Weak Completeness Theorem for Propositional Linear Time Temporal Logic\(^1\)

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**Summary.** We prove weak (finite set of premises) completeness theorem for extended propositional linear time temporal logic with irreflexive version of until-operator. We base it on the proof of completeness for basic propositional linear time temporal logic given in [20] which roughly follows the idea of the Henkin-Hasenjaeger method for classical logic. We show that a temporal model exists for every formula which negation is not derivable (Satisfiability Theorem). The contrapositive of that theorem leads to derivability of every valid formula. We build a tree of consistent and complete PNPs which is used to construct the model.

MML identifier: LTLAXIO4, version: 7.14.01.183.1153

The papers [25], [14], [28], [21], [4], [1], [30], [11], [26], [31], [13], [24], [2], [3], [5], [6], [7], [12], [15], [9], [23], [8], [10], [19], [27], [29], [22], [16], [17], and [18] provide the notation and terminology for this paper.

\(^1\)The author is the winner of the Mizar Prize for Young Researchers in 2012 for this article. This work has been supported by the Polish Ministry of Science and Higher Education project “Managing a Large Repository of Computer-verified Mathematical Knowledge” (N N519 385136).

\(^2\)I would like to thank Prof. Dr. Stephan Merz for valuable hints which helped me to prove the theorem. I would particularly like to thank Dr. Artur Korniłowicz who patiently answered a lot of my questions regarding writing this article. I would like to thank Dr. Josef Urban for discussions and encouragement to write the article. I would like to thank Prof. Andrzej Trybulec, Dr. Adam Naumowicz, Dr. Grzegorz Bancerek and Karol Pąk for their help in preparation of the article.
1. Preliminaries

For simplicity, we use the following convention: $A, B, p, q$ denote elements of the LTLB-WFF, $M$ denotes a LTL Model, $j, k, n$ denote elements of $\mathbb{N}$, $i$ denotes a natural number, $X$ denotes a subset of the LTLB-WFF, $F$ denotes a finite subset of the LTLB-WFF, $f$ denotes a finite sequence of elements of the LTLB-WFF, and $P, Q, R$ denote positive-negative pairs.

Let $X$ be a finite set. We see that the enumeration of $X$ is a one-to-one finite sequence of elements of $X$.

Let $E$ be a set and let $F$ be a finite subset of $E$. We see that the enumeration of $F$ is a one-to-one finite sequence of elements of $E$.

Let $D$ be a set. One can verify that there exists a set of finite sequences of $D$ which is non empty and finite.

We now state the proposition

(1) Let $X$ be a set and $G$ be a non empty finite set of finite sequences of $X$.

Then there exists a finite sequence $A$ such that $A \in G$ and for every finite sequence $B$ such that $B \in G$ holds $\text{len}(B) \leq \text{len}(A)$.

Let $T$ be a decorated tree, let us consider $n$, and let $t$ be a node of $T$. Then $t \upharpoonright n$ is a node of $T$.

We now state the proposition

(2) $p$ is a finite sequence of elements of $\mathbb{N}$.

Let us consider $A$. We introduce $A$ is s-until as a synonym of $A$ is conjunctive.

Let us consider $A$. Let us assume that $A$ is s-until. The right argument of $A$ yields an element of the LTLB-WFF and is defined by:

(Def. 1) There exists $p$ such that $p \cup$ the right argument of $A = A$.

Let us consider $A$. We say that $A$ is satisfiable if and only if:

(Def. 2) There exist $M, n$ such that $\text{SAT}_M(\langle n, A \rangle) = 1$.

We now state four propositions:

(3) $\emptyset_{\text{the LTLB-WFF}} \models A$ iff $\neg A$ is not satisfiable.

(4) If $\top_t \& A$ is satisfiable, then $A$ is satisfiable.

(5) Let $i$ be an element of $\mathbb{N}$. Then $\text{SAT}_M(\langle i, p \cup q \rangle) = 1$ if and only if there exists $j$ such that $j > i$ and $\text{SAT}_M(\langle j, q \rangle) = 1$ and for every $k$ such that $i < k < j$ holds $\text{SAT}_M(\langle k, p \rangle) = 1$.

(6) $\text{SAT}_M(\langle n, (\text{conjunction } f)_{\text{len(conjunction } f)} \rangle) = 1$ iff for every $i$ such that $i \in \text{dom } f$ holds $\text{SAT}_M(\langle n, f_i \rangle) = 1$.

One can prove the following three propositions:

(7) $\hat{W} = \top_t \& \neg A$, where $W = \langle e_{\text{the LTLB-WFF}}, \langle A \rangle \rangle$.

(8) For every complete positive-negative pair $P$ such that $\text{UN}(A, B) \in \text{rng } P$ holds $A, B, A \cup B \in \text{rng } P$.

(9) $\text{rng } P \subseteq \bigcup \sigma(\text{rng } P)$.
2. Set of PNP-formulas. Completions of Formulas and PNPs

In the sequel $P$ is an element of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$.
Let $F$ be a subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$. The functor $\hat{F}$ yields a subset of the LTLB-WFF and is defined by:

(Def. 3) \[
\hat{F} = \{ \hat{P} : P \in F \}.
\]

Let $F$ be a non empty subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$.
Note that $\hat{F}$ is non empty.

Let $F$ be a finite subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$.
Observe that $\hat{F}$ is finite.
We now state the proposition

(10) For all subsets $F$, $G$ of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ holds $\hat{F} \cup \hat{G} = \hat{F} \cup \hat{G}$.

One can prove the following proposition

(11) $\tilde{W} = \{ \top_t \& \& \top_t \}$, where $W = \{ (\varepsilon_{\text{the LTLB-WFF}}), (\varepsilon_{\text{the LTLB-WFF}}) \}$.

In the sequel $Q$ denotes a positive-negative pair.

Let $F$ be a finite subset of the LTLB-WFF. The functor $\text{comp} F$ yielding a non empty finite subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ is defined as follows:

(Def. 4) $\text{comp} F = \{ Q : \text{rng} Q = \tau(F) \land \text{rng}(Q_1) \text{ misses rng}(Q_2) \}$.

Let $F$ be a finite subset of the LTLB-WFF. Note that every element of $\text{comp} F$ is complete.

One can prove the following proposition

(12) $\text{comp}(\emptyset_{\text{the LTLB-WFF}}) = \{ (\varepsilon_{\text{the LTLB-WFF}}), (\varepsilon_{\text{the LTLB-WFF}}) \}$.

Let us consider $P$, $Q$. We say that $Q$ is completion of $P$ if and only if:

(Def. 5) $\text{rng}(P_1) \subseteq \text{rng}(Q_1)$ and $\text{rng}(P_2) \subseteq \text{rng}(Q_2)$ and $\tau(\text{rng} P) = \text{rng} Q$.

We now state the proposition

(13) If $Q$ is completion of $P$, then $Q$ is complete.

We now state the proposition

(14) If $Q$ is complete, then $Q$ is consistent.

In the sequel $Q$ is a consistent positive-negative pair.

Let us consider $P$. The functor $\text{comp} P$ yields a finite subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ and is defined by:

(Def. 6) $\text{comp} P = \{ Q : Q \text{ is completion of } P \}$.

Let $P$ be a consistent positive-negative pair. One can check that $\text{comp} P$ is non empty. Observe that every element of $\text{comp} P$ is consistent.

In the sequel $P$ denotes an element of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$.

Let $X$ be a subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$. The functor $\text{comp} X$ yields a subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ and is defined by:
(Def. 7) \( \text{comp } X = \bigcup \{ \text{comp } P : P \in X \} \).

Let \( X \) be a finite subset of \((\text{the LTLB-WFF})_{1-1}^* \times (\text{the LTLB-WFF})_{1-1}^* \). One can check that \( \text{comp } X \) is finite.

We now state four propositions:

(14) For every non empty subset \( X \) of \((\text{the LTLB-WFF})_{1-1}^* \times (\text{the LTLB-WFF})_{1-1}^* \) such that \( Q \in X \) holds \( \text{comp } Q \subseteq \text{comp } X \).

(15) For every non empty finite subset \( F \) of the LTLB-WFF there exists \( p \) such that \( p \in \tau(F) \) and \( \tau(\tau(F) \setminus \{p\}) = \tau(F) \setminus \{p\} \).

(16) Let \( F \) be a finite subset of the LTLB-WFF and \( f \) be a finite sequence of elements of the LTLB-WFF. If \( \text{rng } f = \text{comp } F \), then \( \emptyset \vdash_{\text{LTLB-WFF}} \neg((\text{conjunction negation } f)_{\text{len conjunction negation }}) \).

(17) Let \( P \) be a consistent positive-negative pair and \( f \) be a finite sequence of elements of the LTLB-WFF. If \( \text{rng } f = \text{comp } P \), then \( \emptyset \vdash_{\text{LTLB-WFF}} \neg((\text{conjunction negation } f)_{\text{len conjunction negation }}) \).

3. Set of Possible Next-State PNPs

In the sequel \( A, B \) denote elements of the LTLB-WFF.

Let us consider \( X \). The functor \( \text{UN}(X) \) yields a subset of the LTLB-WFF and is defined as follows:

(Def. 8) \( \text{UN}(X) = \{ \text{UN}(A, B) : A U B \in X \} \).

Let \( X \) be a finite subset of the LTLB-WFF. One can check that \( \text{UN}(X) \) is finite.

Let us consider \( P \). The functor \( \text{UN}(P) \) yielding a non empty finite subset of \((\text{the LTLB-WFF})_{1-1}^* \times (\text{the LTLB-WFF})_{1-1}^* \) is defined by:

(Def. 9) \( \text{UN}(P) = \{ Q : Q \text{ ranges over positive-negative pairs: } \text{rng}(Q_1) = \text{UN}(\text{rng}(P_1)) \land \text{rng}(Q_2) = \text{UN}(\text{rng}(P_2)) \} \).

One can prove the following proposition

(18) For every element \( Q \) of \( \text{UN}(P) \) holds \( \emptyset \vdash_{\text{LTLB-WFF}} \tilde{P} \Rightarrow \chi \tilde{Q} \).

Let \( P \) be a consistent positive-negative pair. Note that every element of \( \text{UN}(P) \) is consistent. In the sequel \( Q \) denotes an element of \((\text{the LTLB-WFF})_{1-1}^* \times (\text{the LTLB-WFF})_{1-1}^* \).

Let us consider \( P \). The next completion of \( P \) yielding a finite subset of \((\text{the LTLB-WFF})_{1-1}^* \times (\text{the LTLB-WFF})_{1-1}^* \) is defined by:

(Def. 10) \( \text{The next completion of } P = \{ Q : Q \in \text{comp UN}(P) \} \).

Let \( P \) be a consistent positive-negative pair. One can verify that the next completion of \( P \) is non empty.
Let $P$ be a consistent positive-negative pair. One can check that every element of the next completion of $P$ is consistent.

Next we state two propositions:

(19) If $Q \in$ the next completion of $P$ and $R \in \text{UN}(P)$, then $Q$ is completion of $R$.

(20) If $Q \in$ the next completion of $P$, then $Q$ is complete.

Let $P$ be a consistent positive-negative pair. One can verify that every element of the next completion of $P$ is complete.

Next we state several propositions:

(21) If $A \cup B \in \text{rng}(P_2)$ and $Q \in$ the next completion of $P$, then $\text{UN}(A,B) \in \text{rng}(Q_2)$.

(22) If $A \cup B \in \text{rng}(P_1)$ and $Q \in$ the next completion of $P$, then $\text{UN}(A,B) \in \text{rng}(Q_1)$.

(23) If $R \in$ the next completion of $Q$ and $\text{rng}Q \subseteq \bigcup \sigma(\text{rng}P)$, then $\text{rng}R \subseteq \bigcup \sigma(\text{rng}P)$.

(24) Let $P$ be a consistent complete positive-negative pair and $Q$ be an element of the next completion of $P$. If $A \cup B \in \text{rng}(P_2)$, then $B \in \text{rng}(Q_2)$ but $A \in \text{rng}(Q_2)$ or $A \cup B \in \text{rng}(Q_2)$.

(25) Let $P$ be a consistent complete positive-negative pair and $Q$ be an element of the next completion of $P$. If $A \cup B \in \text{rng}(P_1)$, then $B \in \text{rng}(Q_1)$ or $A, A \cup B \in \text{rng}(Q_1)$.

4. A PNP-Tree and its Properties

Let us consider $P$. A finite-branching tree decorated with elements of $(\text{the LTLB-WFF})_{1-1}^* \times (\text{the LTLB-WFF})_{1-1}^*$ is said to be a tree of positive-negative pairs of $P$ if it satisfies the conditions (Def. 11).

(Def. 11)(i) It($\emptyset$) = $P$, and

(ii) for every element $t$ of dom it and for every element $w$ of $(\text{the LTLB-WFF})_{1-1}^* \times (\text{the LTLB-WFF})_{1-1}^*$ such that $w = \text{it}(t)$ holds $\text{succ}(\text{it}(t), t) =$ the enumeration of the next completion of $w$.

In the sequel $T$ is a tree of positive-negative pairs of $P$ and $t$ is a node of $T$. Let us consider $P$, $T$, $t$. Then $T|t$ is a tree of positive-negative pairs of $T(t)$.

Next we state two propositions:

(26) For every natural number $n$ such that $t \in \langle n \rangle \in \text{dom}T$ holds $T(t \in \langle n \rangle) \in$ the next completion of $T(t)$.

(27) If $Q \in \text{rng}T$, then $\text{rng}Q \subseteq \bigcup \sigma(\text{rng}P)$.

Let us consider $P$, $T$. One can check that $\text{rng}T$ is non empty and finite.

Let $P$ be a consistent positive-negative pair and let $T$ be a tree of positive-negative pairs of $P$. One can check that every element of $\text{rng}T$ is consistent.
Let $P$ be a consistent complete positive-negative pair and let $T$ be a tree of positive-negative pairs of $P$. One can verify that every element of $\text{rng} T$ is complete.

Let $P$ be a consistent complete positive-negative pair, let $T$ be a tree of positive-negative pairs of $P$, and let $t$ be a node of $T$. Observe that $T(t)$ is consistent and complete as a positive-negative pair.

Let $P$ be a consistent positive-negative pair, let $T$ be a tree of positive-negative pairs of $P$, and let $t$ be an element of $\text{dom} T$. Observe that $\text{succ} t$ is non empty.

Let us consider $P, T$. The range of $T$ except the root node yields a finite subset of $(\text{the LTLB-WFF})^*_1 \times (\text{the LTLB-WFF})^*_1$ and is defined as follows:

(Def. 12) The range of $T$ except the root node $= \{ T(t); t \text{ ranges over nodes of } T; t \neq \emptyset \}$.

Let $P$ be a consistent positive-negative pair and let $T$ be a tree of positive-negative pairs of $P$. One can verify that the range of $T$ except the root node is non empty.

One can prove the following proposition

(28) If $R \in \text{rng} T$ and $Q \in \text{UN}(R)$, then $\text{comp} Q \subseteq \text{the range of } T$ except the root node.

One can prove the following proposition

(29) Let $P$ be a consistent complete positive-negative pair, $T$ be a tree of positive-negative pairs of $P$, and $f$ be a finite sequence of elements of the LTLB-WFF. If $\text{rng} f = \hat{J}$, then $\emptyset \text{ the LTLB-WFF } \vdash \neg((\text{conjunction negation } f)_{\text{len conjunction negation } f}) \Rightarrow X \neg((\text{conjunction negation } f)_{\text{len conjunction negation } f})$, where $J = \text{the range of } T$ except the root node.

5. A Path in PNP-Tree and its Properties. Existence of Temporal Model for a Consistent PNP. Weak Completeness Theorem

Let $P$ be a consistent positive-negative pair and let $T$ be a tree of positive-negative pairs of $P$. A sequence of $\text{dom} T$ is called a path of $T$ if:

(Def. 13) $\text{It}(0) = \emptyset$ and for every natural number $k$ holds $\text{it}(k+1) \in \text{succ} \text{it}(k)$.

Let $P$ be a consistent complete positive-negative pair, let $T$ be a tree of positive-negative pairs of $P$, let $t$ be a path of $T$, and let us consider $i$. Then $t(i)$ is a node of $T$.

Next we state three propositions:

(30) Let $P$ be a consistent complete positive-negative pair, $T$ be a tree of positive-negative pairs of $P$, and $t$ be a path of $T$. Suppose $A \cup B \in \text{rng} (T(t(i)))$. Let given $j$. If $j > i$, then $B \in \text{rng} (T(t(j)))$ or there exists $k$ such that $i < k < j$ and $A \in \text{rng} (T(t(k)))$. 
(31) Let $P$ be a consistent complete positive-negative pair and $T$ be a tree of positive-negative pairs of $P$. Suppose $A \cup B \in \text{rng}(P_1)$ and for every element $Q$ of the range of $T$ except the root node holds $B \notin \text{rng}(Q_1)$. Let $Q$ be an element of the range of $T$ except the root node. Then $B \in \text{rng}(Q_2)$ and $A \cup B \in \text{rng}(Q_1)$.

(32) Let $P$ be a consistent complete positive-negative pair and $T$ be a tree of positive-negative pairs of $P$. Suppose $A \cup B \in \text{rng}(P_1)$. Then there exists an element $R$ of the range of $T$ except the root node such that $B \in \text{rng}(R_1)$.

Let $P$ be a consistent positive-negative pair, let $T$ be a tree of positive-negative pairs of $P$, and let $t$ be a path of $T$. We say that $t$ is complete if and only if the condition (Def. 14) is satisfied.

(Def. 14) Let given $i$. Suppose $A \cup B \in \text{rng}(T(t(i)))_1$. Then there exists $j$ such that $j > i$ and $B \in \text{rng}(T(t(j)))_1$ and for every $k$ such that $i < k < j$ holds $A \in \text{rng}(T(t(k)))_1$.

Let $P$ be a consistent complete positive-negative pair and let $T$ be a tree of positive-negative pairs of $P$. Note that there exists a path of $T$ which is complete.

Let $P$ be a consistent positive-negative pair. Observe that $\tilde{P}$ is satisfiable.

One can prove the following proposition

(33) If $F \models A$, then $F \vdash A$.

References


*Weak completeness theorem of basic propositional linear temporal logic extended with $\mathcal{U}$ operator (LTLB).*
Received May 7, 2012