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Routh's, Menelaus' and Generalized Ceva's Theorems

Boris A. Shminke
Shakhtyorskaya 2
453850 Meleuz, Russia

Summary. The goal of this article is to formalize Ceva's theorem that is in the [8] on the web. Alongside with it formalizations of Routh's, Menelaus' and generalized form of Ceva's theorem itself are provided.

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The papers [1], [4], [3], [6], [5], [2], [7], and [9] provide the notation and terminology for this paper.

1. Some Properties of the Area of Triangle

We use the following convention: A, B, C, A_1, B_1, C_1, A_2, B_2, C_2 are points of $\mathcal{E}^2_{\rm T}$, l_1, m_1, n_1 are real numbers, and X, Y, Z are subsets of $\mathcal{E}^2_{\rm T}$.

Let us consider X, Y. We introduce X is parallel to Y as a synonym of X misses Y.

Let us consider X, Y, Z. We say that X, Y, Z are concurrent if and only if:

(Def. 1) X is parallel to Y and Y is parallel to Z and Z is parallel to X or there exists A such that $A \in X$ and $A \in Y$ and $A \in Z$.

One can prove the following propositions:

(1) $(A+B)_{\mathbf{1}} = A_{\mathbf{1}} + B_{\mathbf{1}}$ and $(A+B)_{\mathbf{2}} = A_{\mathbf{2}} + B_{\mathbf{2}}$.

(2) $(l_1 \cdot A)_{\mathbf{1}} = l_1 \cdot A_{\mathbf{1}}$ and $(l_1 \cdot A)_{\mathbf{2}} = l_1 \cdot A_{\mathbf{2}}$.

(3) $(-A)_{\mathbf{1}} = -A_{\mathbf{1}}$ and $(-A)_{\mathbf{2}} = -A_{\mathbf{2}}$.

(4) $(l_1 \cdot A + m_1 \cdot B)_{\mathbf{1}} = l_1 \cdot A_{\mathbf{1}} + m_1 \cdot B_{\mathbf{1}}$ and $(l_1 \cdot A + m_1 \cdot B)_{\mathbf{2}} = l_1 \cdot A_{\mathbf{2}} + m_1 \cdot B_{\mathbf{2}}$.

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(5) $((-l_1) \cdot A)_{\mathbf{1}} = -l_1 \cdot A_{\mathbf{1}}$ and $((-l_1) \cdot A)_{\mathbf{2}} = -l_1 \cdot A_{\mathbf{2}}$.

(6) $(l_1 \cdot A - m_1 \cdot B)_{\mathbf{1}} = l_1 \cdot A_{\mathbf{1}} - m_1 \cdot B_{\mathbf{1}}$ and $(l_1 \cdot A - m_1 \cdot B)_{\mathbf{2}} = l_1 \cdot A_{\mathbf{2}} - m_1 \cdot B_{\mathbf{2}}$.

(7) The area of $\triangle((1-l_1)\cdot A+l_1\cdot A_1, B, C) = (1-l_1)\cdot$the area of $\triangle(A, B, C)+ l_1 \cdot$ the area of $\triangle(A_1, B, C)$.

(8) If $\measuredangle(A, B, C) = 0$ and A, B, C are mutually different, then $\measuredangle(B, C, A) = \pi$ or $\measuredangle(B, A, C) = \pi$.

(9) A, B and C are collinear iff the area of $\triangle(A, B, C) = 0$.

(10) The area of $\triangle(0_{\mathcal{E}^2_{\mathrm{T}}}, B, C) = \frac{B_{\mathbf{1}} \cdot C_{\mathbf{2}} - C_{\mathbf{1}} \cdot B_{\mathbf{2}}}{2}$.

(11) The area of $\triangle(A + A_1, B, C) =$ ((the area of $\triangle(A, B, C)$) + (the area of $\triangle(A_1, B, C)$)) − the area of $\triangle(0_{\mathcal{E}^2_{\mathrm{T}}}, B, C)$.

(12) If $A \in \mathcal{L}(B, C)$, then $A \in \mathrm{Line}(B, C)$.

(13) If $B \neq C$, then A, B and C are collinear iff $A \in \mathrm{Line}(B, C)$.

(14) If A, B, C form a triangle and $A_1 = (1 - l_1) \cdot B + l_1 \cdot C$, then $A \neq A_1$.

(15) Suppose A, B, C form a triangle. Then
 (i) A, C, B form a triangle,
 (ii) B, A, C form a triangle,
 (iii) B, C, A form a triangle,
 (iv) C, A, B form a triangle, and
 (v) C, B, A form a triangle.

(16) Suppose A, B, C form a triangle and $A_1 = (1 - l_1) \cdot B + l_1 \cdot C$ and $B_1 = (1 - m_1) \cdot C + m_1 \cdot A$ and $m_1 \neq 1$. Then $(1 - m_1) + l_1 \cdot m_1 \neq 0$ if and only if $\mathrm{Line}(A, A_1)$ is not parallel to $\mathrm{Line}(B, B_1)$.

2. Ceva's Theorem and Others

The following propositions are true:

(17) Suppose $A_1 = (1 - l_1) \cdot B + l_1 \cdot C$ and $B_1 = (1 - m_1) \cdot C + m_1 \cdot A$ and $C_1 = (1 - n_1) \cdot A + n_1 \cdot B$. Then the area of $\triangle(A_1, B_1, C_1) = ((1 - l_1) \cdot (1 - m_1) \cdot (1 - n_1) + l_1 \cdot m_1 \cdot n_1) \cdot$ the area of $\triangle(A, B, C)$.

(18) Suppose A, B, C form a triangle and $A_1 = (1 - l_1) \cdot B + l_1 \cdot C$ and $B_1 = (1 - m_1) \cdot C + m_1 \cdot A$ and $C_1 = (1 - n_1) \cdot A + n_1 \cdot B$ and $l_1 \neq 1$ and $m_1 \neq 1$ and $n_1 \neq 1$. Then A_1, B_1 and C_1 are collinear if and only if $\frac{l_1}{1-l_1} \cdot \frac{m_1}{1-m_1} \cdot \frac{n_1}{1-n_1} = -1$.

(19) Suppose that A, B, C form a triangle and $A_1 = (1 - l_1) \cdot B + l_1 \cdot C$ and $B_1 = (1 - m_1) \cdot C + m_1 \cdot A$ and $C_1 = (1 - n_1) \cdot A + n_1 \cdot B$ and $l_1 \neq 1$ and $m_1 \neq 1$ and $n_1 \neq 1$ and A, A_1 and C_2 are collinear and B, B_1 and C_2 are collinear and B, B_1 and A_2 are collinear and C, C_1 and A_2 are collinear and A, A_1 and B_2 are collinear and C, C_1 and B_2 are collinear. Then

(i) $((1-m_1)+l_1\cdot m_1)\cdot((1-l_1)+n_1\cdot l_1)\cdot((1-n_1)+m_1\cdot n_1)\neq 0$, and

(ii) the area of $\triangle(A_2,B_2,C_2)=\frac{(m_1\cdot n_1\cdot l_1-(1-m_1)\cdot(1-n_1)\cdot(1-l_1))^{\mathbf{2}}}{((1-m_1)+l_1\cdot m_1)\cdot((1-l_1)+n_1\cdot l_1)\cdot((1-n_1)+m_1\cdot n_1)}\cdot$ the area of $\triangle(A,B,C)$.

(20) Suppose that A, B, C form a triangle and $A_1=\frac{2}{3}\cdot B+\frac{1}{3}\cdot C$ and $B_1=\frac{2}{3}\cdot C+\frac{1}{3}\cdot A$ and $C_1=\frac{2}{3}\cdot A+\frac{1}{3}\cdot B$ and A, A_1 and C_2 are collinear and B, B_1 and C_2 are collinear and B, B_1 and A_2 are collinear and C, C_1 and A_2 are collinear and A, A_1 and B_2 are collinear and C, C_1 and B_2 are collinear. Then the area of $\triangle(A_2,B_2,C_2)=\frac{\text{the area of }\triangle(A,B,C)}{7}$.

(21) Suppose that A, B, C form a triangle and $A_1=(1-l_1)\cdot B+l_1\cdot C$ and $B_1=(1-m_1)\cdot C+m_1\cdot A$ and $C_1=(1-n_1)\cdot A+n_1\cdot B$ and $l_1\neq 1$ and $m_1\neq 1$ and $n_1\neq 1$ and $(1-m_1)+l_1\cdot m_1\neq 0$ and $(1-l_1)+n_1\cdot l_1\neq 0$ and $(1-n_1)+m_1\cdot n_1\neq 0$. Then $\frac{l_1}{1-l_1}\cdot\frac{m_1}{1-m_1}\cdot\frac{n_1}{1-n_1}=1$ if and only if there exists A_2 such that A, A_1 and A_2 are collinear and B, B_1 and A_2 are collinear and C, C_1 and A_2 are collinear.

(22) Suppose A, B, C form a triangle and $A_1=(1-l_1)\cdot B+l_1\cdot C$ and $B_1=(1-m_1)\cdot C+m_1\cdot A$ and $C_1=(1-n_1)\cdot A+n_1\cdot B$ and $l_1\neq 1$ and $m_1\neq 1$ and $n_1\neq 1$. Then $\frac{l_1}{1-l_1}\cdot\frac{m_1}{1-m_1}\cdot\frac{n_1}{1-n_1}=1$ if and only if $\mathrm{Line}(A,A_1)$, $\mathrm{Line}(B,B_1)$, $\mathrm{Line}(C,C_1)$ are concurrent.

References

[1] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[3] Akihiro Kubo. Lines in n-dimensional Euclidean spaces. *Formalized Mathematics*, 11(**4**):371–376, 2003.
[4] Akihiro Kubo and Yatsuka Nakamura. Angle and triangle in Euclidian topological space. *Formalized Mathematics*, 11(**3**):281–287, 2003.
[5] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[6] Marco Riccardi. Heron's formula and Ptolemy's theorem. *Formalized Mathematics*, 16(**2**):97–101, 2008, doi:10.2478/v10037-008-0014-2.
[7] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[8] Freek Wiedijk. Formalizing 100 theorems. `http://www.cs.ru.nl/~freek/100/`.
[9] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

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