

# The Properties of Sets of Temporal Logic Subformulas<sup>1</sup>

Mariusz Giero<sup>2</sup>

Department of Logic, Informatics and Philosophy of Science  
University of Białystok  
Plac Uniwersytecki 1, 15-420 Białystok, Poland

**Summary.** This is a second preliminary article to prove the completeness theorem of an extension of basic propositional temporal logic. We base it on the proof of completeness for basic propositional temporal logic given in [17]. We introduce two modified definitions of a subformula. In the former one we treat until-formula as indivisible. In the latter one, we extend the set of subformulas of until-formulas by a special disjunctive formula. This is needed to construct a temporal model. We also define an ordered positive-negative pair of finite sequences of formulas (PNP). PNPs represent states of a temporal model.

MML identifier: LTLAXI03, version: 7.14.01 4.183.1153

The notation and terminology used here have been introduced in the following papers: [21], [11], [24], [18], [4], [1], [26], [8], [22], [27], [10], [20], [2], [3], [5], [9], [12], [19], [6], [7], [16], [15], [23], [25], [13], and [14].

## 1. PRELIMINARIES

For simplicity, we adopt the following convention:  $A, B, p, q, r$  are elements of the LTLB-WFF,  $n$  is an element of  $\mathbb{N}$ ,  $X$  is a subset of the LTLB-WFF,  $g$  is a function from the LTLB-WFF into *Boolean*, and  $x$  is a set.

Next we state two propositions:

---

<sup>1</sup>This work has been supported by the Polish Ministry of Science and Higher Education project “Managing a Large Repository of Computer-verified Mathematical Knowledge” (N N519 385136).

<sup>2</sup>I would like to thank Prof. Andrzej Trybulec, Dr. Artur Kornilowicz, Dr. Adam Naumowicz and Karol Pąk for their help in preparation of the article.

- (1) Let  $X$  be a non empty set,  $t$  be a finite sequence of elements of  $X$ , and  $k$  be a natural number. If  $k + 1 \leq \text{len } t$ , then  $t_{|k} = \langle t(k + 1) \rangle \wedge (t_{|k+1})$ .
- (2)  $\mathbb{N} \mapsto \emptyset$  is a LTL Model.

Let us consider  $X$ . We say that  $X$  is without implication if and only if:

- (Def. 1) For every  $p$  such that  $p \in X$  holds  $p$  is not conditional.

Let  $D$  be a set. The functor  $D_{1-1}^*$  yielding a set is defined by:

- (Def. 2) For every  $x$  holds  $x \in D_{1-1}^*$  iff  $x$  is a one-to-one finite sequence of elements of  $D$ .

Let  $D$  be a set. One can verify that  $D_{1-1}^*$  is non empty.

Let  $D$  be a finite set. Observe that  $D_{1-1}^*$  is finite.

We now state the proposition

- (3) For all sets  $D_1, D_2$  such that  $D_1 \subseteq D_2$  holds  $D_{1-1}^* \subseteq D_{2-1}^*$ .

Let  $a_1$  be a set and let  $a_2$  be a subset of  $a_1$ . Then  $a_{2-1}^*$  is a non empty subset of  $a_{1-1}^*$ .

Next we state the proposition

- (4) For all one-to-one finite sequences  $F, G$  such that  $\text{rng } F$  misses  $\text{rng } G$  holds  $F \wedge G$  is one-to-one.

Let  $X$  be a set and let  $f, g$  be one-to-one finite sequences of elements of  $X$ .

Let us assume that  $\text{rng } f$  misses  $\text{rng } g$ . The functor  $f \wedge g$  yielding a one-to-one finite sequence of elements of  $X$  is defined as follows:

- (Def. 3)  $f \wedge g = f \wedge g$ .

## 2. SET OF SUBFORMULAS WHERE AN UNTIL-FORMULA IS TREATED AS INDIVISIBLE AND ITS PROPERTIES

The function  $\dot{\tau}$  from the LTLB-WFF into  $2^{\text{the LTLB-WFF}}$  is defined as follows:

- (Def. 4)  $\dot{\tau}(\perp_t) = \{\perp_t\}$  and  $\dot{\tau}(\text{prop } n) = \{\text{prop } n\}$  and  $\dot{\tau}(A \Rightarrow B) = \{A \Rightarrow B\} \cup \dot{\tau}(A) \cup \dot{\tau}(B)$  and  $\dot{\tau}(A \mathcal{U} B) = \{A \mathcal{U} B\}$ .

One can prove the following propositions:

- (5) If  $A$  is not conditional, then  $\dot{\tau}(A) = \{A\}$ .
- (6)  $p \in \dot{\tau}(p)$ .

Let us consider  $p$ . Observe that  $\dot{\tau}(p)$  is non empty and finite.

One can prove the following propositions:

- (7) If  $p \Rightarrow q \in \dot{\tau}(r)$ , then  $p, q \in \dot{\tau}(r)$ .
- (8) If  $p \in \dot{\tau}(q)$ , then  $\dot{\tau}(p) \subseteq \dot{\tau}(q)$ .
- (9) If  $p \mathcal{U} q \in \dot{\tau}(\neg A)$ , then  $p \mathcal{U} q \in \dot{\tau}(A)$ .
- (10) If  $p \mathcal{U} q \in \dot{\tau}(A \&\& B)$ , then  $p \mathcal{U} q \in \dot{\tau}(A)$  or  $p \mathcal{U} q \in \dot{\tau}(B)$ .
- (11) If  $p \in \dot{\tau}(q)$  and  $p \neq q$ , then  $\text{len } p < \text{len } q$ .

- (12)  $\dot{\tau}(p) \subseteq \dot{\tau}(\neg p)$ .
- (13)  $\dot{\tau}(q) \subseteq \dot{\tau}(p \&\& q)$ .
- (14)  $\dot{\tau}(q) \subseteq \dot{\tau}(p \vee q)$ .

Let us consider  $X$ . The functor  $\tau(X)$  yields a subset of the LTLB-WFF and is defined as follows:

(Def. 5)  $x \in \tau(X)$  iff there exists  $A$  such that  $A \in X$  and  $x \in \dot{\tau}(A)$ .

We now state two propositions:

- (15)  $\tau(X) = \bigcup \{\dot{\tau}(p); p \text{ ranges over elements of the LTLB-WFF: } p \in X\}$ .
- (16)  $X \subseteq \tau(X)$ .

Let  $X$  be an empty subset of the LTLB-WFF. One can check that  $\tau(X)$  is empty.

Let  $X$  be a finite subset of the LTLB-WFF. Note that  $\tau(X)$  is finite.

Let  $X$  be a non empty subset of the LTLB-WFF. One can verify that  $\tau(X)$  is non empty.

The following propositions are true:

- (17)  $\tau(\tau(X)) = \tau(X)$ .
- (18) If  $X$  is without implication, then  $\tau(X) = X$ .
- (19) If  $p \Rightarrow q \in \tau(X)$ , then  $p, q \in \tau(X)$ .
- (20) If  $p \&\& q \in \tau(X)$ , then  $p, q \in \tau(X)$ .
- (21) If  $p \vee q \in \tau(X)$ , then  $p, q \in \tau(X)$ .
- (22) If  $\text{UN}(p, q) \in \tau(X)$ , then  $p, q, p \mathcal{U} q \in \tau(X)$ .
- (23) If  $p \in \tau(X)$ , then  $\dot{\tau}(p) \subseteq \tau(X)$ .

### 3. EXTENDED SET OF SUBFORMULAS AND ITS PROPERTIES

The function  $\dot{\sigma}$  from the LTLB-WFF into  $2^{\text{the LTLB-WFF}}$  is defined by:

(Def. 6)  $\dot{\sigma}(\perp_t) = \{\perp_t\}$  and  $\dot{\sigma}(\text{prop } n) = \{\text{prop } n\}$  and  $\dot{\sigma}(A \Rightarrow B) = \{A \Rightarrow B\} \cup \dot{\sigma}(A) \cup \dot{\sigma}(B)$  and  $\dot{\sigma}(A \mathcal{U} B) = \dot{\tau}(\text{UN}(A, B)) \cup \dot{\sigma}(A) \cup \dot{\sigma}(B)$ .

One can prove the following propositions:

- (24)  $p \mathcal{U} q \in \dot{\sigma}(p \mathcal{U} q)$ .
- (25)  $\dot{\tau}(p) \subseteq \dot{\sigma}(p)$ .

Let us consider  $p$ . Note that  $\dot{\sigma}(p)$  is non empty and finite.

The following proposition is true

- (26) If  $p \in \dot{\sigma}(A \mathcal{U} B)$ , then if  $A \mathcal{U} B \in \dot{\sigma}(q)$ , then  $p \in \dot{\sigma}(q)$ .

Let us consider  $X$ . The functor  $\sigma(X)$  yielding a subset of  $2^{\text{the LTLB-WFF}}$  is defined as follows:

(Def. 7)  $\sigma(X) = \{\dot{\sigma}(A); A \text{ ranges over elements of the LTLB-WFF: } A \in X\}$ .

Let  $X$  be a finite subset of the LTLB-WFF. Note that  $\sigma(X)$  is finite and finite-membered.

#### 4. AN ORDERED PAIR OF FINITE SEQUENCES OF FORMULAS. PNP-FORMULA, CONSISTENT PNP AND COMPLETE PNP

A positive-negative pair is an element of  
(the LTLB-WFF) $_{1-1}^* \times$  (the LTLB-WFF) $_{1-1}^*$ .

In the sequel  $P, Q, P_1, R$  are positive-negative pairs.

Let us consider  $P$ . Then  $P_1$  is a one-to-one finite sequence of elements of the LTLB-WFF. Then  $P_2$  is a one-to-one finite sequence of elements of the LTLB-WFF.

Let us consider  $P$ . The functor  $\text{rng } P$  yielding a finite subset of the LTLB-WFF is defined by:

$$\text{(Def. 8) } \text{rng } P = \text{rng}(P_1) \cup \text{rng}(P_2).$$

Let  $f_1, f_2$  be one-to-one finite sequences of elements of the LTLB-WFF. Then  $\langle f_1, f_2 \rangle$  is a positive-negative pair.

Let us consider  $P$ . The functor  $\widehat{P}$  yielding an element of the LTLB-WFF is defined by:

$$\text{(Def. 9) } \widehat{P} = (\text{conjunction}(P_1))_{\text{len conjunction}(P_1)} \&\& (\text{conjunction negation}(P_2))_{\text{len conjunction negation}(P_2)}.$$

We now state three propositions:

$$(27) \quad \widehat{F} = \top_t \&\& \top_t, \text{ where } F = \langle \varepsilon_{(\text{the LTLB-WFF})}, \varepsilon_{(\text{the LTLB-WFF})} \rangle.$$

$$(28) \quad \text{If } A \in \text{rng}(P_1), \text{ then } \emptyset_{\text{the LTLB-WFF}} \vdash \widehat{P} \Rightarrow A.$$

$$(29) \quad \text{If } A \in \text{rng}(P_2), \text{ then } \emptyset_{\text{the LTLB-WFF}} \vdash \widehat{P} \Rightarrow \neg A.$$

Let us consider  $P$ . We say that  $P$  is inconsistent if and only if:

$$\text{(Def. 10) } \emptyset_{\text{the LTLB-WFF}} \vdash \neg \widehat{P}.$$

Let us consider  $P$ . We introduce  $P$  is consistent as an antonym of  $P$  is inconsistent.

We say that  $P$  is complete if and only if:

$$\text{(Def. 11) } \tau(\text{rng } P) = \text{rng } P.$$

One can check that  $\langle \varepsilon_{(\text{the LTLB-WFF})}, \varepsilon_{(\text{the LTLB-WFF})} \rangle$  is consistent as a positive-negative pair.

Let us observe that  $\langle \varepsilon_{(\text{the LTLB-WFF})}, \varepsilon_{(\text{the LTLB-WFF})} \rangle$  is complete as a positive-negative pair.

One can check that there exists a positive-negative pair which is consistent and complete.

Let  $P$  be a consistent positive-negative pair. Observe that  $\langle P_1, P_2 \rangle$  is consistent as a positive-negative pair.

## 5. THE PROPERTIES OF CONSISTENT PNPs

One can prove the following propositions:

- (30) For every consistent positive-negative pair  $P$  holds  $\text{rng}(P_1)$  misses  $\text{rng}(P_2)$ .
- (31) Let  $P$  be a consistent positive-negative pair. If  $A \notin \text{rng } P$ , then  $\langle (P_1) \frown \langle A \rangle, P_2 \rangle$  is consistent or  $\langle P_1, (P_2) \frown \langle A \rangle \rangle$  is consistent.
- (32) For every consistent positive-negative pair  $P$  holds  $\perp_t \notin \text{rng}(P_1)$ .
- (33) Let  $P$  be a consistent positive-negative pair. Suppose  $A, B, A \Rightarrow B \in \text{rng } P$ . Then  $A \Rightarrow B \in \text{rng}(P_1)$  if and only if  $A \in \text{rng}(P_2)$  or  $B \in \text{rng}(P_1)$ .
- (34) Let  $P$  be a consistent positive-negative pair. Then there exists a consistent positive-negative pair  $P_1$  such that  $\text{rng}(P_1) \subseteq \text{rng}((P_1)_1)$  and  $\text{rng}(P_2) \subseteq \text{rng}((P_1)_2)$  and  $\tau(\text{rng } P) = \text{rng } P_1$ .

## REFERENCES

- [1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(2):377–382, 1990.
- [2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [3] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(2):421–427, 1990.
- [4] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [5] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(3):397–402, 1991.
- [6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(3):529–536, 1990.
- [8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [10] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [11] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [13] Mariusz Giero. The axiomatization of propositional linear time temporal logic. *Formalized Mathematics*, 19(2):113–119, 2011, doi: 10.2478/v10037-011-0018-1.
- [14] Mariusz Giero. The derivations of temporal logic formulas. *Formalized Mathematics*, 20(3):215–219, 2012, doi: 10.2478/v10037-012-0025-x.
- [15] Adam Grabowski. Hilbert positive propositional calculus. *Formalized Mathematics*, 8(1):69–72, 1999.
- [16] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(2):275–278, 1992.
- [17] Fred Kröger and Stephan Merz. *Temporal Logic and State Systems*. Springer-Verlag, 2008.
- [18] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [19] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [20] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(1):115–122, 1990.
- [21] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(1):25–34, 1990.
- [22] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(1):97–105, 1990.

- [23] Andrzej Trybulec. Defining by structural induction in the positive propositional language. *Formalized Mathematics*, 8(1):133–137, 1999.
- [24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [25] Edmund Woronowicz. Many argument relations. *Formalized Mathematics*, 1(4):733–737, 1990.
- [26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

*Received May 7, 2012*

---