

Isomorphisms of Direct Products of Finite Commutative Groups¹

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Summary. We have been working on the formalization of groups. In [1], we encoded some theorems concerning the product of cyclic groups. In this article, we present the generalized formalization of [1]. First, we show that every finite commutative group which order is composite number is isomorphic to a direct product of finite commutative groups which orders are relatively prime. Next, we describe finite direct products of finite commutative groups.

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The notation and terminology used in this paper have been introduced in the following articles: [2], [3], [19], [7], [13], [20], [8], [9], [10], [23], [24], [25], [26], [27], [14], [22], [17], [4], [5], [15], [16], [6], [11], [21], [18], [29], [28], and [12].

1. PRELIMINARIES

Now we state the propositions:

- (1) Let us consider sets A , B , A_1 , B_1 . Suppose
- (i) A misses B , and
 - (ii) $A_1 \subseteq A$, and
 - (iii) $B_1 \subseteq B$, and
 - (iv) $A_1 \cup B_1 = A \cup B$.

Then

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(v) $A_1 = A$, and

(vi) $B_1 = B$.

PROOF: $A \subseteq A_1$. $B \subseteq B_1$. \square

(2) Let us consider non empty finite sets H, K . Then $\overline{\prod \langle H, K \rangle} = \overline{H} \cdot \overline{K}$.

Let us consider bags p_2, p_1, f of Prime and a natural number q . Now we state the propositions:

(3) If support p_2 misses support p_1 and $f = p_2 + p_1$ and $q \in \text{support } p_2$, then $p_2(q) = f(q)$.

(4) If support p_2 misses support p_1 and $f = p_2 + p_1$ and $q \in \text{support } p_1$, then $p_1(q) = f(q)$.

Now we state the propositions:

(5) Let us consider a non zero natural number h and a prime number q . If q and h are not relatively prime, then $q \mid h$.

(6) Let us consider non zero natural numbers h, s . Suppose a prime number q . Suppose $q \in \text{support PrimeFactorization}(s)$. Then q and h are not relatively prime. Then $\text{support PrimeFactorization}(s) \subseteq \text{support PrimeFactorization}(h)$. The theorem is a consequence of (5).

(7) Let us consider non zero natural numbers h, k, s, t . Suppose

(i) h and k are relatively prime, and

(ii) $s \cdot t = h \cdot k$, and

(iii) for every prime number q such that $q \in \text{support PrimeFactorization}(s)$ holds q and h are not relatively prime, and

(iv) for every prime number q such that $q \in \text{support PrimeFactorization}(t)$ holds q and k are not relatively prime.

Then

(v) $s = h$, and

(vi) $t = k$.

The theorem is a consequence of (6), (1), (3), and (4). PROOF: Set $p_2 = \text{PrimeFactorization}(s)$. Set $p_1 = \text{PrimeFactorization}(t)$. For every natural number p such that $p \in \text{support PFEExp}(h)$ holds $p_2(p) = p^{p\text{-count}(h)}$. For every natural number p such that $p \in \text{support PFEExp}(k)$ holds $p_1(p) = p^{p\text{-count}(k)}$. \square

Let G be a non empty multiplicative magma, I be a finite set, and b be a (the carrier of G)-valued total I -defined function. The functor $\prod b$ yielding an element of G is defined by

(Def. 1) There exists a finite sequence f of elements of G such that

(i) $it = \prod f$, and

$$(ii) f = b \cdot \text{CFS}(I).$$

Now we state the propositions:

- (8) Let us consider a commutative group G , non empty finite sets A, B , a (the carrier of G)-valued total A -defined function F_3 , a (the carrier of G)-valued total B -defined function F_2 , and a (the carrier of G)-valued total $A \cup B$ -defined function F_1 . Suppose

$$(i) A \text{ misses } B, \text{ and}$$

$$(ii) F_1 = F_3 + \cdot F_2.$$

$$\text{Then } \prod F_1 = \prod F_3 \cdot \prod F_2.$$

- (9) Let us consider a non empty multiplicative magma G , a set q , an element z of G , and a (the carrier of G)-valued total $\{q\}$ -defined function f . If $f = q \mapsto z$, then $\prod f = z$.

2. DIRECT PRODUCT OF FINITE COMMUTATIVE GROUPS

Now we state the propositions:

- (10) Let us consider non empty multiplicative magmas X, Y . Then the carrier of $\prod \langle X, Y \rangle = \prod \langle \text{the carrier of } X, \text{ the carrier of } Y \rangle$. PROOF: Set $\text{Carr}X = \text{the carrier of } X$. Set $\text{Carr}Y = \text{the carrier of } Y$. For every element a such that $a \in \text{dom the support of } \langle X, Y \rangle$ holds (the support of $\langle X, Y \rangle$)(a) = $\langle \text{the carrier of } X, \text{ the carrier of } Y \rangle$ (a). \square
- (11) Let us consider a group G and normal subgroups A, B of G . Suppose $(\text{the carrier of } A) \cap (\text{the carrier of } B) = \{\mathbf{1}_G\}$. Let us consider elements a, b of G . If $a \in A$ and $b \in B$, then $a \cdot b = b \cdot a$.
- (12) Let us consider a group G and normal subgroups A, B of G . Suppose
- (i) for every element x of G , there exist elements a, b of G such that $a \in A$ and $b \in B$ and $x = a \cdot b$, and
 - (ii) $(\text{the carrier of } A) \cap (\text{the carrier of } B) = \{\mathbf{1}_G\}$.

Then there exists a homomorphism h from $\prod \langle A, B \rangle$ to G such that

$$(iii) h \text{ is bijective, and}$$

$$(iv) \text{ for every elements } a, b \text{ of } G \text{ such that } a \in A \text{ and } b \in B \text{ holds } h(\langle a, b \rangle) = a \cdot b.$$

The theorem is a consequence of (11). PROOF: Define $\mathcal{P}[\text{set}, \text{set}] \equiv$ there exists an element x of G and there exists an element y of G such that $x \in A$ and $y \in B$ and $\$1 = \langle x, y \rangle$ and $\$2 = x \cdot y$. For every element z of $\prod \langle A, B \rangle$, there exists an element w of G such that $\mathcal{P}[z, w]$. Consider h being a function from $\prod \langle A, B \rangle$ into G such that for every element z of $\prod \langle A, B \rangle$, $\mathcal{P}[z, h(z)]$. For every elements a, b of G such that $a \in A$ and $b \in B$ holds

$h(\langle a, b \rangle) = a \cdot b$. For every elements z, w of $\prod \langle A, B \rangle$, $h(z \cdot w) = h(z) \cdot h(w)$.
□

Let us consider a finite commutative group G , a natural number m , and a subset A of G . Now we state the propositions:

- (13) Suppose $A = \{x \text{ where } x \text{ is an element of } G : x^m = \mathbf{1}_G\}$. Then
- (i) $A \neq \emptyset$, and
 - (ii) for every elements g_1, g_2 of G such that $g_1, g_2 \in A$ holds $g_1 \cdot g_2 \in A$, and
 - (iii) for every element g of G such that $g \in A$ holds $g^{-1} \in A$.
- (14) Suppose $A = \{x \text{ where } x \text{ is an element of } G : x^m = \mathbf{1}_G\}$. Then there exists a strict finite subgroup H of G such that
- (i) the carrier of $H = A$, and
 - (ii) H is commutative and normal.

Now we state the propositions:

- (15) Let us consider a finite commutative group G , a natural number m , and a finite subgroup H of G . Suppose the carrier of $H = \{x \text{ where } x \text{ is an element of } G : x^m = \mathbf{1}_G\}$. Let us consider a prime number q . Suppose $q \in \text{support PrimeFactorization}(\overline{H})$. Then q and m are not relatively prime.
- (16) Let us consider a finite commutative group G and natural numbers h, k . Suppose
- (i) $\overline{G} = h \cdot k$, and
 - (ii) h and k are relatively prime.

Then there exist strict finite subgroups H, K of G such that

- (iii) the carrier of $H = \{x \text{ where } x \text{ is an element of } G : x^h = \mathbf{1}_G\}$, and
- (iv) the carrier of $K = \{x \text{ where } x \text{ is an element of } G : x^k = \mathbf{1}_G\}$, and
- (v) H is normal, and
- (vi) K is normal, and
- (vii) for every element x of G , there exist elements a, b of G such that $a \in H$ and $b \in K$ and $x = a \cdot b$, and
- (viii) $(\text{the carrier of } H) \cap (\text{the carrier of } K) = \{\mathbf{1}_G\}$.

The theorem is a consequence of (14). PROOF: Set $A = \{x \text{ where } x \text{ is an element of } G : x^h = \mathbf{1}_G\}$. Set $B = \{x \text{ where } x \text{ is an element of } G : x^k = \mathbf{1}_G\}$. $A \subseteq \text{the carrier of } G$. $B \subseteq \text{the carrier of } G$. Consider H being a strict finite subgroup of G such that the carrier of $H = A$ and H is commutative and H is normal. Consider K being a strict finite subgroup of G such that the carrier of $K = B$ and K is commutative and K is

normal. Consider a, b being integers such that $a \cdot h + b \cdot k = 1$. (The carrier of H) \cap (the carrier of K) $\subseteq \{1_G\}$. For every element x of G , there exist elements s, t of G such that $s \in H$ and $t \in K$ and $x = s \cdot t$. \square

(17) Let us consider finite groups H, K . Then $\overline{\prod \langle H, K \rangle} = \overline{H} \cdot \overline{K}$. The theorem is a consequence of (10) and (2).

(18) Let us consider a finite commutative group G and non zero natural numbers h, k . Suppose

- (i) $\overline{G} = h \cdot k$, and
- (ii) h and k are relatively prime.

Then there exist strict finite subgroups H, K of G such that

- (iii) $\overline{H} = h$, and
- (iv) $\overline{K} = k$, and
- (v) (the carrier of H) \cap (the carrier of K) = $\{1_G\}$, and
- (vi) there exists a homomorphism F from $\prod \langle H, K \rangle$ to G such that F is bijective and for every elements a, b of G such that $a \in H$ and $b \in K$ holds $F(\langle a, b \rangle) = a \cdot b$.

The theorem is a consequence of (16), (12), (17), (15), and (7).

3. FINITE DIRECT PRODUCTS OF FINITE COMMUTATIVE GROUPS

Let us consider a group G , a set q , an associative group-like multiplicative magma family F of $\{q\}$, and a function f from G into $\prod F$. Now we state the propositions:

- (19) If $F = q \dashrightarrow G$ and for every element x of G , $f(x) = q \dashrightarrow x$, then f is a homomorphism from G to $\prod F$.
- (20) If $F = q \dashrightarrow G$ and for every element x of G , $f(x) = q \dashrightarrow x$, then f is bijective.

Now we state the propositions:

(21) Let us consider a set q , an associative group-like multiplicative magma family F of $\{q\}$, and a group G . Suppose $F = q \dashrightarrow G$. Then there exists a homomorphism I from G to $\prod F$ such that

- (i) I is bijective, and
- (ii) for every element x of G , $I(x) = q \dashrightarrow x$.

The theorem is a consequence of (19) and (20). PROOF: Define $\mathcal{P}[\text{set}, \text{set}] \equiv \mathcal{S}_2 = q \dashrightarrow \mathcal{S}_1$. For every element z of G , there exists an element w of $\prod F$ such that $\mathcal{P}[z, w]$. Consider I being a function from G into $\prod F$ such that for every element x of G , $\mathcal{P}[x, I(x)]$. \square

(22) Let us consider non empty finite sets I_0, I , an associative group-like multiplicative magma family F_0 of I_0 , an associative group-like multiplicative magma family F of I , groups H, K , an element q of I , an element k of K , and a function g . Suppose

- (i) $g \in$ the carrier of $\prod F_0$, and
- (ii) $q \notin I_0$, and
- (iii) $I = I_0 \cup \{q\}$, and
- (iv) $F = F_0 + \cdot (q \dashrightarrow K)$.

Then $g + \cdot (q \dashrightarrow k) \in$ the carrier of $\prod F$. PROOF: Set $HK = \langle H, K \rangle$. Set $w = g + \cdot (q \dashrightarrow k)$. For every element x such that $x \in$ dom the support of F holds $w(x) \in$ (the support of F)(x). \square

Let us consider non empty finite sets I_0, I , an associative group-like multiplicative magma family F_0 of I_0 , an associative group-like multiplicative magma family F of I , groups H, K , an element q of I , a function G_0 from H into $\prod F_0$, and a function G from $\prod \langle H, K \rangle$ into $\prod F$. Now we state the propositions:

- (23) Suppose G_0 is a homomorphism from H to $\prod F_0$ and G_0 is bijective and $q \notin I_0$ and $I = I_0 \cup \{q\}$ and $F = F_0 + \cdot (q \dashrightarrow K)$. Then suppose for every element h of H and for every element k of K , there exists a function g such that $g = G_0(h)$ and $G(\langle h, k \rangle) = g + \cdot (q \dashrightarrow k)$. Then G is a homomorphism from $\prod \langle H, K \rangle$ to $\prod F$.
- (24) Suppose G_0 is a homomorphism from H to $\prod F_0$ and G_0 is bijective and $q \notin I_0$ and $I = I_0 \cup \{q\}$ and $F = F_0 + \cdot (q \dashrightarrow K)$. Then suppose for every element h of H and for every element k of K , there exists a function g such that $g = G_0(h)$ and $G(\langle h, k \rangle) = g + \cdot (q \dashrightarrow k)$. Then G is bijective.

Now we state the propositions:

- (25) Let us consider a set q , a multiplicative magma family F of $\{q\}$, and a non empty multiplicative magma G . Suppose $F = q \dashrightarrow G$. Let us consider a (the carrier of G)-valued total $\{q\}$ -defined function y . Then
 - (i) $y \in$ the carrier of $\prod F$, and
 - (ii) $y(q) \in$ the carrier of G , and
 - (iii) $y = q \dashrightarrow y(q)$.
- (26) Let us consider a set q , an associative group-like multiplicative magma family F of $\{q\}$, and a group G . Suppose $F = q \dashrightarrow G$. Then there exists a homomorphism H_0 from $\prod F$ to G such that
 - (i) H_0 is bijective, and
 - (ii) for every (the carrier of G)-valued total $\{q\}$ -defined function x , $H_0(x) = \prod x$.

The theorem is a consequence of (21), (25), and (9). PROOF: Consider I being a homomorphism from G to $\prod F$ such that I is bijective and for every element x of G , $I(x) = q \cdot x$. Set $H_0 = I^{-1}$. For every (the carrier of G)-valued total $\{q\}$ -defined function y , $H_0(y) = \prod y$. \square

- (27) Let us consider non empty finite sets I_0, I , an associative group-like multiplicative magma family F_0 of I_0 , an associative group-like multiplicative magma family F of I , groups H, K , an element q of I , and a homomorphism G_0 from H to $\prod F_0$. Suppose

- (i) $q \notin I_0$, and
- (ii) $I = I_0 \cup \{q\}$, and
- (iii) $F = F_0 + \cdot (q \cdot K)$, and
- (iv) G_0 is bijective.

Then there exists a homomorphism G from $\prod \langle H, K \rangle$ to $\prod F$ such that

- (v) G is bijective, and
- (vi) for every element h of H and for every element k of K , there exists a function g such that $g = G_0(h)$ and $G(\langle h, k \rangle) = g + \cdot (q \cdot k)$.

The theorem is a consequence of (22), (23), and (24). PROOF: Set $HK = \langle H, K \rangle$. Define $\mathcal{P}[\text{set}, \text{set}] \equiv$ there exists an element h of H and there exists an element k of K and there exists a function g such that $\$1 = \langle h, k \rangle$ and $g = G_0(h)$ and $\$2 = g + \cdot (q \cdot k)$. For every element z of $\prod \langle H, K \rangle$, there exists an element w of the carrier of $\prod F$ such that $\mathcal{P}[z, w]$. Consider G being a function from $\prod \langle H, K \rangle$ into $\prod F$ such that for every element x of $\prod \langle H, K \rangle$, $\mathcal{P}[x, G(x)]$. For every element h of H and for every element k of K , there exists a function g such that $g = G_0(h)$ and $G(\langle h, k \rangle) = g + \cdot (q \cdot k)$. \square

- (28) Let us consider non empty finite sets I_0, I , an associative group-like multiplicative magma family F_0 of I_0 , an associative group-like multiplicative magma family F of I , groups H, K , an element q of I , and a homomorphism G_0 from $\prod F_0$ to H . Suppose

- (i) $q \notin I_0$, and
- (ii) $I = I_0 \cup \{q\}$, and
- (iii) $F = F_0 + \cdot (q \cdot K)$, and
- (iv) G_0 is bijective.

Then there exists a homomorphism G from $\prod F$ to $\prod \langle H, K \rangle$ such that

- (v) G is bijective, and
- (vi) for every function x_0 and for every element k of K and for every element h of H such that $h = G_0(x_0)$ and $x_0 \in \prod F_0$ holds $G(x_0 + \cdot (q \cdot k)) = \langle h, k \rangle$.

The theorem is a consequence of (27). PROOF: Set $L0 = G_0^{-1}$. Consider L being a homomorphism from $\prod\langle H, K \rangle$ to $\prod F$ such that L is bijective and for every element h of H and for every element k of K , there exists a function g such that $g = L0(h)$ and $L(\langle h, k \rangle) = g + \cdot (q \mapsto k)$. Set $G = L^{-1}$. For every function x_0 and for every element k of K and for every element h of H such that $h = G_0(x_0)$ and $x_0 \in \prod F_0$ holds $G(x_0 + \cdot (q \mapsto k)) = \langle h, k \rangle$. \square

(29) Let us consider a non empty finite set I , an associative group-like multiplicative magma family F of I , and a total I -defined function x . Suppose an element p of I . Then $x(p) \in F(p)$. Then $x \in$ the carrier of $\prod F$.

(30) Let us consider non empty finite sets I_0, I , an associative group-like multiplicative magma family F_0 of I_0 , an associative group-like multiplicative magma family F of I , a group K , an element q of I , and an element x of $\prod F$. Suppose

- (i) $q \notin I_0$, and
- (ii) $I = I_0 \cup \{q\}$, and
- (iii) $F = F_0 + \cdot (q \mapsto K)$.

Then there exists a total I_0 -defined function x_0 and there exists an element k of K such that $x_0 \in \prod F_0$ and $x = x_0 + \cdot (q \mapsto k)$ and for every element p of I_0 , $x_0(p) \in F_0(p)$. PROOF: Reconsider $y = x$ as a total I -defined function. Reconsider $k = y(q)$ as an element of K . Reconsider $y_0 = y|_{I_0}$ as an I_0 -defined function. For every element i of I_0 , $y_0(i) \in$ (the support of F_0)(i) and $y_0(i) \in F_0(i)$. \square

(31) Let us consider a group G , a subgroup H of G , a finite sequence f of elements of G , and a finite sequence g of elements of H . If $f = g$, then $\prod f = \prod g$. PROOF: Define \mathcal{P} [natural number] \equiv for every finite sequence f of elements of G for every finite sequence g of elements of H such that $\$1 = \text{len } f$ and $f = g$ holds $\prod f = \prod g$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. \square

(32) Let us consider a non empty finite set I , a group G , a subgroup H of G , a (the carrier of G)-valued total I -defined function x , and a (the carrier of H)-valued total I -defined function x_0 . If $x = x_0$, then $\prod x = \prod x_0$. The theorem is a consequence of (31).

(33) Let us consider a commutative group G , non empty finite sets I_0, I , an element q of I , a (the carrier of G)-valued total I -defined function x , a (the carrier of G)-valued total I_0 -defined function x_0 , and an element k of G . Suppose

- (i) $q \notin I_0$, and
- (ii) $I = I_0 \cup \{q\}$, and

(iii) $x = x_0 + \cdot (q \mapsto k)$.

Then $\prod x = \prod x_0 \cdot k$. The theorem is a consequence of (8) and (9). PROOF: Reconsider $y = q \mapsto k$ as a (the carrier of G)-valued total $\{q\}$ -defined function. I_0 misses $\{q\}$. \square

Let us consider a finite commutative group G . Now we state the propositions:

- (34) Suppose $\overline{G} > 1$. Then there exists a non empty finite set I and there exists an associative group-like commutative multiplicative magma family F of I and there exists a homomorphism H_0 from $\prod F$ to G such that $I = \text{support PrimeFactorization}(\overline{G})$ and for every element p of I , $F(p)$ is a subgroup of G and $\overline{F(p)} = (\text{PrimeFactorization}(\overline{G}))(p)$ and for every elements p, q of I such that $p \neq q$ holds $(\text{the carrier of } F(p)) \cap (\text{the carrier of } F(q)) = \{1_G\}$ and H_0 is bijective and for every (the carrier of G)-valued total I -defined function x such that for every element p of I , $x(p) \in F(p)$ holds $x \in \prod F$ and $H_0(x) = \prod x$.
- (35) Suppose $\overline{G} > 1$. Then there exists a non empty finite set I and there exists an associative group-like commutative multiplicative magma family F of I such that $I = \text{support PrimeFactorization}(\overline{G})$ and for every element p of I , $F(p)$ is a subgroup of G and $\overline{F(p)} = (\text{PrimeFactorization}(\overline{G}))(p)$ and for every elements p, q of I such that $p \neq q$ holds $(\text{the carrier of } F(p)) \cap (\text{the carrier of } F(q)) = \{1_G\}$ and for every element y of G , there exists a (the carrier of G)-valued total I -defined function x such that for every element p of I , $x(p) \in F(p)$ and $y = \prod x$ and for every (the carrier of G)-valued total I -defined functions x_1, x_2 such that for every element p of I , $x_1(p) \in F(p)$ and for every element p of I , $x_2(p) \in F(p)$ and $\prod x_1 = \prod x_2$ holds $x_1 = x_2$.

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