The Borsuk-Ulam Theorem

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Summary. The Borsuk-Ulam theorem about antipodals is proven, [18, pp. 32–33].

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The notation and terminology used here have been introduced in the following papers: [33], [36], [15], [16], [2], [5], [28], [35], [13], [26], [20], [30], [4], [34], [6], [7], [8], [38], [27], [1], [3], [9], [29], [31], [19], [41], [42], [39], [11], [43], [37], [40], [25], [32], [14], [23], [24], [22], [12], [21], [17], and [10].

1. Preliminaries

For simplicity, we adopt the following rules: $a, b, x, y, z, X, Y, Z$ denote sets, $n$ denotes a natural number, $i$ denotes an integer, $r, r_1, r_2, r_3, s$ denote real numbers, $c, c_1, c_2$ denote complex numbers, and $p$ denotes a point of $E^2_T$.

Let us observe that every element of $\mathbb{Q}$ is irrational.

Next we state a number of propositions:

(1) If $0 \leq r$ and $0 \leq s$ and $r^2 = s^2$, then $r = s$.

(2) If $\frac{r}{s} \geq \frac{s}{s}$, then $\frac{r}{r} = \frac{r}{r} - \frac{s}{s}$.

(3) If $\frac{r}{s} < \frac{s}{s}$, then $\frac{r}{r} = (\frac{r}{r} - \frac{s}{s}) + 1$.

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There exists \( i \) such that \( \text{frac}(r - s) = (\text{frac} r - \text{frac} s) + i \) but \( i = 0 \) or \( i = 1 \).

If \( \sin r = 0 \), then \( r = 2 \cdot \pi \cdot \lfloor \frac{r}{2\pi} \rfloor \) or \( r = \pi + 2 \cdot \pi \cdot \lfloor \frac{r}{2\pi} \rfloor \).

If \( \cos r = 0 \), then \( r = \frac{\pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2\pi} \rfloor \) or \( r = \frac{3\pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2\pi} \rfloor \).

If \( \sin r = 0 \), then there exists \( i \) such that \( r = \pi \cdot i \).

If \( \cos r = 0 \), then there exists \( i \) such that \( r = \frac{\pi}{2} + \pi \cdot i \).

If \( \sin r = \sin s \), then there exists \( i \) such that \( r = s + 2 \cdot \pi \cdot i \) or \( r = (\pi - s) + 2 \cdot \pi \cdot i \).

If \( \cos r = \cos s \), then there exists \( i \) such that \( r = s + 2 \cdot \pi \cdot i \) or \( r = -s + 2 \cdot \pi \cdot i \).

If \( |c_1| = |c_2| \) and \( \text{Arg} c_1 = \text{Arg} c_2 + 2 \cdot \pi \cdot i \), then \( c_1 = c_2 \).

Let \( f \) be a one-to-one complex-valued function and let us consider \( c \). One can verify that \( f + c \) is one-to-one.

Let \( f \) be a one-to-one complex-valued function and let us consider \( c \). Note that \( f - c \) is one-to-one.

One can prove the following propositions:

13. For every complex-valued finite sequence \( f \) holds \( \text{len}(-f) = \text{len} f \).

14. \( -\langle 0, \ldots, 0 \rangle = \langle 0, \ldots, 0 \rangle \).

15. For every complex-valued function \( f \) such that \( f \neq \langle 0, \ldots, 0 \rangle \) holds \( -f \neq \langle 0, \ldots, 0 \rangle \).

16. \( 2^2 \langle r_1, r_2, r_3 \rangle = \langle r_1^2, r_2^2, r_3^2 \rangle \).

17. \( \sum^2(r_1, r_2, r_3) = r_1^2 + r_2^2 + r_3^2 \).

18. For every complex-valued finite sequence \( f \) holds \( (c \cdot f)^2 = c^2 \cdot f^2 \).

19. For every complex-valued finite sequence \( f \) holds \( (f/c)^2 = f^2/c^2 \).

20. For every real-valued finite sequence \( f \) such that \( \sum f \neq 0 \) holds \( \sum (f/\sum f) = 1 \).

Let \( a, b, c, x, y, z \) be sets. The functor \( [a \mapsto x, b \mapsto y, c \mapsto z] \) is defined by:

(Def. 1) \( [a \mapsto x, b \mapsto y, c \mapsto z] = [a \mapsto x, b \mapsto y] + (c \mapsto z) \).

Let \( a, b, c, x, y, z \) be sets. One can check that \( [a \mapsto x, b \mapsto y, c \mapsto z] \) is function-like and relation-like.

The following propositions are true:

21. \( \text{dom}([a \mapsto x, b \mapsto y, c \mapsto z]) = \{a, b, c\} \).

22. \( \text{rng}([a \mapsto x, b \mapsto y, c \mapsto z]) \subseteq \{x, y, z\} \).

23. \( [a \mapsto x, a \mapsto y, a \mapsto z] = [a \mapsto z] \).

24. \( [a \mapsto x, a \mapsto y, b \mapsto z] = [a \mapsto y, b \mapsto z] \).

25. If \( a \neq b \), then \( [a \mapsto x, b \mapsto y, a \mapsto z] = [a \mapsto z, b \mapsto y] \).
(26) \([a \mapsto x, b \mapsto y, b \mapsto z] = [a \mapsto x, b \mapsto z].\)

(27) If \(a \neq b\) and \(a \neq c\), then \(((a \mapsto x, b \mapsto y, c \mapsto z))(a) = x.\)

(28) If \(a, b, c\) are mutually different, then \(((a \mapsto x, b \mapsto y, c \mapsto z))(a) = x\) and \(((a \mapsto x, b \mapsto y, c \mapsto z))(b) = y\) and \(((a \mapsto x, b \mapsto y, c \mapsto z))(c) = z.\)

(29) For every function \(f\) such that \(\text{dom } f = \{a, b, c\}\) and \(f(a) = x\) and \(f(b) = y\) and \(f(c) = z\) holds \(f = [a \mapsto x, b \mapsto y, c \mapsto z].\)

(30) \((a, b, c) = [1 \mapsto a, 2 \mapsto b, 3 \mapsto c].\)

(31) If \(a, b, c\) are mutually different, then \(\prod([a \mapsto \{x\}, b \mapsto \{y\}, c \mapsto \{z\}]) =\)
\(\{[a \mapsto x, b \mapsto y, c \mapsto z]\}.\)

(32) For all sets \(A, B, C, D, E, F\) such that \(A \subseteq B\) and \(C \subseteq D\) and \(E \subseteq F\) holds \(\prod([a \mapsto A, b \mapsto C, c \mapsto E]) \subseteq \prod([a \mapsto B, b \mapsto D, c \mapsto F]).\)

(33) If \(a, b, c\) are mutually different and \(x \in X\) and \(y \in Y\) and \(z \in Z\), then \(\{a \mapsto x, b \mapsto y, c \mapsto z\} \in \prod([a \mapsto X, b \mapsto Y, c \mapsto Z]).\)

Let \(f\) be a function. We say that \(f\) is odd if and only if:

(Def. 2) For all complex-valued functions \(x, y\) such that \(x, -x \in \text{dom } f\) and \(y = f(x)\) holds \(f(-x) = -y.\)

Let us mention that \(\emptyset\) is odd.

Let us observe that there exists a function which is odd and complex-functions-valued.

The following propositions are true:

(34) For every point \(p\) of \(\mathbb{C}^3\) holds \(\sum^2 p = \langle(p_1)^2, (p_2)^2, (p_3)^2\rangle.\)

(35) For every point \(p\) of \(\mathbb{C}^3\) holds \(\sum^2 p = (p_1)^2 + (p_2)^2 + (p_3)^2.\)

The following two propositions are true:

(36) For every subset \(S\) of \(\mathbb{R}^1\) such that \(S = \mathbb{Q}\) holds \(\mathbb{Q} \cap ]-\infty, r[\) is an open subset of \(\mathbb{R}^1|S.\)

(37) For every subset \(S\) of \(\mathbb{R}^1\) such that \(S = \mathbb{Q}\) holds \(\mathbb{Q} \cap ]r, +\infty[\) is an open subset of \(\mathbb{R}^1|S.\)

Let \(X\) be a connected non empty topological space, let \(Y\) be a non empty topological space, and let \(f\) be a continuous function from \(X\) into \(Y\). Note that \(\text{Im } f\) is connected.

Next we state two propositions:

(38) Let \(S\) be a subset of \(\mathbb{R}^1.\) Suppose \(S = \mathbb{Q}.\) Let \(T\) be a connected topological space and \(f\) be a function from \(T\) into \(\mathbb{R}^1|S.\) If \(f\) is continuous, then \(f\) is constant.

(39) Let \(a, b\) be real numbers, \(f\) be a continuous function from \([a, b]\) into \(\mathbb{R}^1,\) and \(g\) be a partial function from \(\mathbb{R}\) to \(\mathbb{R}.\) If \(a \leq b\) and \(f = g,\) then \(g\) is continuous.

Let \(s\) be a point of \(\mathbb{R}^1\) and let \(r\) be a real number. Then \(s + r\) is a point of \(\mathbb{R}^1.\)
Let \( s \) be a point of \( \mathbb{R}^1 \) and let \( r \) be a real number. Then \( s - r \) is a point of \( \mathbb{R}^1 \).

Let \( X \) be a set, let \( f \) be a function from \( X \) into \( \mathbb{R}^1 \), and let us consider \( r \). Then \( f + r \) is a function from \( X \) into \( \mathbb{R}^1 \).

Let \( X \) be a set, let \( f \) be a function from \( X \) into \( \mathbb{R}^1 \), and let us consider \( r \). Then \( f - r \) is a function from \( X \) into \( \mathbb{R}^1 \).

Let \( s, t \) be points of \( \mathbb{R}^1 \), let \( f \) be a path from \( s \) to \( t \), and let \( r \) be a real number. Then \( f + r \) is a path from \( s + r \) to \( t + r \). Then \( f - r \) is a path from \( s - r \) to \( t - r \).

The point \( c[100] \) of TopUnitCircle3 is defined by:

\[
(\text{Def. 3}) \quad c[100] = [1, 0, 0].
\]

The point \( c[-100] \) of TopUnitCircle3 is defined by:

\[
(\text{Def. 4}) \quad c[-100] = [-1, 0, 0].
\]

Next we state several propositions:

\[
(40) \quad -c[100] = c[-100].
\]

\[
(41) \quad -c[-100] = c[100].
\]

\[
(42) \quad c[100] - c[-100] = [2, 0, 0].
\]

\[
(43) \quad \text{For every point } p \text{ of } \mathcal{E}_1^3 \text{ holds } p_1 = |p| \cdot \cos \text{Arg } p \text{ and } p_2 = |p| \cdot \sin \text{Arg } p.
\]

\[
(44) \quad \text{For every point } p \text{ of } \mathcal{E}_1^3 \text{ holds } p = \text{cpx2euc}(|p| \cdot \cos \text{Arg } p + |p| \cdot \sin \text{Arg } p \cdot i).
\]

\[
(45) \quad \text{For all points } p_1, p_2 \text{ of } \mathcal{E}_1^3 \text{ such that } |p_1| = |p_2| \text{ and } \text{Arg } p_1 = \text{Arg } p_2 + 2 \cdot \pi \cdot i \text{ holds } p_1 = p_2.
\]

One can prove the following propositions:

\[
(46) \quad \text{For every point } p \text{ of } \mathcal{E}_1^3 \text{ such that } p = \text{CircleMap}(r) \text{ holds } \text{Arg } p = 2 \cdot \pi \cdot \frac{r}{2}.
\]

\[
(47) \quad \text{Let } p_1, p_2 \text{ be points of } \mathcal{E}_1^3 \text{ and } u_1, u_2 \text{ be points of } \mathcal{E}_3. \text{ If } u_1 = p_1 \text{ and } u_2 = p_2, \text{ then } \rho^3(u_1, u_2) = \sqrt{(p_11 - p_21)^2 + (p_12 - p_22)^2 + (p_13 - p_23)^2}.
\]

\[
(48) \quad \text{Let } p \text{ be a point of } \mathcal{E}_1^3 \text{ and } e \text{ be a point of } \mathcal{E}_3. \text{ If } p = e \text{ and } p_3 = 0, \text{ then } \Pi([1 \mapsto p_1 - \frac{r}{\sqrt{2}}, p_1 + \frac{r}{\sqrt{2}}[2 \mapsto p_2 - \frac{r}{\sqrt{2}}, p_2 + \frac{r}{\sqrt{2}}[3 \mapsto \{0\}]) \subseteq \text{Ball}(e, r).
\]

\[
(49) \quad \text{For every real number } s \text{ holds } c \cup s = c \cup s \cup 2 \cdot \pi \cdot i.
\]

\[
(50) \quad \text{For every real number } s \text{ holds } \text{Rotate } s = \text{Rotate}(s + 2 \cdot \pi \cdot i).
\]

\[
(51) \quad \text{For every real number } s \text{ and for every point } p \text{ of } \mathcal{E}_1^3 \text{ holds } |(\text{Rotate } s)(p)| = |p|.
\]

\[
(52) \quad \text{For every real number } s \text{ and for every point } p \text{ of } \mathcal{E}_1^3 \text{ holds } \text{Arg} \text{Rotate } s(p) = \text{Arg} \text{euc2cpx} (p) \cup s.
\]

\[
(53) \quad \text{For every real number } s \text{ and for every point } p \text{ of } \mathcal{E}_1^3 \text{ such that } p \neq 0_{\mathcal{E}_1^3} \text{ there exists } i \text{ such that } \text{Arg} \text{Rotate } s(p) = s + \text{Arg } p + 2 \cdot \pi \cdot i.
\]

\[
(54) \quad \text{For every real number } s \text{ holds } \text{Rotate } s(0_{\mathcal{E}_1^3}) = 0_{\mathcal{E}_1^3}.
\]
(55) For every real number $s$ and for every point $p$ of $\mathcal{E}^2_T$ such that $(\text{Rotate } s)(p) = 0_{\mathcal{E}^2_T}$ holds $p = 0_{\mathcal{E}^2_T}$.

(56) For every real number $s$ and for every point $p$ of $\mathcal{E}^2_T$ holds $(\text{Rotate } s)((\text{Rotate } (-s))(p)) = p$.

(57) For every real number $s$ holds $\text{Rotate } s \cdot \text{Rotate } (-s) = \text{id}_{\mathcal{E}^2_T}$.

(58) For every real number $s$ and for every point $p$ of $\mathcal{E}^2_T$ holds $p \in \text{Sphere}(0_{\mathcal{E}^2_T}, r)$ iff $(\text{Rotate } s)(p) \in \text{Sphere}(0_{\mathcal{E}^2_T}, r)$.

(59) For every non negative real number $r$ and for every real number $s$ holds $(\text{Rotate } s)^{\circ} \text{Sphere}(0_{\mathcal{E}^2_T}, r) = \text{Sphere}(0_{\mathcal{E}^2_T}, r)$.

Let $r$ be a non negative real number and let $s$ be a real number. The functor $\text{RotateCircle}(r, s)$ yields a function from $\text{Tcircle}(0_{\mathcal{E}^2_T}, r)$ into $\text{Tcircle}(0_{\mathcal{E}^2_T}, r)$ and is defined by:

(Def. 5) $\text{RotateCircle}(r, s) = \text{Rotate } s | \text{Tcircle}(0_{\mathcal{E}^2_T}, r)$.

Let $r$ be a non negative real number and let $s$ be a real number. Note that $\text{RotateCircle}(r, s)$ is homeomorphism.

One can prove the following proposition

(60) For every point $p$ of $\mathcal{E}^2_T$ such that $p = \text{CircleMap}(r_2)$ holds $(\text{RotateCircle}(1, (-\text{Arg } p)))(\text{CircleMap}(r_1)) = \text{CircleMap}(r_1 - r_2)$.

2. ON THE ANTIPODALS

Let $n$ be a non empty natural number, let $p$ be a point of $\mathcal{E}^n_T$, and let $r$ be a non negative real number. The functor $\text{CircleIso}(p, r)$ yields a function from $\text{TopUnitCircle} n$ into $\text{Tcircle}(p, r)$ and is defined as follows:

(Def. 6) For every point $a$ of $\text{TopUnitCircle} n$ and for every point $b$ of $\mathcal{E}^n_T$ such that $a = b$ holds $(\text{CircleIso}(p, r))(a) = r \cdot b + p$.

Let $n$ be a non empty natural number, let $p$ be a point of $\mathcal{E}^n_T$, and let $r$ be a positive real number. Note that $\text{CircleIso}(p, r)$ is homeomorphism.

The function $\text{SphereMap}$ from $\mathbb{R}^1$ into $\text{TopUnitCircle} 3$ is defined by:

(Def. 7) For every real number $x$ holds $(\text{SphereMap})(x) = [\cos(2 \cdot \pi \cdot x), \sin(2 \cdot \pi \cdot x), 0]$.

We now state the proposition

(61) $(\text{SphereMap})(i) = c[100]$.

Let us note that $\text{SphereMap}$ is continuous.

Let $r$ be a real number. The functor $\text{eLoop } r$ yields a function from $\mathbb{I}$ into $\text{TopUnitCircle} 3$ and is defined as follows:

(Def. 8) For every point $x$ of $\mathbb{I}$ holds $(\text{eLoop } r)(x) = [\cos(2 \cdot \pi \cdot r \cdot x), \sin(2 \cdot \pi \cdot r \cdot x), 0]$.

We now state the proposition

(62) $\text{eLoop } r = \text{SphereMap} \cdot \text{ExtendInt } r$. 
Let us consider $i$. Then eLoop $i$ is a loop of $c[100]$. One can check that eLoop $i$ is null-homotopic as a loop of $c[100]$. One can prove the following proposition

(63) If $p \neq 0_{E_T}$, then $|p/p| = 1$.

Let $n$ be a natural number and let $p$ be a point of $E^n_T$. Let us assume that $p \neq 0_{E_T}$. The functor $(R^n \to S^1)^n$ yields a point of $Tcircle(0_{E^n}, 1)$ and is defined by:

(Def. 9) $(R^n \to S^1)^n p = p/|p|.$

Let $n$ be a non zero natural number and let $f$ be a function from $Tcircle(0, 1)$ into $E^n_T$. The functor $(S^{n+1} \to S^n) f$ yielding a function from $TopUnitCircle(n + 1)$ into $TopUnitCircle n$ is defined as follows:

(Def. 10) For all points $x, y$ of $Tcircle(0, 1)$ such that $y = -x$ holds

$\langle (S^{n+1} \to S^n) f\rangle(x) = (R^n \to S^1)(f(x) - f(y)).$

Let $x_0, y_0$ be points of $TopUnitCircle 2$, let $x_1$ be a set, and let $f$ be a path from $x_0$ to $y_0$. Let us assume that $x_1 \in CircleMap^{-1}(\{x_0\})$. The functor liftPath($f, x_1$) yielding a function from $I$ into $R^1$ is defined by the conditions (Def. 11).

(Def. 11)(i) liftPath($f, x_1$))($0$) = $x_1$,

(ii) $f = CircleMap \cdot liftPath(f, x_1),$

(iii) liftPath($f, x_1$) is continuous, and

(iv) for every function $f_1$ from $I$ into $R^1$ such that $f_1$ is continuous and

$f = CircleMap \cdot f_1$ and $f_1(0) = x_1$ holds liftPath($f, x_1$) = $f_1$.

Let $n$ be a natural number, let $p, x, y$ be points of $E^n_T$, and let $r$ be a real number. We say that $x$ and $y$ are antipodals of $p$ and $r$ if and only if:

(Def. 12) $x$ is a point of $Tcircle(p, r)$ and $y$ is a point of $Tcircle(p, r)$ and $p \in \mathcal{L}(x, y)$.

Let $n$ be a natural number, let $p, x, y$ be points of $E^n_T$, let $r$ be a real number, and let $f$ be a function. We say that $x$ and $y$ are antipodals of $p, r$ and $f$ if and only if:

(Def. 13) $x$ and $y$ are antipodals of $p$ and $r$ and $x, y \in dom f$ and $f(x) = f(y)$.

Let $m, n$ be natural numbers, let $p$ be a point of $E^m_T$, let $r$ be a real number, and let $f$ be a function from $Tcircle(p, r)$ into $E^n_T$. We say that $f$ has antipodals if and only if:

(Def. 14) There exist points $x, y$ of $E^m_T$ such that $x$ and $y$ are antipodals of $p, r$ and $f$.

Let $m, n$ be natural numbers, let $p$ be a point of $E^m_T$, let $r$ be a real number, and let $f$ be a function from $Tcircle(p, r)$ into $E^n_T$. We introduce $f$ is without antipodals as an antonym of $f$ has antipodals.

One can prove the following propositions:
(64) Let \( n \) be a non empty natural number, \( r \) be a non negative real number, and \( x \) be a point of \( E^n_T \). Suppose \( x \) is a point of Tcircle(0, \( E^n_T \), \( r \)). Then \( x \) and \(-x\) are antipodals of 0 \( E^n_T \) and \( r \).

(65) Let \( n \) be a non empty natural number, \( p \), \( x \), \( y \), \( x_2 \), \( y_1 \) be points of \( E^n_T \), and \( r \) be a positive real number. Suppose \( x \) and \( y \) are antipodals of 0 \( E^n_T \) and 1 and \( x_2 = (CircleIso(p, r))(x) \) and \( y_1 = (CircleIso(p, r))(y) \). Then \( x_2 \) and \( y_1 \) are antipodals of \( p \) and \( r \).

(66) Let \( f \) be a function from Tcircle(0, \( E^n_{n+1} \), 1) into \( E^n_T \) and \( x \) be a point of Tcircle(0, \( E^n_{n+1} \), 1). If \( f \) is without antipodals, then \( f(x) - f(-x) \neq 0 \) \( E^n_T \).

(67) For every function \( f \) from Tcircle(0, \( E^n_{n+1} \), 1) into \( E^n_T \) such that \( f \) is without antipodals holds \((S^{n+1} \rightarrow S^n) f \) is odd.

(68) Let \( f \) be a function from Tcircle(0, \( E^n_{n+1} \), 1) into \( E^n_T \) and \( g \), \( B_1 \) be functions from Tcircle(0, \( E^n_{n+1} \), 1) into \( E^n_T \). If \( g = f \circ - \) and \( B_1 = f - g \) and \( f \) is without antipodals, then \((S^{n+1} \rightarrow S^n) f = B_1/(n \cdot NormF \cdot B_1) \).

Let us consider \( n \), let \( r \) be a negative real number, and let \( p \) be a point of \( E^n_{n+1} \). Observe that every function from Tcircle(\( p, r \)) into \( E^n_T \) is without antipodals.

Let \( r \) be a non negative real number and let \( p \) be a point of \( E^2_T \). Note that every function from Tcircle(\( p, r \)) into \( E^2_T \) which is continuous also has antipodals.\(^2\)

REFERENCES


\(^2\)The Borsuk-Ulam Theorem

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