The Friendship Theorem

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Summary. In this article we prove the friendship theorem according to the article [1], which states that if a group of people has the property that any pair of persons have exactly one common friend, then there is a universal friend, i.e. a person who is a friend of every other person in the group.

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The papers [3], [2], [6], [7], [11], [8], [9], [15], [14], [4], [13], [5], [17], [18], [12], [16], and [10] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: \( x, y, z \) are sets, \( i, k, n \) are natural numbers, \( R \) is a binary relation, \( P \) is a finite binary relation, and \( p, q \) are finite sequences.

Let us consider \( P, x \). Observe that \( P^2x \) is finite.

We now state several propositions:

1. \( \overline{\overline{R}} = \overline{R^\perp} \).
2. If \( R \) is symmetric, then \( R^2x = R^{-1}(x) \).
3. If \( (p||k) \cap (p||k) = (q||n) \cap (q||n) \) and \( k \leq n \leq \text{len } p \), then \( p = (q||n-k) \cap (q||n-k) \).
4. If \( n \in \text{dom } q \) and \( p = (q||n) \cap (q||n) \), then \( q = (p||\text{len } p-n) \cap (p||\text{len } p-n) \).

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\( (5) \) If \((p|k) \cong (p|k) = (q|n) \cong (q|n)\), then there exists \(i\) such that \(p = (q|i) \cong (q|i)\).

The scheme \( Sch \) deals with a non empty set \( A \), a non zero natural number \( B \), and a unary predicate \( P \), and states that:

There exists a cardinal number \( C \) such that \( B \cdot C = \{ F \in \mathcal{A}^B : P[F] \} \)

provided the following requirements are met:

- For all finite sequences \( p, q \) of elements of \( A \) such that \( p \cong q \) is \( B \)-element and \( P[p \cong q] \) holds \( P[q \cong p] \), and
- For every element \( p \) of \( A^B \) such that \( P[p] \) and for every natural number \( i \) such that \( i < B \) and \( p = (p|i) \cong (p|i) \) holds \( i = 0 \).

One can prove the following propositions:

\( (6) \) Let \( X \) be a non empty set, \( A \) be a non empty finite subset of \( X \), and \( P \) be a function from \( X \) into \( 2^X \). Suppose that for every \( x \) such that \( x \in X \) holds \( P(x) = n \). Then

\[
\{ F \in X^{k+1} : F(1) \in A \land \bigwedge_i (i \in \text{Seg} k \Rightarrow F(i+1) \in P(F(i))) \} = A \cdot n^k.
\]

\( (7) \) If \( \text{len} p \) is prime and there exists \( i \) such that \( 0 < i < \text{len} p \) and \( p = (p|i) \cong (p|i) \), then \( \text{rng} p \subseteq \{p(1)\} \).

\[ 2. \text{The Friendship Graph} \]

Let us consider \( R \) and let \( x \) be an element of field \( R \). We say that \( x \) is universal friend if and only if:

(Def. 1) For every \( y \) such that \( y \in \text{field} R \setminus \{x\} \) holds \( \langle x, y \rangle \in R \).

Let \( R \) be a binary relation. We say that \( R \) has universal friend if and only if:

(Def. 2) There exists an element of field \( R \) which is universal friend.

Let \( R \) be a binary relation. We introduce \( R \) is without universal friend as an antonym of \( R \) has universal friend.

Let \( R \) be a binary relation. We say that \( R \) is friendship graph like if and only if:

(Def. 3) For all \( x, y \) such that \( x, y \in \text{field} R \) and \( x \neq y \) there exists \( z \) such that \( R^x \cap \text{Coim}(R, y) = \{z\} \).

Let us observe that there exists a binary relation which is finite, symmetric, irreflexive, and friendship graph like.

A friendship graph is a finite symmetric irreflexive friendship graph like binary relation.

In the sequel \( F_1 \) is a friendship graph.

The following propositions are true:
The friendship theorem

(8) \( 2 \mid \overline{F_1 \circ x} \).

(9) If \( x, y \in \text{field } F_1 \) and \( \langle x, y \rangle \notin F_1 \), then \( \overline{F_1 \circ x} = \overline{F_1 \circ y} \).

(10) If \( F_1 \) is without universal friend and \( x \in \text{field } F_1 \), then \( \overline{F_1 \circ x} > 2 \).

(11) If \( F_1 \) is without universal friend and \( x, y \in \text{field } F_1 \), then \( \overline{F_1 \circ x} = \overline{F_1 \circ y} \).

(12) If \( F_1 \) is without universal friend and \( x \in \text{field } F_1 \), then \( \text{field } F_1 = 1 + \overline{F_1 \circ x} \cdot (\overline{F_1 \circ x} - 1) \).

(13) For all elements \( x, y \) of field \( F_1 \) such that \( x \) is universal friend and \( x \neq y \) there exists \( z \) such that \( F_1 \circ y = \{x, z\} \) and \( F_1 \circ z = \{x, y\} \).

3. The Friendship Theorem

Next we state the proposition

(14) If \( F_1 \) is non empty, then \( F_1 \) has universal friend.

REFERENCES


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