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The Borsuk-Ulam Theorem

Artur Korniłowicz[1]
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok
Poland

Marco Riccardi
Via del Pero 102
54038 Montignoso
Italy

Summary. The Borsuk-Ulam theorem about antipodals is proven, [18, pp. 32–33].

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The notation and terminology used here have been introduced in the following papers: [33], [36], [15], [16], [2], [5], [28], [35], [13], [26], [20], [30], [4], [34], [6], [7], [8], [38], [27], [1], [3], [9], [29], [31], [19], [41], [42], [39], [11], [43], [37], [40], [25], [32], [14], [23], [24], [22], [12], [21], [17], and [10].

1. Preliminaries

For simplicity, we adopt the following rules: a, b, x, y, z, X, Y, Z denote sets, n denotes a natural number, i denotes an integer, r, r_1, r_2, r_3, s denote real numbers, c, c_1, c_2 denote complex numbers, and p denotes a point of $\mathcal{E}^n_{\mathrm{T}}$.

Let us observe that every element of $\mathbb{IQ}$ is irrational.

Next we state a number of propositions:

(1) If $0 \leq r$ and $0 \leq s$ and $r^{\mathbf{2}} = s^{\mathbf{2}}$, then $r = s$.

(2) If $\mathrm{frac}\, r \geq \mathrm{frac}\, s$, then $\mathrm{frac}(r - s) = \mathrm{frac}\, r - \mathrm{frac}\, s$.

(3) If $\mathrm{frac}\, r < \mathrm{frac}\, s$, then $\mathrm{frac}(r - s) = (\mathrm{frac}\, r - \mathrm{frac}\, s) + 1$.

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(4) There exists i such that $\operatorname{frac}(r-s) = (\operatorname{frac} r - \operatorname{frac} s) + i$ but $i = 0$ or $i = 1$.

(5) If $\sin r = 0$, then $r = 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$ or $r = \pi + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$.

(6) If $\cos r = 0$, then $r = \frac{\pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$ or $r = \frac{3 \cdot \pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$.

(7) If $\sin r = 0$, then there exists i such that $r = \pi \cdot i$.

(8) If $\cos r = 0$, then there exists i such that $r = \frac{\pi}{2} + \pi \cdot i$.

(9) If $\sin r = \sin s$, then there exists i such that $r = s + 2 \cdot \pi \cdot i$ or $r = (\pi - s) + 2 \cdot \pi \cdot i$.

(10) If $\cos r = \cos s$, then there exists i such that $r = s + 2 \cdot \pi \cdot i$ or $r = -s + 2 \cdot \pi \cdot i$.

(11) If $\sin r = \sin s$ and $\cos r = \cos s$, then there exists i such that $r = s + 2 \cdot \pi \cdot i$.

(12) If $|c_1| = |c_2|$ and $\operatorname{Arg} c_1 = \operatorname{Arg} c_2 + 2 \cdot \pi \cdot i$, then $c_1 = c_2$.

Let f be a one-to-one complex-valued function and let us consider c. One can verify that $f + c$ is one-to-one.

Let f be a one-to-one complex-valued function and let us consider c. Note that $f - c$ is one-to-one.

One can prove the following propositions:

(13) For every complex-valued finite sequence f holds $\operatorname{len}(-f) = \operatorname{len} f$.

(14) $-\langle \underbrace{0, \ldots, 0}_{n} \rangle = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(15) For every complex-valued function f such that $f \neq \langle \underbrace{0, \ldots, 0}_{n} \rangle$ holds $-f \neq \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(16) ${}^{2}\langle r_1, r_2, r_3 \rangle = \langle {r_1}^{\mathbf{2}}, {r_2}^{\mathbf{2}}, {r_3}^{\mathbf{2}} \rangle$.

(17) $\sum^{2}\langle r_1, r_2, r_3 \rangle = {r_1}^{\mathbf{2}} + {r_2}^{\mathbf{2}} + {r_3}^{\mathbf{2}}$.

(18) For every complex-valued finite sequence f holds $(c \cdot f)^{\mathbf{2}} = c^{\mathbf{2}} \cdot f^{\mathbf{2}}$.

(19) For every complex-valued finite sequence f holds $(f/c)^{\mathbf{2}} = f^{\mathbf{2}}/c^{\mathbf{2}}$.

(20) For every real-valued finite sequence f such that $\sum f \neq 0$ holds $\sum(f/\sum f) = 1$.

Let a, b, c, x, y, z be sets. The functor $[a \mapsto x, b \mapsto y, c \mapsto z]$ is defined by:

(Def. 1) $[a \mapsto x, b \mapsto y, c \mapsto z] = [a \longmapsto x, b \longmapsto y] +\cdot (c \dot{\longmapsto} z)$.

Let a, b, c, x, y, z be sets. One can check that $[a \mapsto x, b \mapsto y, c \mapsto z]$ is function-like and relation-like.

The following propositions are true:

(21) $\operatorname{dom}([a \mapsto x, b \mapsto y, c \mapsto z]) = \{a, b, c\}$.

(22) $\operatorname{rng}([a \mapsto x, b \mapsto y, c \mapsto z]) \subseteq \{x, y, z\}$.

(23) $[a \mapsto x, a \mapsto y, a \mapsto z] = a \dot{\longmapsto} z$.

(24) $[a \mapsto x, a \mapsto y, b \mapsto z] = [a \longmapsto y, b \longmapsto z]$.

(25) If $a \neq b$, then $[a \mapsto x, b \mapsto y, a \mapsto z] = [a \longmapsto z, b \longmapsto y]$.

(26) $[a \mapsto x, b \mapsto y, b \mapsto z] = [a \longmapsto x, b \longmapsto z]$.

(27) If $a \neq b$ and $a \neq c$, then $([a \mapsto x, b \mapsto y, c \mapsto z])(a) = x$.

(28) If a, b, c are mutually different, then $([a \mapsto x, b \mapsto y, c \mapsto z])(a) = x$ and $([a \mapsto x, b \mapsto y, c \mapsto z])(b) = y$ and $([a \mapsto x, b \mapsto y, c \mapsto z])(c) = z$.

(29) For every function f such that $\mathrm{dom}\, f = \{a, b, c\}$ and $f(a) = x$ and $f(b) = y$ and $f(c) = z$ holds $f = [a \mapsto x, b \mapsto y, c \mapsto z]$.

(30) $\langle a, b, c\rangle = [1 \mapsto a, 2 \mapsto b, 3 \mapsto c]$.

(31) If a, b, c are mutually different, then $\prod([a \mapsto \{x\}, b \mapsto \{y\}, c \mapsto \{z\}]) = \{[a \mapsto x, b \mapsto y, c \mapsto z]\}$.

(32) For all sets A, B, C, D, E, F such that $A \subseteq B$ and $C \subseteq D$ and $E \subseteq F$ holds $\prod([a \mapsto A, b \mapsto C, c \mapsto E]) \subseteq \prod([a \mapsto B, b \mapsto D, c \mapsto F])$.

(33) If a, b, c are mutually different and $x \in X$ and $y \in Y$ and $z \in Z$, then $[a \mapsto x, b \mapsto y, c \mapsto z] \in \prod([a \mapsto X, b \mapsto Y, c \mapsto Z])$.

Let f be a function. We say that f is odd if and only if:

(Def. 2) For all complex-valued functions x, y such that x, $-x \in \mathrm{dom}\, f$ and $y = f(x)$ holds $f(-x) = -y$.

Let us mention that $\emptyset$ is odd.

Let us observe that there exists a function which is odd and complex-functions-valued.

The following propositions are true:

(34) For every point p of $\mathcal{E}^3_{\mathrm{T}}$ holds ${}^2p = \langle (p_{\mathbf{1}})^{\mathbf{2}}, (p_{\mathbf{2}})^{\mathbf{2}}, (p_{\mathbf{3}})^{\mathbf{2}}\rangle$.

(35) For every point p of $\mathcal{E}^3_{\mathrm{T}}$ holds $\sum {}^2p = (p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}} + (p_{\mathbf{3}})^{\mathbf{2}}$.

The following two propositions are true:

(36) For every subset S of $\mathbb{R}^{\mathbf{1}}$ such that $S = \mathbb{Q}$ holds $\mathbb{Q} \cap]-\infty, r[$ is an open subset of $\mathbb{R}^{\mathbf{1}} \restriction S$.

(37) For every subset S of $\mathbb{R}^{\mathbf{1}}$ such that $S = \mathbb{Q}$ holds $\mathbb{Q} \cap]r, +\infty[$ is an open subset of $\mathbb{R}^{\mathbf{1}} \restriction S$.

Let X be a connected non empty topological space, let Y be a non empty topological space, and let f be a continuous function from X into Y. Note that $\mathrm{Im}\, f$ is connected.

Next we state two propositions:

(38) Let S be a subset of $\mathbb{R}^{\mathbf{1}}$. Suppose $S = \mathbb{Q}$. Let T be a connected topological space and f be a function from T into $\mathbb{R}^{\mathbf{1}} \restriction S$. If f is continuous, then f is constant.

(39) Let a, b be real numbers, f be a continuous function from $[a, b]_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$, and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. If $a \leq b$ and $f = g$, then g is continuous.

Let s be a point of $\mathbb{R}^{\mathbf{1}}$ and let r be a real number. Then $s + r$ is a point of $\mathbb{R}^{\mathbf{1}}$.

Let s be a point of $\mathbb{R}^{\mathbf{1}}$ and let r be a real number. Then $s-r$ is a point of $\mathbb{R}^{\mathbf{1}}$.

Let X be a set, let f be a function from X into $\mathbb{R}^{\mathbf{1}}$, and let us consider r. Then $f+r$ is a function from X into $\mathbb{R}^{\mathbf{1}}$.

Let X be a set, let f be a function from X into $\mathbb{R}^{\mathbf{1}}$, and let us consider r. Then $f-r$ is a function from X into $\mathbb{R}^{\mathbf{1}}$.

Let s, t be points of $\mathbb{R}^{\mathbf{1}}$, let f be a path from s to t, and let r be a real number. Then $f+r$ is a path from $s+r$ to $t+r$. Then $f-r$ is a path from $s-r$ to $t-r$.

The point c[100] of TopUnitCircle 3 is defined by:

(Def. 3) $\mathrm{c}[100]=[1,0,0]$.

The point c[−100] of TopUnitCircle 3 is defined by:

(Def. 4) $\mathrm{c}[-100]=[-1,0,0]$.

Next we state several propositions:

(40) $-\mathrm{c}[100]=\mathrm{c}[-100]$.

(41) $-\mathrm{c}[-100]=\mathrm{c}[100]$.

(42) $\mathrm{c}[100]-\mathrm{c}[-100]=[2,0,0]$.

(43) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $p_{\mathbf{1}}=|p|\cdot\cos\operatorname{Arg}p$ and $p_{\mathbf{2}}=|p|\cdot\sin\operatorname{Arg}p$.

(44) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $p=\mathrm{cpx2euc}(|p|\cdot\cos\operatorname{Arg}p+|p|\cdot\sin\operatorname{Arg}p\cdot i)$.

(45) For all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $|p_1|=|p_2|$ and $\operatorname{Arg}p_1=\operatorname{Arg}p_2+2\cdot\pi\cdot i$ holds $p_1=p_2$.

One can prove the following propositions:

(46) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p=\mathrm{CircleMap}(r)$ holds $\operatorname{Arg}p=2\cdot\pi\cdot\operatorname{frac}r$.

(47) Let p_1, p_2 be points of $\mathcal{E}^3_{\mathrm{T}}$ and u_1, u_2 be points of $\mathcal{E}^3$. If $u_1=p_1$ and $u_2=p_2$, then $\rho^3(u_1,u_2)=\sqrt{((p_1)_{\mathbf{1}}-(p_2)_{\mathbf{1}})^{\mathbf{2}}+((p_1)_{\mathbf{2}}-(p_2)_{\mathbf{2}})^{\mathbf{2}}+((p_1)_{\mathbf{3}}-(p_2)_{\mathbf{3}})^{\mathbf{2}}}$.

(48) Let p be a point of $\mathcal{E}^3_{\mathrm{T}}$ and e be a point of $\mathcal{E}^3$. If $p=e$ and $p_{\mathbf{3}}=0$, then $\prod([1\mapsto]p_{\mathbf{1}}-\frac{r}{\sqrt{2}},p_{\mathbf{1}}+\frac{r}{\sqrt{2}}[,2\mapsto]p_{\mathbf{2}}-\frac{r}{\sqrt{2}},p_{\mathbf{2}}+\frac{r}{\sqrt{2}}[,3\mapsto\{0\}])\subseteq\mathrm{Ball}(e,r)$.

(49) For every real number s holds $c\circlearrowleft s=c\circlearrowleft s+2\cdot\pi\cdot i$.

(50) For every real number s holds $\mathrm{Rotate}\,s=\mathrm{Rotate}(s+2\cdot\pi\cdot i)$.

(51) For every real number s and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $|(\mathrm{Rotate}\,s)(p)|=|p|$.

(52) For every real number s and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $\operatorname{Arg}(\mathrm{Rotate}\,s)(p)=\operatorname{Arg}(\mathrm{euc2cpx}(p)\circlearrowleft s)$.

(53) For every real number s and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p\neq0_{\mathcal{E}^2_{\mathrm{T}}}$ there exists i such that $\operatorname{Arg}(\mathrm{Rotate}\,s)(p)=s+\operatorname{Arg}p+2\cdot\pi\cdot i$.

(54) For every real number s holds $(\mathrm{Rotate}\,s)(0_{\mathcal{E}^2_{\mathrm{T}}})=0_{\mathcal{E}^2_{\mathrm{T}}}$.

(55) For every real number s and for every point p of $\mathcal{E}^2_{\rm T}$ such that $({\rm Rotate}\, s)(p) = 0_{\mathcal{E}^2_{\rm T}}$ holds $p = 0_{\mathcal{E}^2_{\rm T}}$.

(56) For every real number s and for every point p of $\mathcal{E}^2_{\rm T}$ holds $({\rm Rotate}\, s)(({\rm Rotate}(-s))(p)) = p$.

(57) For every real number s holds ${\rm Rotate}\, s \cdot {\rm Rotate}(-s) = {\rm id}_{\mathcal{E}^2_{\rm T}}$.

(58) For every real number s and for every point p of $\mathcal{E}^2_{\rm T}$ holds $p \in {\rm Sphere}((0_{\mathcal{E}^2_{\rm T}}), r)$ iff $({\rm Rotate}\, s)(p) \in {\rm Sphere}((0_{\mathcal{E}^2_{\rm T}}), r)$.

(59) For every non negative real number r and for every real number s holds $({\rm Rotate}\, s)^\circ\, {\rm Sphere}((0_{\mathcal{E}^2_{\rm T}}), r) = {\rm Sphere}((0_{\mathcal{E}^2_{\rm T}}), r)$.

Let r be a non negative real number and let s be a real number. The functor ${\rm RotateCircle}(r, s)$ yields a function from ${\rm Tcircle}(0_{\mathcal{E}^2_{\rm T}}, r)$ into ${\rm Tcircle}(0_{\mathcal{E}^2_{\rm T}}, r)$ and is defined by:

(Def. 5) ${\rm RotateCircle}(r, s) = {\rm Rotate}\, s \restriction {\rm Tcircle}(0_{\mathcal{E}^2_{\rm T}}, r)$.

Let r be a non negative real number and let s be a real number. Note that ${\rm RotateCircle}(r, s)$ is homeomorphism.

One can prove the following proposition

(60) For every point p of $\mathcal{E}^2_{\rm T}$ such that $p = {\rm CircleMap}(r_2)$ holds $({\rm RotateCircle}(1, (-{\rm Arg}\, p)))({\rm CircleMap}(r_1)) = {\rm CircleMap}(r_1 - r_2)$.

2. On the Antipodals

Let n be a non empty natural number, let p be a point of $\mathcal{E}^n_{\rm T}$, and let r be a non negative real number. The functor ${\rm CircleIso}(p, r)$ yields a function from ${\rm TopUnitCircle}\, n$ into ${\rm Tcircle}(p, r)$ and is defined as follows:

(Def. 6) For every point a of ${\rm TopUnitCircle}\, n$ and for every point b of $\mathcal{E}^n_{\rm T}$ such that $a = b$ holds $({\rm CircleIso}(p, r))(a) = r \cdot b + p$.

Let n be a non empty natural number, let p be a point of $\mathcal{E}^n_{\rm T}$, and let r be a positive real number. Note that ${\rm CircleIso}(p, r)$ is homeomorphism.

The function SphereMap from $\mathbb{R}^{\mathbf{1}}$ into ${\rm TopUnitCircle}\, 3$ is defined by:

(Def. 7) For every real number x holds $({\rm SphereMap})(x) = [\cos(2 \cdot \pi \cdot x), \sin(2 \cdot \pi \cdot x), 0]$.

We now state the proposition

(61) $({\rm SphereMap})(i) = {\rm c}[100]$.

Let us note that SphereMap is continuous.

Let r be a real number. The functor ${\rm eLoop}\, r$ yields a function from $\mathbb{I}$ into ${\rm TopUnitCircle}\, 3$ and is defined as follows:

(Def. 8) For every point x of $\mathbb{I}$ holds $({\rm eLoop}\, r)(x) = [\cos(2 \cdot \pi \cdot r \cdot x), \sin(2 \cdot \pi \cdot r \cdot x), 0]$.

We now state the proposition

(62) ${\rm eLoop}\, r = {\rm SphereMap} \cdot {\rm ExtendInt}\, r$.

Let us consider i. Then eLoop i is a loop of c[100].

One can check that eLoop i is null-homotopic as a loop of c[100].

One can prove the following proposition

(63) If $p \neq 0_{\mathcal{E}^n_{\rm T}}$, then $|p/|p|| = 1$.

Let n be a natural number and let p be a point of $\mathcal{E}^n_{\rm T}$. Let us assume that $p \neq 0_{\mathcal{E}^n_{\rm T}}$. The functor $(R^n \to S^1)\,p$ yields a point of Tcircle$(0_{\mathcal{E}^n_{\rm T}}, 1)$ and is defined by:

(Def. 9) $(R^n \to S^1)\,p = p/|p|$.

Let n be a non zero natural number and let f be a function from Tcircle$(0_{\mathcal{E}^{n+1}_{\rm T}}, 1)$ into $\mathcal{E}^n_{\rm T}$. The functor $(S^{n+1} \to S^n)\,f$ yielding a function from TopUnitCircle$(n+1)$ into TopUnitCircle n is defined as follows:

(Def. 10) For all points x, y of Tcircle$(0_{\mathcal{E}^{n+1}_{\rm T}}, 1)$ such that $y = -x$ holds $((S^{n+1} \to S^n)\,f)(x) = (R^n \to S^1)(f(x) - f(y))$.

Let x_0, y_0 be points of TopUnitCircle 2, let x_1 be a set, and let f be a path from x_0 to y_0. Let us assume that $x_1 \in \text{CircleMap}^{-1}(\{x_0\})$. The functor liftPath$(f, x_1)$ yielding a function from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ is defined by the conditions (Def. 11).

(Def. 11)(i) $(\text{liftPath}(f, x_1))(0) = x_1$,

(ii) $f = \text{CircleMap} \cdot \text{liftPath}(f, x_1)$,

(iii) liftPath(f, x_1) is continuous, and

(iv) for every function f_1 from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that f_1 is continuous and $f = \text{CircleMap} \cdot f_1$ and $f_1(0) = x_1$ holds liftPath$(f, x_1) = f_1$.

Let n be a natural number, let p, x, y be points of $\mathcal{E}^n_{\rm T}$, and let r be a real number. We say that x and y are antipodals of p and r if and only if:

(Def. 12) x is a point of Tcircle(p, r) and y is a point of Tcircle(p, r) and $p \in \mathcal{L}(x, y)$.

Let n be a natural number, let p, x, y be points of $\mathcal{E}^n_{\rm T}$, let r be a real number, and let f be a function. We say that x and y are antipodals of p, r and f if and only if:

(Def. 13) x and y are antipodals of p and r and $x, y \in \text{dom}\, f$ and $f(x) = f(y)$.

Let m, n be natural numbers, let p be a point of $\mathcal{E}^m_{\rm T}$, let r be a real number, and let f be a function from Tcircle(p, r) into $\mathcal{E}^n_{\rm T}$. We say that f has antipodals if and only if:

(Def. 14) There exist points x, y of $\mathcal{E}^m_{\rm T}$ such that x and y are antipodals of p, r and f.

Let m, n be natural numbers, let p be a point of $\mathcal{E}^m_{\rm T}$, let r be a real number, and let f be a function from Tcircle(p, r) into $\mathcal{E}^n_{\rm T}$. We introduce f is without antipodals as an antonym of f has antipodals.

One can prove the following propositions:

(64) Let n be a non empty natural number, r be a non negative real number, and x be a point of $\mathcal{E}^n_{\rm T}$. Suppose x is a point of $\mathrm{Tcircle}(0_{\mathcal{E}^n_{\rm T}}, r)$. Then x and $-x$ are antipodals of $0_{\mathcal{E}^n_{\rm T}}$ and r.

(65) Let n be a non empty natural number, p, x, y, x_2, y_1 be points of $\mathcal{E}^n_{\rm T}$, and r be a positive real number. Suppose x and y are antipodals of $0_{\mathcal{E}^n_{\rm T}}$ and 1 and $x_2 = (\mathrm{CircleIso}(p, r))(x)$ and $y_1 = (\mathrm{CircleIso}(p, r))(y)$. Then x_2 and y_1 are antipodals of p and r.

(66) Let f be a function from $\mathrm{Tcircle}(0_{\mathcal{E}^{n+1}_{\rm T}}, 1)$ into $\mathcal{E}^n_{\rm T}$ and x be a point of $\mathrm{Tcircle}(0_{\mathcal{E}^{n+1}_{\rm T}}, 1)$. If f is without antipodals, then $f(x) - f(-x) \neq 0_{\mathcal{E}^n_{\rm T}}$.

(67) For every function f from $\mathrm{Tcircle}(0_{\mathcal{E}^{n+1}_{\rm T}}, 1)$ into $\mathcal{E}^n_{\rm T}$ such that f is without antipodals holds $(S^{n+1} \to S^n)\, f$ is odd.

(68) Let f be a function from $\mathrm{Tcircle}(0_{\mathcal{E}^{n+1}_{\rm T}}, 1)$ into $\mathcal{E}^n_{\rm T}$ and g, B_1 be functions from $\mathrm{Tcircle}(0_{\mathcal{E}^{n+1}_{\rm T}}, 1)$ into $\mathcal{E}^n_{\rm T}$. If $g = f \circ -$ and $B_1 = f - g$ and f is without antipodals, then $(S^{n+1} \to S^n)\, f = B_1/(n\, \mathrm{NormF} \cdot B_1)$.

Let us consider n, let r be a negative real number, and let p be a point of $\mathcal{E}^{n+1}_{\rm T}$. Observe that every function from $\mathrm{Tcircle}(p, r)$ into $\mathcal{E}^n_{\rm T}$ is without antipodals.

Let r be a non negative real number and let p be a point of $\mathcal{E}^3_{\rm T}$. Note that every function from $\mathrm{Tcircle}(p, r)$ into $\mathcal{E}^2_{\rm T}$ which is continuous also has antipodals.[2]

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