{diff}_Z(g,\$_1)$. $\mathcal{P}[0]$ by [21, (30)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [2, (11)], [11, (40)], [8, (5)]. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. $\square$

(23) Let us consider a natural number n and partial functions f, g from S to T. Suppose

(i) $1 \leqslant n$, and

(ii) f is differentiable n times on Z, and

(iii) g is differentiable n times on Z.

Then $f-g$ is differentiable n times on Z. The theorem is a consequence of (18), (14), (19), and (22). PROOF: For every natural number i such that $i \leqslant n-1$ holds $\mathrm{diff}_Z(f-g,i)$ is differentiable on Z by [11, (40)]. $\square$

(24) Let us consider a natural number n, a real number r, and a partial function f from S to T. Suppose

(i) $1 \leqslant n$, and

(ii) f is differentiable n times on Z.

Let us consider a natural number i. If $i \leqslant n$, then $\operatorname{diff}_Z(r \cdot f, i) = r \cdot \operatorname{diff}_Z(f, i)$. The theorem is a consequence of (18), (14), (19), (10), and (13). Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant n$, then $\operatorname{diff}_Z(r \cdot f, \$_1) = r \cdot \operatorname{diff}_Z(f, \$_1)$. $\mathcal{P}[0]$ by [21, (31)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [2, (11)], [11, (41)]. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. $\square$

(25) Let us consider a natural number n, a real number r, and a partial function f from S to T. Suppose

(i) $1 \leqslant n$, and

(ii) f is differentiable n times on Z.

Then $r \cdot f$ is differentiable n times on Z. The theorem is a consequence of (18), (14), (24), and (19). Proof: For every natural number i such that $i \leqslant n - 1$ holds $\operatorname{diff}_Z(r \cdot f, i)$ is differentiable on Z by [11, (41)]. $\square$

(26) Let us consider a natural number n and a partial function f from S to T. Suppose

(i) $1 \leqslant n$, and

(ii) f is differentiable n times on Z.

Let us consider a natural number i. Suppose $i \leqslant n$. Then $\operatorname{diff}_Z(-f, i) = -\operatorname{diff}_Z(f, i)$. The theorem is a consequence of (24).

(27) Let us consider a natural number n and a partial function f from S to T. Suppose

(i) $1 \leqslant n$, and

(ii) f is differentiable n times on Z.

Then $-f$ is differentiable n times on Z. The theorem is a consequence of (25).

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Received May 19, 2013