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Weak Completeness Theorem for Propositional Linear Time Temporal Logic[1]

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Summary. We prove weak (finite set of premises) completeness theorem for extended propositional linear time temporal logic with irreflexive version of until-operator. We base it on the proof of completeness for basic propositional linear time temporal logic given in [20] which roughly follows the idea of the Henkin-Hasenjaeger method for classical logic. We show that a temporal model exists for every formula which negation is not derivable (Satisfiability Theorem). The contrapositive of that theorem leads to derivability of every valid formula. We build a tree of consistent and complete PNPs which is used to construct the model.

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The papers [25], [14], [28], [21], [4], [1], [30], [11], [26], [31], [13], [24], [2], [3], [5], [6], [7], [12], [15], [9], [23], [8], [10], [19], [27], [29], [22], [16], [17], and [18] provide the notation and terminology for this paper.

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1. Preliminaries

For simplicity, we use the following convention: A, B, p, q denote elements of the LTLB-WFF, M denotes a LTL Model, j, k, n denote elements of $\mathbb{N}$, i denotes a natural number, X denotes a subset of the LTLB-WFF, F denotes a finite subset of the LTLB-WFF, f denotes a finite sequence of elements of the LTLB-WFF, and P, Q, R denote positive-negative pairs.

Let X be a finite set. We see that the enumeration of X is a one-to-one finite sequence of elements of X.

Let E be a set and let F be a finite subset of E. We see that the enumeration of F is a one-to-one finite sequence of elements of E.

Let D be a set. One can verify that there exists a set of finite sequences of D which is non empty and finite.

We now state the proposition

(1) Let X be a set and G be a non empty finite set of finite sequences of X. Then there exists a finite sequence A such that $A \in G$ and for every finite sequence B such that $B \in G$ holds $\operatorname{len} B \leq \operatorname{len} A$.

Let T be a decorated tree, let us consider n, and let t be a node of T. Then $t{\restriction}n$ is a node of T.

We now state the proposition

(2) p is a finite sequence of elements of $\mathbb{N}$.

Let us consider A. We introduce A is s-until as a synonym of A is conjunctive.

Let us consider A. Let us assume that A is s-until. The right argument of A yields an element of the LTLB-WFF and is defined by:

(Def. 1) There exists p such that $p\,\mathcal{U}$ the right argument of $A = A$.

Let us consider A. We say that A is satisfiable if and only if:

(Def. 2) There exist M, n such that $\mathrm{SAT}_M(\langle n, A\rangle) = 1$.

We now state four propositions:

(3) $\emptyset_{\text{the LTLB-WFF}} \models A$ iff $\neg A$ is not satisfiable.

(4) If $\top_t \mathbin{\&\&} A$ is satisfiable, then A is satisfiable.

(5) Let i be an element of $\mathbb{N}$. Then $\mathrm{SAT}_M(\langle i, p\,\mathcal{U}\,q\rangle) = 1$ if and only if there exists j such that $j > i$ and $\mathrm{SAT}_M(\langle j, q\rangle) = 1$ and for every k such that $i < k < j$ holds $\mathrm{SAT}_M(\langle k, p\rangle) = 1$.

(6) $\mathrm{SAT}_M(\langle n, (\text{conjunction } f)_{\text{len conjunction } f}\rangle) = 1$ iff for every i such that $i \in \operatorname{dom} f$ holds $\mathrm{SAT}_M(\langle n, f_i\rangle) = 1$.

One can prove the following three propositions:

(7) $\widehat{W} = \top_t \mathbin{\&\&} \neg A$, where $W = \langle \varepsilon_{(\text{the LTLB-WFF})}, \langle A\rangle\rangle$.

(8) For every complete positive-negative pair P such that $\mathrm{UN}(A, B) \in \operatorname{rng} P$ holds A, B, $A\,\mathcal{U}\,B \in \operatorname{rng} P$.

(9) $\operatorname{rng} P \subseteq \bigcup \sigma(\operatorname{rng} P)$.

2. Set of PNP-formulas. Completions of Formulas and PNPs

In the sequel P is an element of (the LTLB-WFF)$^*_{1-1}\times$(the LTLB-WFF)$^*_{1-1}$.

Let F be a subset of (the LTLB-WFF)$^*_{1-1}\times$(the LTLB-WFF)$^*_{1-1}$. The functor $\widehat{F}$ yields a subset of the LTLB-WFF and is defined by:

(Def. 3) $\widehat{F}=\{\widehat{P}: P\in F\}$.

Let F be a non empty subset of (the LTLB-WFF)$^*_{1-1}\times$(the LTLB-WFF)$^*_{1-1}$. Note that $\widehat{F}$ is non empty.

Let F be a finite subset of (the LTLB-WFF)$^*_{1-1}\times$ (the LTLB-WFF)$^*_{1-1}$. Observe that $\widehat{F}$ is finite.

We now state the proposition

(10) For all subsets F, G of (the LTLB-WFF)$^*_{1-1}\times$ (the LTLB-WFF)$^*_{1-1}$ holds $\widehat{F\cup G}=\widehat{F}\cup\widehat{G}$.

One can prove the following proposition

(11) $\widehat{W}=\{\top_t \mathbin{\&\&} \top_t\}$, where $W=\{\langle\varepsilon_{(\text{the LTLB-WFF})},\varepsilon_{(\text{the LTLB-WFF})}\rangle\}$.

In the sequel Q denotes a positive-negative pair.

Let F be a finite subset of the LTLB-WFF. The functor $\operatorname{comp} F$ yielding a non empty finite subset of (the LTLB-WFF)$^*_{1-1}\times$ (the LTLB-WFF)$^*_{1-1}$ is defined as follows:

(Def. 4) $\operatorname{comp} F=\{Q:\operatorname{rng} Q=\tau(F)\ \wedge\ \operatorname{rng}(Q_{\mathbf{1}})\text{ misses }\operatorname{rng}(Q_{\mathbf{2}})\}$.

Let F be a finite subset of the LTLB-WFF. Note that every element of $\operatorname{comp} F$ is complete.

One can prove the following proposition

(12) $\operatorname{comp}(\emptyset_{\text{the LTLB-WFF}})=\{\langle\varepsilon_{(\text{the LTLB-WFF})},\varepsilon_{(\text{the LTLB-WFF})}\rangle\}$.

Let us consider P, Q. We say that Q is completion of P if and only if:

(Def. 5) $\operatorname{rng}(P_{\mathbf{1}})\subseteq\operatorname{rng}(Q_{\mathbf{1}})$ and $\operatorname{rng}(P_{\mathbf{2}})\subseteq\operatorname{rng}(Q_{\mathbf{2}})$ and $\tau(\operatorname{rng} P)=\operatorname{rng} Q$.

We now state the proposition

(13) If Q is completion of P, then Q is complete.

In the sequel Q is a consistent positive-negative pair.

Let us consider P. The functor $\operatorname{comp} P$ yields a finite subset of (the LTLB-WFF)$^*_{1-1}\times$ (the LTLB-WFF)$^*_{1-1}$ and is defined by:

(Def. 6) $\operatorname{comp} P=\{Q: Q\text{ is completion of }P\}$.

Let P be a consistent positive-negative pair. One can check that $\operatorname{comp} P$ is non empty. Observe that every element of $\operatorname{comp} P$ is consistent.

In the sequel P denotes an element of (the LTLB-WFF)$^*_{1-1}\times$ (the LTLB-WFF)$^*_{1-1}$.

Let X be a subset of (the LTLB-WFF)$^*_{1-1}\times$(the LTLB-WFF)$^*_{1-1}$. The functor $\operatorname{comp} X$ yields a subset of (the LTLB-WFF)$^*_{1-1}\times$ (the LTLB-WFF)$^*_{1-1}$ and is defined by:

(Def. 7) $\operatorname{comp} X = \bigcup\{\operatorname{comp} P : P \in X\}$.

Let X be a finite subset of (the LTLB-WFF)$^*_{1-1}\times$(the LTLB-WFF)$^*_{1-1}$. One can check that $\operatorname{comp} X$ is finite.

We now state four propositions:

(14) For every non empty subset X of (the LTLB-WFF)$^*_{1-1}$ $\times$ (the LTLB-WFF)$^*_{1-1}$ such that $Q \in X$ holds $\operatorname{comp} Q \subseteq \operatorname{comp} X$.

(15) For every non empty finite subset F of the LTLB-WFF there exists p such that $p \in \tau(F)$ and $\tau(\tau(F) \setminus \{p\}) = \tau(F) \setminus \{p\}$.

(16) Let F be a finite subset of the LTLB-WFF and f be a finite sequence of elements of the LTLB-WFF. If $\operatorname{rng} f = \widehat{\operatorname{comp} F}$, then $\emptyset_{\text{the LTLB-WFF}} \vdash \neg((\text{conjunction negation } f)_{\text{len conjunction negation } f})$.

(17) Let P be a consistent positive-negative pair and f be a finite sequence of elements of the LTLB-WFF. If $\operatorname{rng} f = \widehat{\operatorname{comp} P}$, then $\emptyset_{\text{the LTLB-WFF}} \vdash \widehat{P} \Rightarrow \neg((\text{conjunction negation } f)_{\text{len conjunction negation } f})$.

3. Set of Possible Next-State PNPs

In the sequel A, B denote elements of the LTLB-WFF.

Let us consider X. The functor $\mathrm{UN}(X)$ yields a subset of the LTLB-WFF and is defined as follows:

(Def. 8) $\mathrm{UN}(X) = \{\mathrm{UN}(A, B) : A\,\mathcal{U}\,B \in X\}$.

Let X be a finite subset of the LTLB-WFF. One can check that $\mathrm{UN}(X)$ is finite.

Let us consider P. The functor $\mathrm{UN}(P)$ yielding a non empty finite subset of (the LTLB-WFF)$^*_{1-1}$ $\times$ (the LTLB-WFF)$^*_{1-1}$ is defined by:

(Def. 9) $\mathrm{UN}(P) = \{Q; Q$ ranges over positive-negative pairs: $\operatorname{rng}(Q_{\mathbf{1}}) = \mathrm{UN}(\operatorname{rng}(P_{\mathbf{1}})) \ \wedge\ \operatorname{rng}(Q_{\mathbf{2}}) = \mathrm{UN}(\operatorname{rng}(P_{\mathbf{2}}))\}$.

One can prove the following proposition

(18) For every element Q of $\mathrm{UN}(P)$ holds $\emptyset_{\text{the LTLB-WFF}} \vdash \widehat{P} \Rightarrow \mathcal{X}\,\widehat{Q}$.

Let P be a consistent positive-negative pair. Note that every element of $\mathrm{UN}(P)$ is consistent. In the sequel Q denotes an element of (the LTLB-WFF)$^*_{1-1}$ $\times$ (the LTLB-WFF)$^*_{1-1}$.

Let us consider P. The next completion of P yielding a finite subset of (the LTLB-WFF)$^*_{1-1}$ $\times$ (the LTLB-WFF)$^*_{1-1}$ is defined by:

(Def. 10) The next completion of $P = \{Q : Q \in \operatorname{comp} \mathrm{UN}(P)\}$.

Let P be a consistent positive-negative pair. One can verify that the next completion of P is non empty.

Let P be a consistent positive-negative pair. One can check that every element of the next completion of P is consistent.

Next we state two propositions:

(19) If $Q \in$ the next completion of P and $R \in \mathrm{UN}(P)$, then Q is completion of R.

(20) If $Q \in$ the next completion of P, then Q is complete.

Let P be a consistent positive-negative pair. One can verify that every element of the next completion of P is complete.

Next we state several propositions:

(21) If $A \mathcal{U} B \in \mathrm{rng}(P_{\mathbf{2}})$ and $Q \in$ the next completion of P, then $\mathrm{UN}(A, B) \in \mathrm{rng}(Q_{\mathbf{2}})$.

(22) If $A \mathcal{U} B \in \mathrm{rng}(P_{\mathbf{1}})$ and $Q \in$ the next completion of P, then $\mathrm{UN}(A, B) \in \mathrm{rng}(Q_{\mathbf{1}})$.

(23) If $R \in$ the next completion of Q and $\mathrm{rng}\, Q \subseteq \bigcup \sigma(\mathrm{rng}\, P)$, then $\mathrm{rng}\, R \subseteq \bigcup \sigma(\mathrm{rng}\, P)$.

(24) Let P be a consistent complete positive-negative pair and Q be an element of the next completion of P. If $A \mathcal{U} B \in \mathrm{rng}(P_{\mathbf{2}})$, then $B \in \mathrm{rng}(Q_{\mathbf{2}})$ but $A \in \mathrm{rng}(Q_{\mathbf{2}})$ or $A \mathcal{U} B \in \mathrm{rng}(Q_{\mathbf{2}})$.

(25) Let P be a consistent complete positive-negative pair and Q be an element of the next completion of P. If $A \mathcal{U} B \in \mathrm{rng}(P_{\mathbf{1}})$, then $B \in \mathrm{rng}(Q_{\mathbf{1}})$ or A, $A \mathcal{U} B \in \mathrm{rng}(Q_{\mathbf{1}})$.

4. A PNP-Tree and its Properties

Let us consider P. A finite-branching tree decorated with elements of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ is said to be a tree of positive-negative pairs of P if it satisfies the conditions (Def. 11).

(Def. 11)(i) $\mathrm{It}(\emptyset) = P$, and

(ii) for every element t of $\mathrm{dom}\, \mathrm{it}$ and for every element w of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ such that $w = \mathrm{it}(t)$ holds $\mathrm{succ}(\mathrm{it}, t) =$ the enumeration of the next completion of w.

In the sequel T is a tree of positive-negative pairs of P and t is a node of T.

Let us consider P, T, t. Then $T{\restriction}t$ is a tree of positive-negative pairs of $T(t)$.

Next we state two propositions:

(26) For every natural number n such that $t \frown \langle n \rangle \in \mathrm{dom}\, T$ holds $T(t \frown \langle n \rangle) \in$ the next completion of $T(t)$.

(27) If $Q \in \mathrm{rng}\, T$, then $\mathrm{rng}\, Q \subseteq \bigcup \sigma(\mathrm{rng}\, P)$.

Let us consider P, T. One can check that $\mathrm{rng}\, T$ is non empty and finite.

Let P be a consistent positive-negative pair and let T be a tree of positive-negative pairs of P. One can check that every element of $\mathrm{rng}\, T$ is consistent.

Let P be a consistent complete positive-negative pair and let T be a tree of positive-negative pairs of P. One can verify that every element of $\operatorname{rng} T$ is complete.

Let P be a consistent complete positive-negative pair, let T be a tree of positive-negative pairs of P, and let t be a node of T. Observe that $T(t)$ is consistent and complete as a positive-negative pair.

Let P be a consistent positive-negative pair, let T be a tree of positive-negative pairs of P, and let t be an element of $\operatorname{dom} T$. Observe that $\operatorname{succ} t$ is non empty.

Let us consider P, T. The range of T except the root node yields a finite subset of $(\text{the LTLB-WFF})^*_{1-1} \times (\text{the LTLB-WFF})^*_{1-1}$ and is defined as follows:

(Def. 12) The range of T except the root node $= \{T(t); t$ ranges over nodes of T: $t \neq \emptyset\}$.

Let P be a consistent positive-negative pair and let T be a tree of positive-negative pairs of P. One can verify that the range of T except the root node is non empty.

One can prove the following proposition

(28) If $R \in \operatorname{rng} T$ and $Q \in \mathrm{UN}(R)$, then $\operatorname{comp} Q \subseteq$ the range of T except the root node.

One can prove the following proposition

(29) Let P be a consistent complete positive-negative pair, T be a tree of positive-negative pairs of P, and f be a finite sequence of elements of the LTLB-WFF. If $\operatorname{rng} f = \widehat{J}$, then $\emptyset_{\text{the LTLB-WFF}} \vdash \neg((\text{conjunction negation } f)_{\text{len conjunction negation } f}) \Rightarrow \mathcal{X} \neg((\text{conjunction negation } f)_{\text{len conjunction negation } f})$, where $J =$ the range of T except the root node.

5. A Path in PNP-Tree and its Properties. Existence of Temporal Model for a Consistent PNP. Weak Completeness Theorem

Let P be a consistent positive-negative pair and let T be a tree of positive-negative pairs of P. A sequence of $\operatorname{dom} T$ is called a path of T if:

(Def. 13) $\mathrm{It}(0) = \emptyset$ and for every natural number k holds $\mathrm{it}(k+1) \in \operatorname{succ} \mathrm{it}(k)$.

Let P be a consistent complete positive-negative pair, let T be a tree of positive-negative pairs of P, let t be a path of T, and let us consider i. Then $t(i)$ is a node of T.

Next we state three propositions:

(30) Let P be a consistent complete positive-negative pair, T be a tree of positive-negative pairs of P, and t be a path of T. Suppose $A \,\mathcal{U}\, B \in \operatorname{rng}(T(t(i))_{\mathbf{2}})$. Let given j. If $j > i$, then $B \in \operatorname{rng}(T(t(j))_{\mathbf{2}})$ or there exists k such that $i < k < j$ and $A \in \operatorname{rng}(T(t(k))_{\mathbf{2}})$.

(31) Let P be a consistent complete positive-negative pair and T be a tree of positive-negative pairs of P. Suppose $A\,\mathcal{U}\,B \in \mathrm{rng}(P_{\mathbf{1}})$ and for every element Q of the range of T except the root node holds $B \notin \mathrm{rng}(Q_{\mathbf{1}})$. Let Q be an element of the range of T except the root node. Then $B \in \mathrm{rng}(Q_{\mathbf{2}})$ and $A\,\mathcal{U}\,B \in \mathrm{rng}(Q_{\mathbf{1}})$.

(32) Let P be a consistent complete positive-negative pair and T be a tree of positive-negative pairs of P. Suppose $A\,\mathcal{U}\,B \in \mathrm{rng}(P_{\mathbf{1}})$. Then there exists an element R of the range of T except the root node such that $B \in \mathrm{rng}(R_{\mathbf{1}})$.

Let P be a consistent positive-negative pair, let T be a tree of positive-negative pairs of P, and let t be a path of T. We say that t is complete if and only if the condition (Def. 14) is satisfied.

(Def. 14) Let given i. Suppose $A\,\mathcal{U}\,B \in \mathrm{rng}(T(t(i))_{\mathbf{1}})$. Then there exists j such that $j > i$ and $B \in \mathrm{rng}(T(t(j))_{\mathbf{1}})$ and for every k such that $i < k < j$ holds $A \in \mathrm{rng}(T(t(k))_{\mathbf{1}})$.

Let P be a consistent complete positive-negative pair and let T be a tree of positive-negative pairs of P. Note that there exists a path of T which is complete.

Let P be a consistent positive-negative pair. Observe that $\widehat{P}$ is satisfiable.

One can prove the following proposition

(33)[3] If $F \models A$, then $F \vdash A$.

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