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Banach's Continuous Inverse Theorem and Closed Graph Theorem[1]

Hideki Sakurai
406-3, Haneo, Naganohara
Agatuma, Gunma, Japan

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article we formalize one of the most important theorems of linear operator theory – the Closed Graph Theorem commonly used in a standard text book such as [10] in Chapter 24.3. It states that a surjective closed linear operator between Banach spaces is bounded.

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The terminology and notation used here have been introduced in the following articles: [3], [4], [2], [15], [11], [14], [1], [5], [13], [12], [19], [20], [16], [7], [17], [8], [18], [9], and [6].

Let X, Y be non empty normed structures, let x be a point of X, and let y be a point of Y. Then $\langle x, y\rangle$ is a point of $X \times Y$.

Let X, Y be non empty normed structures, let s_1 be a sequence of X, and let s_2 be a sequence of Y. Then $\langle s_1, s_2\rangle$ is a sequence of $X \times Y$.

We now state several propositions:

(1) Let X, Y be real linear spaces and T be a linear operator from X into Y. Suppose T is bijective. Then T^{-1} is a linear operator from Y into X and $\mathrm{rng}(T^{-1})$ = the carrier of X.

(2) Let X, Y be non empty linear topological spaces, T be a linear operator from X into Y, and S be a function from Y into X. Suppose T is bijective

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and open and $S = T^{-1}$. Then S is a linear operator from Y into X, onto, and continuous.

(3) For all real normed spaces X, Y and for every linear operator f from X into Y holds $0_Y = f(0_X)$.

(4) Let X, Y be real normed spaces, f be a linear operator from X into Y, and x be a point of X. Then f is continuous in x if and only if f is continuous in 0_X.

(5) Let X, Y be real normed spaces and f be a linear operator from X into Y. Then f is continuous on the carrier of X if and only if f is continuous in 0_X.

(6) Let X, Y be real normed spaces and f be a linear operator from X into Y. Then f is Lipschitzian if and only if f is continuous on the carrier of X.

(7) Let X, Y be real Banach spaces and T be a Lipschitzian linear operator from X into Y. Suppose T is bijective. Then T^{-1} is a Lipschitzian linear operator from Y into X.

(8) Let X, Y be real normed spaces, s_1 be a sequence of X, s_2 be a sequence of Y, x be a point of X, and y be a point of Y. Then s_1 is convergent and $\lim s_1 = x$ and s_2 is convergent and $\lim s_2 = y$ if and only if $\langle s_1, s_2 \rangle$ is convergent and $\lim \langle s_1, s_2 \rangle = \langle\!\langle x, y \rangle\!\rangle$.

Let X, Y be real normed spaces and let T be a partial function from X to Y. The functor graph(T) yields a subset of $X \times Y$ and is defined as follows:

(Def. 1) graph(T) $= T$.

Let X, Y be real normed spaces and let T be a non empty partial function from X to Y. Observe that graph(T) is non empty.

Let X, Y be real normed spaces and let T be a linear operator from X into Y. Note that graph(T) is linearly closed.

Let X, Y be real normed spaces and let T be a linear operator from X into Y. The functor graphNrm(T) yielding a function from graph(T) into $\mathbb{R}$ is defined as follows:

(Def. 2) graphNrm(T) $=$ (the norm of $X \times Y$)$\restriction$graph(T).

Let X, Y be real normed spaces and let T be a partial function from X to Y. We say that T is closed if and only if:

(Def. 3) graph(T) is closed.

Let X, Y be real normed spaces and let T be a linear operator from X into Y. The functor graphNSP(T) yields a non empty normed structure and is defined by:

(Def. 4) graphNSP(T) $= \langle$graph(T), Zero(graph(T), $X \times Y$), Add(graph(T), $X \times Y$), Mult(graph(T), $X \times Y$), graphNrm(T)$\rangle$.

Let X, Y be real normed spaces and let T be a linear operator from X into Y. One can check that graphNSP(T) is Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, and scalar unital.

One can prove the following proposition

(9) For all real normed spaces X, Y and for every linear operator T from X into Y holds graphNSP(T) is a subspace of $X \times Y$.

Let X, Y be real normed spaces and let T be a linear operator from X into Y. Note that graphNSP(T) is reflexive, discernible, and real normed space-like.

We now state several propositions:

(10) Let X be a real normed space, Y be a real Banach space, and X_0 be a subset of Y. Suppose that

- (i) X is a subspace of Y,
- (ii) the carrier of $X = X_0$,
- (iii) the norm of $X =$ (the norm of Y)$\restriction$(the carrier of X), and
- (iv) X_0 is closed.

Then X is complete.

(11) Let X, Y be real Banach spaces and T be a linear operator from X into Y. If T is closed, then graphNSP(T) is complete.

(12) Let X, Y be real normed spaces and T be a non empty partial function from X to Y. Then T is closed if and only if for every sequence s_3 of X such that $\operatorname{rng} s_3 \subseteq \operatorname{dom} T$ and s_3 is convergent and $T_* s_3$ is convergent holds $\lim s_3 \in \operatorname{dom} T$ and $\lim(T_* s_3) = T(\lim s_3)$.

(13) Let X, Y be real normed spaces, T be a non empty partial function from X to Y, and T_0 be a linear operator from X into Y. If T_0 is Lipschitzian and $\operatorname{dom} T$ is closed and $T = T_0$, then T is closed.

(14) Let X, Y be real normed spaces, T be a non empty partial function from X to Y, and S be a non empty partial function from Y to X. If T is closed and one-to-one and $S = T^{-1}$, then S is closed.

(15) For all real normed spaces X, Y and for every point x of X and for every point y of Y holds $\|x\| \leq \|\langle x, y \rangle\|$ and $\|y\| \leq \|\langle x, y \rangle\|$.

Let X, Y be real Banach spaces. Note that every linear operator from X into Y which is closed is also Lipschitzian.

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