

# The Differentiable Functions from $\mathbb{R}$ into $\mathcal{R}^n$

Keiko Narita  
Hirosaki-city  
Aomori, Japan

Artur Kornilowicz  
Institute of Informatics  
University of Białystok  
Sosnowa 64, 15-887 Białystok  
Poland

Yasunari Shidama<sup>1</sup>  
Shinshu University  
Nagano, Japan

**Summary.** In control engineering, differentiable partial functions from  $\mathbb{R}$  into  $\mathcal{R}^n$  play a very important role. In this article, we formalized basic properties of such functions.

MML identifier: NDIFF\_4, version: 7.12.02 4.171.1135

The notation and terminology used in this paper are introduced in the following articles: [25], [26], [6], [2], [27], [8], [7], [24], [1], [4], [3], [5], [9], [22], [20], [28], [21], [10], [23], [17], [13], [11], [12], [15], [19], [18], [16], and [14].

Let us observe that there exists a sequence of real numbers which is convergent to 0 and non-zero.

For simplicity, we adopt the following convention:  $x_0, r$  denote real numbers,  $i, m$  denote elements of  $\mathbb{N}$ ,  $n$  denotes a non empty element of  $\mathbb{N}$ ,  $Y$  denotes a subset of  $\mathbb{R}$ ,  $Z$  denotes an open subset of  $\mathbb{R}$ , and  $f_1, f_2$  denote partial functions from  $\mathbb{R}$  to  $\mathcal{R}^n$ .

The following proposition is true

- (1) For all partial functions  $f_1, f_2$  from  $\mathbb{R}$  to  $\mathcal{R}^m$  holds  $f_1 - f_2 = f_1 + -f_2$ .

---

<sup>1</sup>This work was supported by JSPS KAKENHI 22300285.

Let  $n$  be a non empty element of  $\mathbb{N}$ , let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and let  $x$  be a real number. We say that  $f$  is differentiable in  $x$  if and only if:

- (Def. 1) There exists a partial function  $g$  from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $f = g$  and  $g$  is differentiable in  $x$ .

One can prove the following proposition

- (2) Let  $n$  be a non empty element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ ,  $h$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $x$  be a real number. Suppose  $h = f$ . Then  $f$  is differentiable in  $x$  if and only if  $h$  is differentiable in  $x$ .

Let  $n$  be a non empty element of  $\mathbb{N}$ , let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and let  $x$  be a real number. The functor  $f'(x)$  yields an element of  $\mathcal{R}^n$  and is defined as follows:

- (Def. 2) There exists a partial function  $g$  from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $f = g$  and  $f'(x) = g'(x)$ .

One can prove the following proposition

- (3) Let  $n$  be a non empty element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ ,  $h$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $x$  be a real number. If  $h = f$ , then  $f'(x) = h'(x)$ .

Let us consider  $n, f, X$ . We say that  $f$  is differentiable on  $X$  if and only if:

- (Def. 3)  $X \subseteq \text{dom } f$  and for every  $x$  such that  $x \in X$  holds  $f|_X$  is differentiable in  $x$ .

The following propositions are true:

- (4) If  $f$  is differentiable on  $X$ , then  $X$  is a subset of  $\mathbb{R}$ .  
 (5)  $f$  is differentiable on  $Z$  iff  $Z \subseteq \text{dom } f$  and for every  $x$  such that  $x \in Z$  holds  $f$  is differentiable in  $x$ .  
 (6) If  $f$  is differentiable on  $Y$ , then  $Y$  is open.

Let us consider  $n, f, X$ . Let us assume that  $f$  is differentiable on  $X$ . The functor  $f'_{|_X}$  yields a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  and is defined by:

- (Def. 4)  $\text{dom}(f'_{|_X}) = X$  and for every  $x$  such that  $x \in X$  holds  $f'_{|_X}(x) = f'(x)$ .

One can prove the following propositions:

- (7) Suppose  $Z \subseteq \text{dom } f$  and there exists an element  $r$  of  $\mathcal{R}^n$  such that  $\text{rng } f = \{r\}$ . Then  $f$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(f'_{|_Z})_x = \underbrace{\langle 0, \dots, 0 \rangle}_n$ .  
 (8) Let  $x_0$  be a real number,  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ ,  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $N$  be a neighbourhood of  $x_0$ . Suppose  $f = g$  and  $f$  is differentiable in  $x_0$  and  $N \subseteq \text{dom } f$ . Let given  $h, c$ .

Suppose  $\text{rng } c = \{x_0\}$  and  $\text{rng}(h+c) \subseteq N$ . Then  $h^{-1} \cdot ((g_*(h+c)) - (g_*c))$  is convergent and  $f'(x_0) = \lim(h^{-1} \cdot ((g_*(h+c)) - (g_*c)))$ .

- (9) If  $f$  is differentiable in  $x_0$ , then  $r \cdot f$  is differentiable in  $x_0$  and  $(r \cdot f)'(x_0) = r \cdot f'(x_0)$ .
- (10) If  $f$  is differentiable in  $x_0$ , then  $-f$  is differentiable in  $x_0$  and  $(-f)'(x_0) = -f'(x_0)$ .
- (11) If  $f_1$  is differentiable in  $x_0$  and  $f_2$  is differentiable in  $x_0$ , then  $f_1 + f_2$  is differentiable in  $x_0$  and  $(f_1 + f_2)'(x_0) = f_1'(x_0) + f_2'(x_0)$ .
- (12) If  $f_1$  is differentiable in  $x_0$  and  $f_2$  is differentiable in  $x_0$ , then  $f_1 - f_2$  is differentiable in  $x_0$  and  $(f_1 - f_2)'(x_0) = f_1'(x_0) - f_2'(x_0)$ .
- (13) Suppose  $Z \subseteq \text{dom } f$  and  $f$  is differentiable on  $Z$ . Then  $r \cdot f$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(r \cdot f)'|_Z(x) = r \cdot f'(x)$ .
- (14) If  $Z \subseteq \text{dom } f$  and  $f$  is differentiable on  $Z$ , then  $-f$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(-f)'|_Z(x) = -f'(x)$ .
- (15) Suppose  $Z \subseteq \text{dom}(f_1 + f_2)$  and  $f_1$  is differentiable on  $Z$  and  $f_2$  is differentiable on  $Z$ . Then  $f_1 + f_2$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(f_1 + f_2)'|_Z(x) = f_1'(x) + f_2'(x)$ .
- (16) Suppose  $Z \subseteq \text{dom}(f_1 - f_2)$  and  $f_1$  is differentiable on  $Z$  and  $f_2$  is differentiable on  $Z$ . Then  $f_1 - f_2$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $(f_1 - f_2)'|_Z(x) = f_1'(x) - f_2'(x)$ .
- (17) If  $Z \subseteq \text{dom } f$  and  $f|_Z$  is constant, then  $f$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $f'|_Z(x) = \underbrace{\langle 0, \dots, 0 \rangle}_n$ .
- (18) Let  $r, p$  be elements of  $\mathcal{R}^n$ . Suppose  $Z \subseteq \text{dom } f$  and for every  $x$  such that  $x \in Z$  holds  $f_x = x \cdot r + p$ . Then  $f$  is differentiable on  $Z$  and for every  $x$  such that  $x \in Z$  holds  $f'|_Z(x) = r$ .
- (19) For every real number  $x_0$  such that  $f$  is differentiable in  $x_0$  holds  $f$  is continuous in  $x_0$ .
- (20) If  $f$  is differentiable on  $X$ , then  $f|_X$  is continuous.
- (21) If  $f$  is differentiable on  $X$  and  $Z \subseteq X$ , then  $f$  is differentiable on  $Z$ .

Let  $n$  be a non empty element of  $\mathbb{N}$  and let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . We say that  $f$  is differentiable if and only if:

(Def. 5)  $f$  is differentiable on  $\text{dom } f$ .

Let us consider  $n$ . One can check that  $\mathbb{R} \mapsto \underbrace{\langle 0, \dots, 0 \rangle}_n$  is differentiable.

Let us consider  $n$ . Note that there exists a function from  $\mathbb{R}$  into  $\mathcal{R}^n$  which is differentiable.

One can prove the following proposition

- (22) For every differentiable partial function  $f$  from  $\mathbb{R}$  to  $\mathcal{R}^n$  such that  $Z \subseteq \text{dom } f$  holds  $f$  is differentiable on  $Z$ .

In the sequel  $G_1, R$  are rests of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $D_1, L$  are linears of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ .

Next we state a number of propositions:

- (23) Let  $R$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $R$  is total. Then  $R$  is rest-like if and only if for every real number  $r$  such that  $r > 0$  there exists a real number  $d$  such that  $d > 0$  and for every real number  $z$  such that  $z \neq 0$  and  $|z| < d$  holds  $|z|^{-1} \cdot \|R_z\| < r$ .
- (24) Let  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $x_0$  be a real number. Suppose  $1 \leq i \leq n$  and  $g$  is differentiable in  $x_0$ . Then  $\text{Proj}(i, n) \cdot g$  is differentiable in  $x_0$  and  $(\text{Proj}(i, n))(g'(x_0)) = (\text{Proj}(i, n) \cdot g)'(x_0)$ .
- (25) Let  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $x_0$  be a real number. Then  $g$  is differentiable in  $x_0$  if and only if for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq n$  holds  $\text{Proj}(i, n) \cdot g$  is differentiable in  $x_0$ .
- (26) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  and  $x_0$  be a real number. Suppose  $1 \leq i \leq n$  and  $f$  is differentiable in  $x_0$ . Then  $\text{Proj}(i, n) \cdot f$  is differentiable in  $x_0$  and  $(\text{Proj}(i, n))(f'(x_0)) = (\text{Proj}(i, n) \cdot f)'(x_0)$ .
- (27) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  and  $x_0$  be a real number. Then  $f$  is differentiable in  $x_0$  if and only if for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq n$  holds  $\text{Proj}(i, n) \cdot f$  is differentiable in  $x_0$ .
- (28) Let  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $1 \leq i \leq n$  and  $g$  is differentiable on  $X$ . Then  $\text{Proj}(i, n) \cdot g$  is differentiable on  $X$  and  $\text{Proj}(i, n) \cdot g'_{\upharpoonright X} = (\text{Proj}(i, n) \cdot g)'_{\upharpoonright X}$ .
- (29) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Suppose  $1 \leq i \leq n$  and  $f$  is differentiable on  $X$ . Then  $\text{Proj}(i, n) \cdot f$  is differentiable on  $X$  and  $\text{Proj}(i, n) \cdot f'_{\upharpoonright X} = (\text{Proj}(i, n) \cdot f)'_{\upharpoonright X}$ .
- (30) Let  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Then  $g$  is differentiable on  $X$  if and only if for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq n$  holds  $\text{Proj}(i, n) \cdot g$  is differentiable on  $X$ .
- (31) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Then  $f$  is differentiable on  $X$  if and only if for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq n$  holds  $\text{Proj}(i, n) \cdot f$  is differentiable on  $X$ .
- (32) For every function  $J$  from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  into  $\mathbb{R}$  and for every point  $x_0$  of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  such that  $J = \text{proj}(1, 1)$  holds  $J$  is continuous in  $x_0$ .
- (33) For every function  $I$  from  $\mathbb{R}$  into  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  such that  $I = \text{proj}(1, 1)^{-1}$  holds  $I$  is continuous in  $x_0$ .
- (34) Let  $S, T$  be real normed spaces,  $f_1$  be a partial function from  $S$  to  $\mathbb{R}$ ,  $f_2$  be a partial function from  $\mathbb{R}$  to  $T$ , and  $x_0$  be a point of  $S$ . Suppose  $x_0 \in \text{dom}(f_2 \cdot f_1)$  and  $f_1$  is continuous in  $x_0$  and  $f_2$  is continuous in  $(f_1)_{x_0}$ . Then  $f_2 \cdot f_1$  is continuous in  $x_0$ .
- (35) Let  $J$  be a function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  into  $\mathbb{R}$ ,  $x_0$  be a point of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $y_0$  be an element of  $\mathbb{R}$ ,  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $f$

be a partial function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $J = \text{proj}(1, 1)$  and  $x_0 \in \text{dom } f$  and  $y_0 \in \text{dom } g$  and  $x_0 = \langle y_0 \rangle$  and  $f = g \cdot J$ . Then  $f$  is continuous in  $x_0$  if and only if  $g$  is continuous in  $y_0$ .

(36) Let  $I$  be a function from  $\mathbb{R}$  into  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $x_0$  be a point of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $y_0$  be an element of  $\mathbb{R}$ ,  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $f$  be a partial function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $I = \text{proj}(1, 1)^{-1}$  and  $x_0 \in \text{dom } f$  and  $y_0 \in \text{dom } g$  and  $x_0 = \langle y_0 \rangle$  and  $f \cdot I = g$ . Then  $f$  is continuous in  $x_0$  if and only if  $g$  is continuous in  $y_0$ .

(37) For every function  $I$  from  $\mathbb{R}$  into  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  such that  $I = \text{proj}(1, 1)^{-1}$  holds  $I$  is differentiable in  $x_0$  and  $I'(x_0) = \langle 1 \rangle$ .

Let  $n$  be a non empty element of  $\mathbb{N}$ , let  $f$  be a partial function from  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  to  $\mathbb{R}$ , and let  $x$  be a point of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . We say that  $f$  is differentiable in  $x$  if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a partial function  $g$  from  $\mathcal{R}^n$  to  $\mathbb{R}$  and there exists an element  $y$  of  $\mathcal{R}^n$  such that  $f = g$  and  $x = y$  and  $g$  is differentiable in  $y$ .

Let  $n$  be a non empty element of  $\mathbb{N}$ , let  $f$  be a partial function from  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  to  $\mathbb{R}$ , and let  $x$  be a point of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . The functor  $f'(x)$  yields a function from  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  into  $\mathbb{R}$  and is defined by:

(Def. 7) There exists a partial function  $g$  from  $\mathcal{R}^n$  to  $\mathbb{R}$  and there exists an element  $y$  of  $\mathcal{R}^n$  such that  $f = g$  and  $x = y$  and  $f'(x) = g'(y)$ .

We now state several propositions:

(38) Let  $J$  be a function from  $\mathcal{R}^1$  into  $\mathbb{R}$  and  $x_0$  be an element of  $\mathcal{R}^1$ . If  $J = \text{proj}(1, 1)$ , then  $J$  is differentiable in  $x_0$  and  $J'(x_0) = J$ .

(39) Let  $J$  be a function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  into  $\mathbb{R}$  and  $x_0$  be a point of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ . If  $J = \text{proj}(1, 1)$ , then  $J$  is differentiable in  $x_0$  and  $J'(x_0) = J$ .

(40) Let  $I$  be a function from  $\mathbb{R}$  into  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ . Suppose  $I = \text{proj}(1, 1)^{-1}$ . Then

(i) for every rest  $R$  of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  holds  $R \cdot I$  is a rest of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and

(ii) for every linear operator  $L$  from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  holds  $L \cdot I$  is a linear of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ .

(41) Let  $J$  be a function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  into  $\mathbb{R}$ . Suppose  $J = \text{proj}(1, 1)$ . Then

(i) for every rest  $R$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  holds  $R \cdot J$  is a rest of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and

(ii) for every linear  $L$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  holds  $L \cdot J$  is a bounded linear operator from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ .

(42) Let  $I$  be a function from  $\mathbb{R}$  into  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $x_0$  be a point of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $y_0$  be an element of  $\mathbb{R}$ ,  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $f$  be a partial function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $I = \text{proj}(1, 1)^{-1}$  and  $x_0 \in \text{dom } f$  and  $y_0 \in \text{dom } g$  and  $x_0 = \langle y_0 \rangle$  and  $f \cdot I = g$  and  $f$  is

differentiable in  $x_0$ . Then  $g$  is differentiable in  $y_0$  and  $g'(y_0) = f'(x_0)(\langle 1 \rangle)$  and for every element  $r$  of  $\mathbb{R}$  holds  $f'(x_0)(\langle r \rangle) = r \cdot g'(y_0)$ .

- (43) Let  $I$  be a function from  $\mathbb{R}$  into  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $x_0$  be a point of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $y_0$  be an element of  $\mathbb{R}$ ,  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $f$  be a partial function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $I = \text{proj}(1, 1)^{-1}$  and  $x_0 \in \text{dom } f$  and  $y_0 \in \text{dom } g$  and  $x_0 = \langle y_0 \rangle$  and  $f \cdot I = g$ . Then  $f$  is differentiable in  $x_0$  if and only if  $g$  is differentiable in  $y_0$ .
- (44) Let  $J$  be a function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  into  $\mathbb{R}$ ,  $x_0$  be a point of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $y_0$  be an element of  $\mathbb{R}$ ,  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $f$  be a partial function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $J = \text{proj}(1, 1)$  and  $x_0 \in \text{dom } f$  and  $y_0 \in \text{dom } g$  and  $x_0 = \langle y_0 \rangle$  and  $f = g \cdot J$ . Then  $f$  is differentiable in  $x_0$  if and only if  $g$  is differentiable in  $y_0$ .
- (45) Let  $J$  be a function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  into  $\mathbb{R}$ ,  $x_0$  be a point of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $y_0$  be an element of  $\mathbb{R}$ ,  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $f$  be a partial function from  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $J = \text{proj}(1, 1)$  and  $x_0 \in \text{dom } f$  and  $y_0 \in \text{dom } g$  and  $x_0 = \langle y_0 \rangle$  and  $f = g \cdot J$  and  $g$  is differentiable in  $y_0$ . Then  $f$  is differentiable in  $x_0$  and  $g'(y_0) = f'(x_0)(\langle 1 \rangle)$  and for every element  $r$  of  $\mathbb{R}$  holds  $f'(x_0)(\langle r \rangle) = r \cdot g'(y_0)$ .
- (46) Let  $R$  be a rest of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $R_0 = 0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}$ . Let  $e$  be a real number. Suppose  $e > 0$ . Then there exists a real number  $d$  such that  $d > 0$  and for every real number  $h$  such that  $|h| < d$  holds  $\|R_h\| \leq e \cdot |h|$ .

In the sequel  $m, n$  denote non empty elements of  $\mathbb{N}$ .

One can prove the following propositions:

- (47) For every rest  $R$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and for every bounded linear operator  $L$  from  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  into  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  holds  $L \cdot R$  is a rest of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ .
- (48) Let  $R_1$  be a rest of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $(R_1)_0 = 0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}$ . Let  $R_2$  be a rest of  $\langle \mathcal{E}^n, \|\cdot\| \rangle, \langle \mathcal{E}^m, \|\cdot\| \rangle$ . Suppose  $(R_2)_{0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}} = 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}$ . Let  $L$  be a linear of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Then  $R_2 \cdot (L + R_1)$  is a rest of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ .
- (49) Let  $R_1$  be a rest of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $(R_1)_0 = 0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}$ . Let  $R_2$  be a rest of  $\langle \mathcal{E}^n, \|\cdot\| \rangle, \langle \mathcal{E}^m, \|\cdot\| \rangle$ . Suppose  $(R_2)_{0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}} = 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}$ . Let  $L_1$  be a linear of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $L_2$  be a bounded linear operator from  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  into  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ . Then  $L_2 \cdot R_1 + R_2 \cdot (L_1 + R_1)$  is a rest of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ .
- (50) Let  $x_0$  be an element of  $\mathbb{R}$  and  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $g$  is differentiable in  $x_0$ . Let  $f$  be a partial function from  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ . Suppose  $f$  is differentiable in  $g_{x_0}$ . Then  $f \cdot g$  is differentiable in  $x_0$  and  $(f \cdot g)'(x_0) = f'(g_{x_0})(g'(x_0))$ .

## REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.

- [3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(3):529–536, 1990.
- [6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [9] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(4):661–668, 1990.
- [10] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(4):599–603, 1991.
- [11] Noboru Endou and Yasunari Shidama. Completeness of the real Euclidean space. *Formalized Mathematics*, 13(4):577–580, 2005.
- [12] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. Partial differentiation on normed linear spaces  $\mathcal{R}^n$ . *Formalized Mathematics*, 15(2):65–72, 2007, doi:10.2478/v10037-007-0008-5.
- [13] Hiroshi Imura, Morishige Kimura, and Yasunari Shidama. The differentiable functions on normed linear spaces. *Formalized Mathematics*, 12(3):321–327, 2004.
- [14] Takao Inoué, Adam Naumowicz, Noboru Endou, and Yasunari Shidama. Partial differentiation of vector-valued functions on  $n$ -dimensional real normed linear spaces. *Formalized Mathematics*, 19(1):1–9, 2011, doi: 10.2478/v10037-011-0001-x.
- [15] Keiichi Miyajima and Yasunari Shidama. Riemann integral of functions from  $\mathbb{R}$  into  $\mathcal{R}^n$ . *Formalized Mathematics*, 17(2):179–185, 2009, doi: 10.2478/v10037-009-0021-y.
- [16] Keiko Narita, Artur Kornilowicz, and Yasunari Shidama. More on the continuity of real functions. *Formalized Mathematics*, 19(4):233–239, 2011, doi: 10.2478/v10037-011-0032-3.
- [17] Takaya Nishiyama, Keiji Ohkubo, and Yasunari Shidama. The continuous functions on normed linear spaces. *Formalized Mathematics*, 12(3):269–275, 2004.
- [18] Hiroyuki Okazaki, Noboru Endou, Keiko Narita, and Yasunari Shidama. Differentiable functions into real normed spaces. *Formalized Mathematics*, 19(2):69–72, 2011, doi: 10.2478/v10037-011-0012-7.
- [19] Hiroyuki Okazaki, Noboru Endou, and Yasunari Shidama. More on continuous functions on normed linear spaces. *Formalized Mathematics*, 19(1):45–49, 2011, doi: 10.2478/v10037-011-0008-3.
- [20] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [21] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(1):111–115, 1991.
- [22] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [23] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(1):39–48, 2004.
- [24] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [28] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(2):171–175, 1992.

*Received September 28, 2011*